STABILITY OF REGULAR ROLL WAVES*

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Для неоднородных гиперболических уравнений газодинамического типа исследована устойчивость периодических бегущих волн конечной амплитуды. Критерий устойчивости формулируется как условие гиперболичности уравнений модуляций для периодических волновых пакетов (катящихся волн). Выведены асимптотические формулы для границ нелинейной устойчивости катящихся волн малой амплитуды, а также для катящихся волн максимальной амплитуды. Для наклонных автомодельных каналов построены диаграммы устойчивости периодических течений.

Introduction

Shallow water flows down inclined open channels are of considerable interest in drainage problems. One of the unsteady flow patterns that are frequently encountered by hydraulic engineers and which occur in turbulent flow regimes is the development of roll waves. Roll waves are quasi-periodic waves of finite amplitude generated in a channel due to instability of the uniform steady-state flow. For their appearance the channel must be of sufficient length and the angle of inclination should exceed a critical value. In the initial phase of their development the waves are of small amplitude and the free surface is smooth. After attaining critical conditions they may transform into breaking bores propagating at almost constant speed and followed by smooth long waves. Such behaviour have been observed in laboratory by many investigators (Brock, 1969; Mayer, 1981; Alavian, 1986).

The mathematical theory of roll waves as travelling discontinuous periodic solutions of the shallow water equations have been derived by Dressler (Dressler, 1949). In this theory the length of roll waves is the free parameter, and the flow characteristics of saturated waves, which stop to grow after attaining some critical amplitude, cannot be predicted. Therefore, the question on stability of travelling waves of finite amplitude arises. Dissipative shallow water models (Needham and Merkin, 1981; Kranenburg, 1992; Yu and Kevorkian, 1992) let construct smooth periodic solutions describing the roll wave phenomenon, nevertheless in these models the length of periodic waves is a free parameter again.

*The research reported in this paper was supported in part by CNRS-RAS (project 16364) and by RFBR (project 04-01-00253).

For shallow water equations describing one-layer flow with turbulent friction over an incline a criterion of nonlinear stability of roll waves have been derived in (Liapidevskii, 1998). It is based on modulation equations for periodic wave packets with slowly varying flow parameters along the channel (Whitham, 1974; Bhatnagar, 1979). A roll wave is stable if corresponding modulation equations are hyperbolic. It is shown in (Liapidevskii, 1998) that the system is hyperbolic for wave packets of finite amplitude having corresponding wave numbers in rather narrow band and that the scatter of amplitudes of stable roll waves is very small. For the channels of arbitrary cross-section this approach have been generalized in (Boudlal and Liapidevskii, 2002).

Our paper aims to present a nonlinear study on stability of finite amplitude roll waves occurring in inclined open channels of arbitrary cross-section. Throughout the paper, the averaged equations for one-dimensional flow with turbulent friction on the wall are considered. Starting from these equations, standard roll waves, i. e., periodic travelling waves are constructed by matching smooth solutions with stable hydraulic jumps. As the amplitude and the phase velocity of waves are slowly changing during their propagation downstream a channel, the stability problem raises. Here this problem is solved by deriving modulation equations for wave packets. The stability criterion for nonlinear roll waves is then expressed in terms of hyperbolicity of modulation equations that need calculation of averaged quantities. The main difficulty to establish the stability domain on a roll wave diagram is due to the singularities in the hyperbolicity condition of modulation equations for the waves of infinitesimal and maximal amplitude.

Using an asymptotic analysis, the stability conditions of roll waves of small and maximal amplitude, as well as the approximate position of boundaries of the hyperbolicity domain for the channels of arbitrary cross-section are obtained. Note that for arbitrary shape of the channel the stability condition depends on two governing parameters. Nevertheless, for the class of self–similar channels introduced in the paper, the modulation equations take a rather simple form and the hyperbolicity criterion is reduced to a condition for a function of one variable. Numerical calculations of stability diagrams for self-similar channels corresponding to an inclined plane as well as to a channel of triangular cross-section are presented.

1. Examples of flow

We consider periodic travelling waves for the hyperbolic system of gas dynamic type

$$\begin{aligned} h_t + (hu)_x &= 0, \\ (hu)_t + (hu^2 + P)_x &= f. \end{aligned} \tag{1.1}$$

Here $P = P(h)$ is the given pressure function with $c^2 = P'(h) > 0$ and the function $f = f(h,u)$ describes the interaction between flow acceleration and friction forces. Eqs (1.1) appear in different fields of the continuum mechanics (gas dynamics, hydraulics, elasticity theory etc.). But concerning the problem of roll wave generation they are mostly used in the open channel flow theory. Therefore, we refer to (1.1) as to the shallow water equations for the one-dimensional unsteady flow in channels of arbitrary cross section (Dyment, 1981; Boudlal, 1986). It means that t is the time, x is the coordinate along the channel, h and u are the mean depth and velocity of the liquid layer. It is supposed throughout the paper that the pressure function is convex ($P''(h) > 0$) and the right hand term is chosen in the form

$$f = h\left(\alpha - u^2/\mathcal{R}(h)\right),$$

where $\alpha = \text{const}$ characterizes the acceleration along the channel due to gravity and $\mathcal{R}(h)$ is the hydraulic radius. In this case the flow patterns are similar to the classic roll waves over an incline investigated in (Dressler, 1949), and the nonlinear stability criterion derived in (Liapidevskii, 1998) can be modified for (1.1). We reserve the term “regular roll waves” for the discontinuous periodic waves travelling with a constant velocity $\mathcal{D}$. In the frame of reference moving with the wave velocity the flow is steady and it can be described by the solution of (1.1), which is dependent on the variable $\xi = x - \mathcal{D}t$ (fig. 1). Eqs (1.1) take the form

$$h(\mathcal{D} - u) = y(\mathcal{D} - u_c) = m, \qquad \frac{dG(h)}{d\xi} = F(h). \tag{1.2}$$

Here y and u_c are the critical depth and critical velocity with $\Delta(y) = 0$,

$$G(h) = m^2/h + P(h), \quad F(h) = f(h, \mathcal{D} - m/h), \qquad \Delta(h) = G'(h) = c^2(h) - m^2/h^2. \tag{1.3}$$

For regular roll waves the only jump on the period divides the monotone smooth parts of flow. Therefore, it is necessary for the roll wave existence that the subcritical flow behind the jump ($\Delta > 0$) transforms into the supercritical flow ($\Delta < 0$) before the next jump, and there exists the critical depth y on the period. Equations (1.2) can be rewritten as follows

$$\frac{dh}{d\xi} = \frac{F(h)}{\Delta(h)}. \tag{1.4}$$

Therefore, for the roll wave existence it is necessary that $F(h)$ and $\Delta(h)$ vanish at the critical depth y simultaneously, i. e.,

$$F(y) = 0, \quad \Delta(y) = 0. \tag{1.5}$$

It follows from (1.5) that

$$m(y) = yc(y), \quad f(y, u_c) = F(y) = 0. \tag{1.6}$$

We can also find from (1.5), (1.6) the dependences $u_c = u_c(y)$, $\mathcal{D} = u_c(y) + c(y)$. For the shallow water equations (1.1), (1.2) we have $u_c = \pm\sqrt{\alpha R(y)}$ and the roll wave is defined when

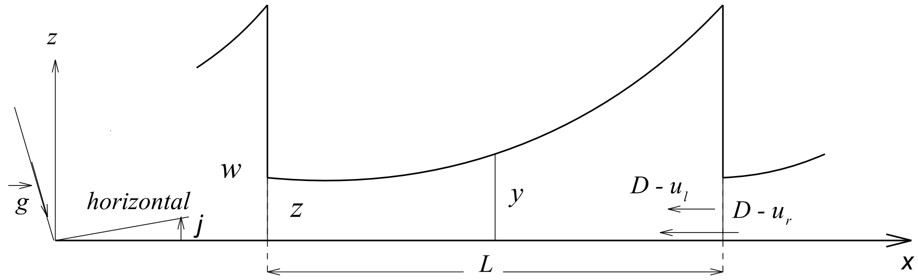

Fig. 1. The structure of roll waves in a regular channel ($P'' > 0$). L is the length of a wave, z, u_r and w, u_l are the depth and velocity respectively upstream and downstream the jump relative to the waves.

the sign of the constant m is chosen. Further we consider waves moving on the right ($0 < u < \mathcal{D}$) with $u_c = \sqrt{\alpha R(y)}$. Nevertheless, the stability criterion derived below can be applied for the right hand term f of the more general form. For regular channels $\Delta'(h) = P''(h) + 2m^2/h^3 > 0$, therefore there is the only one critical depth y on the wave period with $\Delta(h) < 0$ for $h < y$ and $\Delta(h) > 0$ for $h > y$. As was mentioned above, we consider the regular periodic solution of (1.1) with stable jumps. It means that $\Delta(z) < 0$ for the depth z just before the jump and $\Delta(w) > 0$ for the depth w just behind the jump and the wave depth between the jumps is the monotone increasing function of ξ ($z < y < w$). The jump relation for the conjugate depths z and w can be written in the form

$$G(z) = G(w). \tag{1.7}$$

Due to (1.4) it is necessary and sufficient for the existence of roll wave with the jump connecting the conjugate depths z and w ($z < w$) that the following conditions are fulfilled

$$\begin{aligned} F(h) < 0 \quad &\text{for} \quad z < h < y, \\ F(h) > 0 \quad &\text{for} \quad y < h < w. \end{aligned} \tag{1.8}$$

It follows from (1.8) that

$$F'(y) > 0. \tag{1.9}$$

Note that (1.9) is exactly the criterion of instability of a steady-state flow derived by the linear analysis and by the Whitham's method (Boudlal, 1993; Boudlal and Liapidevskii, 2003). For Eqs (1.2) the criterion (1.9) takes the form

$$F'(y) = \alpha y R'(y)/R(y) - 2c(y)\sqrt{\alpha/\mathcal{R}(y)} > 0. \tag{1.10}$$

For the plane case with $P(h) = g_n h^2/2$, $\mathcal{R}(h) = h/c_w$, where g_n is the component of gravity acceleration which is normal to the plane, c_w is the constant friction coefficient, $\alpha = g_n \tan\varphi$, φ is the angle of inclination, the condition (1.10) leads to the well-known criterion of stability (Jeffreys, 1925; Whitham, 1974),

$$\mathrm{Fr} = u_c/\sqrt{g_n y} = \sqrt{\tan\varphi/c_w} > 2. \tag{1.11}$$

Here the Froude number Fr is calculated for the steady flow with the depth y and the velocity $u_c = \sqrt{\alpha y/c_w}$.

The instability condition (1.9) provides the existence of roll waves of infinitesimal amplitude. For roll waves of finite amplitude the conditions (1.8) are not satisfied when $F(z_*) = 0$ or $F(w_*) = 0$ with $z_* < y < w_*$. Let us call the roll wave with the critical depth y regular only if there exists the minimal depth $z_* < y$ and the conjugate maximal depth $w_* > y$ such that $F(z_*) = 0$ or $F(w_*) = 0$. Further in the paper we consider only the case $F(z_*) = 0$, $F'(z_*) \neq 0$.

Let z_* be the closest to the critical depth y ($z_* < y$). Then the conditions (1.8) are satisfied for conjugate depths z, w with $z_* < z < y < w < w_*$. Since $\Delta(z_*) \neq 0$ it follows from (1.4) that the length of the roll wave goes to infinity for $z \to z_*$. Note also that the amplitude of the limiting roll wave is finite.

2. Modulation equations

Every regular periodic solution of (1.1) is defined by two parameters, say, y and z. For any y satisfying (1.9) the admissible conjugate depths z and w in a roll wave belong to the interval

$(z_*(y), w_*(y))$. Therefore, the set of admissible parameters corresponding to regular roll waves represent a domain in $(z, y) - plane$. The problem on nonlinear stability of periodic wave trains with slowly varying governing parameters (z, y) can be solved by analysis of the modulation equations for such trains (Bhatnagar, 1979; Whitham, 1974). When averaging (1.1) over the fixed length scale, which is large enough comparing with the length of the regular roll waves in a train with slowly varying functions z and y, we have the modulation equations

$$\begin{aligned} \bar{h}_t + (\overline{hu})_x &= 0, \\ (\overline{hu})_t + (\overline{hu^2 + P})_x &= \bar{f}. \end{aligned} \tag{2.1}$$

Furthermore, we replace the averaged quantities by the corresponding mean values on the wave period

$$\begin{aligned} \bar{h} &= \frac{1}{L}\int_0^L h(\xi)d\xi = \frac{1}{L}\int_z^w sa(s, y)ds, \\ \bar{G} &= \frac{1}{L}\int_0^L G(h(\xi), y)d\xi = \frac{1}{L}\int_z^w G(s, y)a(s, y)ds, \\ L &= \int_0^L d\xi = \int_z^w a(s, y)ds, \end{aligned} \tag{2.2}$$

where $a(s, y) = \dfrac{d\xi}{ds} = \dfrac{\Delta(s, y)}{F(s, y)} > 0.$

Note that the functions G, Δ, F, a depend not only on s, but on the parameter y too. Therefore we consider them below as the functions of two variables s and y. The functions m and $\mathcal{D}$ in (1.3) are the functions of one variable y. The mean quantities in (2.1) can be presented from (1.3)–(1.6) as the functions of two variables z and y in the following way

$$\begin{aligned} \overline{hu} &= \mathcal{D}\bar{h} - m, \\ \overline{hu^2 + P} &= \mathcal{D}^2\bar{h} - 2\mathcal{D}m + \bar{G}, \\ \bar{f} &= 0. \end{aligned} \tag{2.3}$$

The last relation in (2.3) follows from the fact that the function f is the derivative of the periodic function G and from the jump condition (1.7).

The nonstationary evolution of the governing parameters for periodic wave trains is described by (2.1)–(2.3). We say that the roll waves are stable if the modulation equations (2.1)–(2.3) for corresponding values (y, z) are hyperbolic. The investigation of hyperbolicity of the modulation equations can be performed more easily for the variables y and $\bar{h}$. It can be done by the transformation $\bar{h} = \bar{h}(z, y)$ and $\tilde{G}(\bar{h}, y) = \bar{G}(z, y)$. The modulation equations take the form

$$\begin{aligned} \bar{h}_t + (\mathcal{D}\bar{h} - m)_x &= 0, \\ (\mathcal{D}\bar{h} - m)_t + (\mathcal{D}\bar{h} - 2\mathcal{D}m + \tilde{G})_x &= 0 \end{aligned} \tag{2.4}$$

and the characteristics of (2.4) are

$$\frac{dx}{dt} = \mathcal{D} \pm \frac{\sqrt{R^2 + 4\delta^2\tilde{G}_{\bar{h}}}}{2\delta},$$

with $\delta = \bar{h}\mathcal{D}' - m'$, $R = \tilde{G}_y - 2m\mathcal{D}'$. Here the "prime" denotes the full derivative on the variable y. The hyperbolicity condition for (2.4) is

$$R^2 + 4\delta^2 \tilde{G}_{\bar{h}} > 0. \tag{2.5}$$

For the variables (y, z) the hyperbolicity condition (2.5) can be expressed in the form

$$\left(\bar{G}_y - \bar{G}_z \bar{h}_y/\bar{h}_z - 2m\mathcal{D}'\right)^2 + 4\delta^2 \bar{G}_z/\bar{h}_z > 0. \tag{2.6}$$

To check the stability criterion (2.5) or (2.6) for given roll wave train, we have to resolve the singularities in (2.2) for the waves of infinitesimal ($L \to 0$) and limiting ($L \to \infty$) amplitude. The asymptotic analysis for such waves is performed in the next paragraph. The investigation of the hyperbolicity domain of modulation equations for self-similar channels, in which (2.5) can be reduced to a condition for the function of one variable, is done in §4.

3. Asymptotic analysis of roll waves

In this paragraph the stability conditions for roll waves of infinitesimal and maximal amplitude are given in such form that (2.5) can be verified without calculation of the mean values by the formulae (2.2), as well as the approximation of the boundaries of the hyperbolicity domain for roll waves of near maximal amplitude is presented.

3.1. Stability of roll waves of infinitesimal amplitude

Let y be the given critical depth and the corresponding steady-state flow with the depth y is unstable, i. e.,

$$F_z(z, y)|_{z=y} > 0.$$

Then

$$a(y, y) = \lim_{z \to y} a(z, y) = \lim_{z \to y} \frac{F(z, y)}{\Delta(z, y)} = \frac{F_z(y, y)}{\Delta_z(y, y)} > 0. \tag{3.1}$$

The conjugate depth $w = w(z, y)$ and the corresponding derivatives at $z = y$ can be calculated from (1.7) as follows

$$\begin{gathered} w(y, y) = y, \quad w_z(y, y) = -1, \quad w_y(y, y) = 2, \\ w_{zz}(y, y) = -\frac{2}{3}\frac{G_{zzz}(y, y)}{G_{zz}(y, y)}. \end{gathered} \tag{3.2}$$

For the derivatives of the mean values in (2.2) one obtains

$$\begin{aligned} \bar{h}_z &= \frac{1}{L}\left(a(w, y)(w - \bar{h})w_z - a(z, y)(z - \bar{h})\right) = \frac{H}{L}, \\ \bar{G}_z &= \frac{1}{L}\left(a(w, y)(G(w, y) - \bar{G})w_z - a(z, y)(G(z, y) - \bar{G})\right) = \frac{Q}{L}. \end{aligned} \tag{3.3}$$

With the functions $H = H(z, y)$ and $Q = Q(z, y)$ defined in (3.3) we have

$$\begin{gathered} \bar{h}_z|_{z=y} = \lim\nolimits_{z \to y} \frac{H(z, y)}{L(z, y)} = \frac{H_z(y, y)}{L_z(y, y)}, \\ \bar{G}_z|_{z=y} = \lim\nolimits_{z \to y} \frac{Q(z, y)}{L(z, y)} = \frac{Q_z(y, y)}{L_z(y, y)}, \\ L_z(y, y) = -2a(y, y) < 0. \end{gathered} \tag{3.4}$$

It follows from (3.2)–(3.4) that

$$\bar{h}_z \mid_{z=y} = -\bar{h}_z \mid_{z=y} = 0, \quad \bar{G}_z \mid_{z=y} = -\bar{G}_z \mid_{z=y} = 0. \tag{3.5}$$

The second derivatives of H and Q can be expressed at $z = y$ by formulae

$$\begin{aligned} H_{zz} &= -4a_{zz} + 2a\bar{h}_{zz} - 3aw_{zz}, \\ Q_{zz} &= -2a(G_{zz} - \bar{G}_{zz}). \end{aligned} \tag{3.6}$$

Therefore, at $z = y$ we have from (3.3)

$$\begin{aligned} \bar{h}_{zz} &= \frac{H_{zz}}{L_{zz}} - \bar{h}_{zz}, \\ \bar{G}_{zz} &= \frac{Q_{zz}}{L_{zz}} - \bar{G}_{zz}, \end{aligned} \tag{3.7}$$

and finally, from (3.4)–(3.7) one obtains

$$\begin{aligned} \bar{h}_{zz}(y,y) &= \frac{2}{3}\left(\frac{a_z(y,y)}{a(y,y)} - \frac{1}{2}\frac{G_{zzz}(y,y)}{G_{zz}(y,y)}\right), \\ \bar{G}_{zz}(y,y) &= \frac{1}{3}G_{zz}(y,y), \\ \tilde{G}_{\bar{h}} \mid_{z=y} = \lim_{z\to y} \frac{\bar{G}_z(z,y)}{\bar{h}_z(z,y)} &= \frac{\bar{G}_{zz}(y,y)}{\bar{h}_{zz}(y,y)}. \end{aligned} \tag{3.8}$$

To find the sign of (2.5), we need also the expression of $\bar{h}_y$ and $\bar{G}_y$ at $z = y$. It follows from (2.2) that

$$\begin{aligned} \bar{h}_y(y,y) &= \frac{1}{2}w_y(y,y) = 1, \\ \bar{G}_y(y,y) = G_y(y,y) &= \frac{2m(y)m'(y)}{y}. \end{aligned} \tag{3.9}$$

Now the stability of roll waves of infinitesimal amplitude can be checked by (2.5), (3.8), (3.9) using only values of the known functions $a(s,y)$, $G(s,y)$, $m(y)$ and their derivatives at $s = y$. It will be shown in § 4 that for the wide class of flows in self-similar channels, including the flows over an incline, the infinitesimal roll waves are unstable and that roll waves become stable when they reach the near maximal amplitude. The asymptotic formulae for long roll waves are derived in the next section.

3.2. Stability of limiting roll waves

Suppose that regular roll waves are defined for every critical depth y from an interval. It means that there are the smooth functions $z_* = z_*(y)$ and $w_* = w(z_*(y), y)$ and the conditions (1.8) are satisfied for conjugate depths z, w with $G(z) = G(w)$ and $z_* < z < y < w < w_*$, $F(z_*, y) = 0$,

$F_z(z_*, y) \neq 0$, $F(w_*, y) \neq 0$. When $z \to z_*$ we can use the asymptotic formulae

$$\begin{gathered} a(s,y) = \frac{b(s,y)}{s - z_*}, \quad b(z_*, y) \neq 0, \\ L(z,y) = \int\limits_z^w \frac{(b(s,y) - b(z_*, y))}{s - z_*} ds + b(z_*, y) \ln \frac{w - z_*}{z - z_*}|_{z \to z_*} \to \infty, \\ \bar{h}(z,y) = \frac{1}{L} \int\limits_z^w (s - z_*) a(s,y) ds + z_*|_{z \to z_*} \to z_*, \\ \bar{G}(z,y) = \frac{1}{L} \int\limits_z^w (G(s,y) - G(z_*, y)) a(s,y) ds + G(z_*, y)|_{z \to z_*} \to G(z_*, y). \end{gathered} \tag{3.10}$$

Excluding the wave length L from (3.10), we have

$$\bar{G}(z,y) = G(z_*, y) + \frac{(\bar{h} - z_*)}{\int\limits_z^w b(s,y) ds} \int\limits_z^w \frac{(G(s,y) - G(z_*, y)) b(s,y) ds}{s - z_*}. \tag{3.11}$$

The approximate expression for the function $\tilde{G}(\bar{h}, y) = \bar{G}(z, y)$ is given by the following formulae, in which the limits of integrals in (3.11) are used for $z \to z_*$.

$$\tilde{G}_*(\bar{h}, y) = G(z_*, y) + A_*(\bar{h} - z_*), \tag{3.12}$$

$$A_* = A_*(y) = \frac{\int\limits_{z_*}^{w_*} \frac{(G(s,y) - G(z_*, y)) b(s,y) ds}{s - z_*}}{\int\limits_{z_*}^{w_*} b(s,y) ds}. \tag{3.13}$$

The approximation (3.12) can be applied for calculations of the hyperbolicity domain of the modulation equations (2.4). For that we replace the function $\tilde{G}(\bar{h}, y)$ by $\tilde{G}_*(\bar{h}, y)$. The criterion of hyperbolicity takes the form

$$Disk_* = (\tilde{G}_{*y} - 2m\mathcal{D}')^2 + 4\delta^2 A_* > 0. \tag{3.14}$$

Due to the linear dependence $\tilde{G}_*$ on $\bar{h}$ in (3.12) the boundaries $\bar{h}^{\pm} = \bar{h}^{\pm}(y)$ of the hyperbolicity domain can be calculated from the quadratic equation $Disk_*(\bar{h}, y) = 0$ relative to $\bar{h}$ in the explicit form

$$\bar{h}^{\pm} = \frac{(A_* z_* - G(z_*, y))' \pm 2\sqrt{-A_*} m'}{A_*' \pm 2\sqrt{-A_*} \mathcal{D}'}. \tag{3.15}$$

Here the "prime" denotes the full derivative on y. In the next paragraph the approximate presentation of the hyperbolicity domain on a roll wave diagram is demonstrated for self-similar channels. The effectiveness of the approach is due to the fact that for self-similar channels the roll waves, which are stable according to the criterion (2.5), have the near maximal amplitude.

4. Stability of roll waves in self-similar channels

Consider the particular case of (1.1)

$$
\begin{aligned}
&h_t + (hu)_x = 0,\\
&(hu)_t + \left(hu^2 + \frac{1}{\beta}h^\beta\right)_x = \alpha h - h^{2-\beta}u^2.
\end{aligned} \tag{4.1}
$$

By choosing the special parametrisation (Boudlal and Liapidevskii, 2002) Eqs. (4.1) describe the flows over an incline ($\beta = 2$) and in inclined channels of triangular shape ($\beta = 1.5$). We choose the term self-similar channels for (4.1) since due to the geometrical similarity, flow patterns do not depend on the flow depth after proper flow scaling. The modulation equations (2.4) for (4.1) take more simple form and the hyperbolicity criterion (2.5) can be rewritten for the function of one variable. For the case of an inclined plane ($\beta = 2$) stability of roll waves has been investigated in (Liapidevskii, 1998).

Eqs. (1.2) take the form

$$
\begin{aligned}
\frac{dh}{d\xi} &= \frac{F(h,y)}{\Delta(h,y)},\\
F(h) &= \alpha h - h^{2-\beta}(\mathcal{D} - m/h)^2,\\
G(h,y) &= m^2/h + h^\beta/\beta,\\
\Delta(h,y) &= h^{\beta-1} - m^2/h^2.
\end{aligned} \tag{4.2}
$$

Here y is the critical depth and $m = yc(y) = y^{(\beta+1)/2}$, $\mathcal{D} = \mathcal{D}_1 y^{(\beta-1)/2}$, $\mathcal{D}_1 = 1 + \sqrt{\alpha}$. With the variable $s = h/y$ (4.2) transforms as follows

$$
\frac{d\xi}{dh} = y^{\beta-2}a(s), \tag{4.3}
$$

where $a(s) = (s^{\beta-1} - s^{-2})/\theta(s)$, $\theta(s) = \alpha s - s^{2-\beta}(\mathcal{D}_1 - 1/s)$. The necessary condition for roll wave existence is

$$
\theta'(1) = (\beta - 1)\alpha - 2\sqrt{\alpha} > 0. \tag{4.4}
$$

For $\beta = 2$ the condition (4.4) gives the well-known criterion $\alpha > 4$ of instability of steady-state flow (Jeffreys, 1925). The jump condition for the conjugate depths z, w ($z < y < w$) with the variables $\zeta = z/y$, $\omega = w/y$ takes the form

$$
g(\zeta) = g(\omega), \tag{4.5}
$$

where $g(s) = 1/s + s^\beta/\beta$. It follows from (4.5) that $\omega = \omega(\zeta)$ ($\zeta < 1 < \omega$), and with variables y, ζ we can find the dependence of all mean quantities in (2.2) on y

$$
\begin{aligned}
&L = y^{\beta-1}\lambda(\zeta), \quad \lambda(\zeta) = \int_\zeta^\omega a(s)ds,\\
&\bar{h} = y\bar{\zeta}, \quad \bar{\zeta} = \frac{1}{\lambda}\int_\zeta^\omega sa(s)ds = \varphi(\zeta),\\
&\bar{G} = y^\beta \bar{g}(\zeta), \quad \bar{g} = \frac{1}{\lambda}\int_\zeta^\omega g(s)a(s)ds = \psi(\bar{\zeta}),\\
&\psi(\bar{\zeta}) = \bar{g}(\varphi^{-1}(\bar{\zeta})).
\end{aligned} \tag{4.6}
$$

Now the stability condition can be described as the condition on the function ψ and its derivative. Due to (4.2) and (4.6) we have for $\bar\zeta = \bar h/y$

$$
\begin{gathered}
\delta = \bar h \mathcal{D}' - m' = \frac{1}{2} y^{(\beta-1)/2} \left((\beta-1)\mathcal{D}_1\bar\zeta - \beta - 1\right), \\
\tilde G(\bar h, y) = y^\beta \psi(\bar\zeta), \\
\tilde G_y(\bar h, y) = y^{\beta-1}\left(\frac{1}{2}(\beta-1)\psi(\bar\zeta) - \bar\zeta\psi'(\bar\zeta)\right), \\
\tilde G_{\bar h}(\bar h, y) = y^{\beta-1}\psi'(\bar\zeta), \\
R = y^{2(\beta-1)}\left(\beta\psi(\bar\zeta) - \bar\zeta\psi'(\bar\zeta) - (\beta-1)\mathcal{D}_1\right).
\end{gathered} \tag{4.7}
$$

The hyperbolicity condition (2.5) gives the inequality for the function ψ of the one variable $\bar\zeta$

$$
\left(\beta\psi - \bar\zeta\psi' - (\beta-1)\mathcal{D}_1\right)^2 + \psi'\left((\beta-1)\mathcal{D}_1\bar\zeta - \beta - 1\right)^2 > 0. \tag{4.8}
$$

Stability of infinitesimal roll waves can be investigated by (3.6)–(3.9), which for (4.1) are expressed through the functions $g(s)$ and $\theta(s)$

$$
\begin{gathered}
\psi(1) = g(1), \quad \psi'(1) = \frac{\bar g''(1)}{\varphi''(1)}, \\
\bar g''(1) = \frac{1}{3} g''(1), \quad \varphi''(1) = \frac{2}{3}\left(\frac{a'(1)}{a(1)} - \frac{1}{2}\frac{g'''(1)}{g''(1)}\right), \\
a(1) = \frac{g''(1)}{\theta'(1)}, \quad a'(1) = \frac{g'''(1)\theta'(1) - g''(1)\theta''(1)}{2(\theta'(1))^2}.
\end{gathered} \tag{4.9}
$$

For investigation of stability of roll waves of finite amplitude it is necessary to calculate the mean values on the period according to (4.6) and then to apply (4.8). Let Fr be the Froude number of steady-state flow governed by (4.1), y be the critical depth, $u_c = \sqrt{\alpha} y^{(\beta-1)/2}$ be the critical velocity and

$$
\mathrm{Fr} = \frac{u_c}{c(y)} = \sqrt{\alpha}.
$$

It follows from (4.4) that on the (Fr, y) – *plane* the existence domain for roll waves lies in the half-plane $\mathrm{Fr} > \mathrm{Fr}_s = 2/(\beta-1)$. Furthermore, for given $\mathrm{Fr} > \mathrm{Fr}_s$ the roll waves exist for $\zeta_* < \bar\zeta < 1$, where ζ_* is the closest to 1 root of the equation

$$
\theta(s) = \alpha s - s^{2-\beta}(\mathcal{D}_1 - 1/s)^2.
$$

The existence $\zeta_* < 1$ with $\theta(\zeta_*) = 0$ is provided by the inequalities (4.4) and $\theta(1/\mathcal{D}_1) > 0$. The value $\bar\zeta = 1$ corresponds to roll waves of infinitesimal amplitude, and for $\bar\zeta \to \zeta_*$ the wave length goes to infinity ($\lambda \to \infty$). Therefore, for the roll waves of near maximal amplitude, the linear approximation (3.11) for $\tilde G(\bar h, y)$ can be used. It gives for (4.1)

$$
\psi_*(\bar\zeta) = g(\zeta_*) + \psi_*'(\bar\zeta - \zeta_*),
$$

$$
\psi_*' = \frac{\displaystyle\int_{\zeta_*}^{\omega_*} (g(s) - g(\zeta_*))a(s)ds}{\displaystyle\int_{\zeta_*}^{\omega_*} (s - \zeta_*)a(s)ds}.
$$

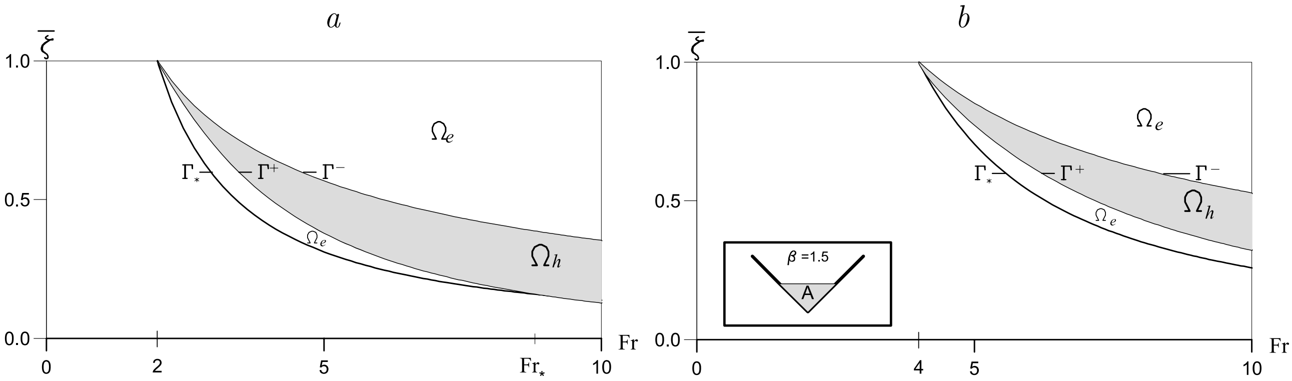

Fig. 2. Stability diagram for roll waves: a) rectangular cross-section, b) triangular cross-section. Curve Γ_* is the lower boundary of roll waves existing domain ($\bar\zeta = \zeta_*(\mathrm{Fr})$). Curves Γ^+ and Γ^- are the boundaries of the hyperbolicity region Ω_h ($\bar\zeta = \zeta^{\pm}(\mathrm{Fr})$). In the ellipticity domains Ω_e roll waves are unstable. The boundary $\bar\zeta = 1$ corresponds to roll waves of infinitesimal amplitude.

Here $\omega_* = \omega(\zeta_*)$. Analogously to (3.15) the condition $Disk_* = 0$ can be used to find the approximate boundaries of the hyperbolicity domain $\bar\zeta = \bar\zeta^{\pm}$ as the solution of the quadratic equation.

$$\Big(\beta\psi_* + \bar\zeta\psi_*' - (\beta - 1)\mathcal{D}_1\Big)^2 + \psi_*'\left((\beta - 1)\mathcal{D}_1\bar\zeta - \beta - 1\right)^2 = 0.$$

The roll wave diagram for $\beta = 2$ is shown in fig. 2. The curve Γ_* describing the roll waves of maximal amplitude is given by the function $\bar\zeta = \zeta_*(\mathrm{Fr})$. For $\beta = 2$ we have $\mathrm{Fr}_s = 2$ and the existence domain for roll waves is bounded by the line $\bar\zeta = 1$ and by the curve Γ_* for $\mathrm{Fr} > 2$. The stability domain Ω_h calculated by (4.8) is shown as shaded area. As it is seen from fig. 2, the roll waves of infinitesimal amplitude are unstable, and only the waves of finite amplitude become stable. The boundaries $\Gamma^{\pm}$ defined by the dependence $\bar\zeta = \zeta^{\pm}(\mathrm{Fr})$ give very good approximation for the stability domain Ω_h. Moreover, the point of the intersection of the curves Γ^+ and Γ_* gives the exact value Fr_*, at which the waves of maximal amplitude start to be stable with increasing the channel inclination.

Conclusion

The criterion (2.5) of nonlinear stability of periodic travelling waves (roll waves) considered in the paper is based on the analysis of hyperbolicity of modulation equations for governing flow parameters (Whitham, 1974). For shallow water equations with turbulent friction (1.1), roll waves are described by the two-parameter family of travelling waves, therefore the modulation system consists only of two equations and the hyperbolicity conditions can be expressed in the explicit form. The main problem in defining the roll wave stability domain on the plane of governing parameters is in the singularity of expressions for mean values and their derivatives in (2.2) when the wave length tends to zero or infinity. The asymptotic analysis performed in paragraph 4 gives the stability criteria for roll waves of small and maximal amplitude as well as the approximate position of boundaries of the stability domain.

For the class of self-similar channels, the stability condition can be expressed by the function of one variable. In this case it can be shown that the roll waves of infinitesimal amplitude are always unstable and that the stable waves have the amplitude high enough, so that the approximation derived for roll waves of maximal amplitude gives very effective formulae for the stability domain prediction.

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Received for publication July 20, 2004