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PRICING AND HEDGING OF SWAPTIONS*

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Целью данной статьи является пояснение некоторых приемов формирования процентной ставки при валютных операциях, в котором мы обсуждаем оценку валютного обмена в соответствии с модифицированной Black model. Обсуждаются параметры риска и стратегии ограничений в применении к валютному обмену.

Introduction

Interest rate swaps (IRS) have been widely used by the larger corporates for some time as an efficient method to manage interest rate exposure. In this way, a floating-rate borrower who expects a rise in interest rates can swap his floating rate obligation to a fixed rate obligation, thus locking in his future cost. Should he subsequently decide that rates have peaked, and that the trend is reversing, the interest obligation could be swapped back to a floating rate basis, thereby gaining advantage from the anticipated fall in rates.

Swaptions first came into vogue in the mid-1980s in the US on the back of structured bonds tagged with a callable option issued by borrowers [13]. With a callable bond, a borrower issues a fixed-rate bond which he may call at par from the investor at a specific date(s) in the future. In return for the issuer having the right to call the bond issue at par, investors are offered an enhanced yield. Bond issuers often issue an IRS in conjunction with the bond issue in order to change their interest profile from fixed to floating. Swaptions are then required by the issuer as protection to terminate all or part of the original IRS in the event of the bonds being put or called.

There are two main reasons why the Black model (see, e. g., [9]) is widely accepted and used in the financial market for the valuation of european interest rate options such as caps, floors or swaptions: On the one hand, the only value to be computed for input in this model is the

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volatility of the underlying, which in most cases is computed from market data. On the other hand, under the assumption of lognormal distribution of the underlying, the basis instrument is modelled directly.

However, this elegance of the Black model has its own shortcoming: The assumption of constant volatility and risk-free interest rate limits its applicability to a certain extent. Furthermore, the Black model is only appropriate for valuing european options.

Alternatives to the Black model are, e.g., arbitrage-free binomial models, such as, amongst others, the *Black-Derman-Toy model* [3] and the *Ho-Lee model* [7]. Binomial tree based modelling founds on stochastically changing risk-free interest rates at each time interval. Such a model is appropriate for valuing interest rate derivatives of all maturities, particularly american options.

This contribution seeks to illustrate the valuation of swaptions from a mathematical perspective, on the basis of the Black model. Furthermore, there is a quick review of swaps, decause they form the underlying instrument of swaptions and the basis upon which swaptions are modelled and valued. Finally, swaption oriented hedging strategies are discussed.

1. The Black Model

The *Black model (1976)* represents a modification of the Black-Scholes model [4] for the valuation of equity options, having futures contracts as underlying instrument. *Black* prices an european option as though its value at maturity T did not depend on the spot price of the underlying instrument, but rather on its future price.

Notation

T	–	maturity of option
t	–	present time
S	–	strike price
i	–	risk-free interest rate
P	–	spot price of the underlying at time t
P_T	–	price of the underlying at time T (future preis)
σ	–	volatility
$\Phi(\cdot)$	–	Gaussian distribution function

If continuous interest is assumed as is usually the case in models, then we have the discount factor e^{-iT}, and $\mathrm{e}^{-i(t-T)}$ respectively. Other discount factors are possible and frequently used. In what follows, we assume without loss of generality that $t = 0$.

Definition 1. *A* (Put)Call option *(see Peter Reißner (1991), [12], page 23) gives its holder the right without obligation to (sell)purchase an underlying asset (to)from the writer at a predetermined price (strike price, S), on or before an agreed future date (expiry date T).*

In the case that the aforementioned right is exercised before the expiry date, then it is an *american* call or put option. Otherwise, it is an *european* option, whose prices we denote with P_{Call} and P_{Put}. It is easily seen that the following boundary conditions hold at maturity

$$P_{\text{Call}} = \max\{P_T - S; 0\}, \qquad P_{\text{Put}} = \max\{S - P_T; 0\}\,.$$

In addition to the assumptions[1] of the Black-Scholes model (see [4, 12]) for equity options, the **main assumption** of the Black model is the lognormal distribution[2] of the future price P_T of the underlying at maturity date T of the option.

The Black-Scholes formula takes the following form (see, e. g., [4]):

$$P_{\text{Call}} = P\Phi(d_1) - Sd_T\Phi(d_2),$$

$$P_{\text{Put}} = Sd_T\Phi(-d_2) - P\Phi(-d_1),$$

$$d_T = \mathrm{e}^{-iT},$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{P}{S} + \left(\frac{1}{2}\sigma^2 + i\right)T\right], \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

The following relationship exists between the future price P_T and the spot price P:

$$P_T \cdot \mathrm{e}^{-iT} = P,$$

assuming the above security yields no dividends or other payments in $[t, T]$. In this way, the Black-Scholes formula can be transformed as follows (see [1]):

$$P_{\text{Call}} = \mathrm{e}^{-iT}[P_T\Phi(d_1) - S\Phi(d_2)], \tag{1}$$

$$P_{\text{Put}} = \mathrm{e}^{-iT}[S\Phi(-d_2) - P_T\Phi(-d_1)] \tag{2}$$

with

$$d_1 = \frac{\ln\frac{P_T}{S} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\ln\frac{P_T}{S} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}\,. \tag{3}$$

2. Interest Rate Swaps

2.1. Concept and Application

Definition 2. *An* interest rate swap (IRS[3]) *is a contractual agreement entered into between two counterparties under which they agree to exchange fixed for variable interest rates (mostly LIBOR[4]) periodically, for an agreed period of time based upon a notional amount of principal. The principal amount is notional because there is no need to exchange actual amounts of principal. Equally, however, a notional amount of principal is required in order to compute the actual cash amounts that will be periodically exchanged.*

An IRS is an agreement of specified duration between two parties (Swap partners or Counterparties) for the exchange of interest rate payments relative to a nominal value (Notional Principal Amount) at predetermined periods of time. That is, Counterparty A makes fixed

[1] These assumptions are:
(A1) It is assumed that the financial market is frictionless.
(A2) Trading in the financial market is possible on a continuous time scale.
(A3) Security prices follow a continuous-time Markov process.
(A4) There are no arbitrage possibilities.

[2] A random variable X is said to be lognormally distributed if the variable $Y = \ln X$ has normal distribution.

[3] In this paper the terms Interest Rate Swap and Swap are used interchangeably.

[4] London Interbank Offered Rate

interest payments to Counterparty B at specific time intervals. On the other hand, A receives from B variable payments relative to an agreed reference interest rate[5].

Counterparty B receives fixed interest payments (receiver position), That is to say, B pays variable. On the other hand, A makes fixed interest payments (payer position). That is, A receives variable payments in the swap. Variable and fixed coupon payments occur in predetermined time intervals. It is however stressed here that in practical terms, the dates of variable and fixed payments may not always coincide. If one denotes the variable cashflow period[6] with Δt, the interest rate of reference would be agreed upon at time t, with the cashflow occurring at time $t+\Delta t$. Cashflows on both sides coinciding would mean the net value being paid to the beneficiary. Swaps are particularly useful in the restructuring of risk in an investment. Eventual interest rate risks can be hedged away with swaps. It is for this reason that swaps have become so important in financial management.

2.2. Pricing Interest Rate Swaps

Pricing[7] a swap means determining the fixed interest rate i_{fix} of the swap (swap rate) such that the value of the swap is zero at time $t = 0$ (see [1, 10]). It is clear from the definition that a swap is equivalent to a portfolio of two bonds, one short and the other long, one a fixed-rate bond and the other a floating rate bond[8].

Let $0 < t_1 < \cdots < t_n$ represent the reset dates of the swap. It can be shown that, given a constant notional capital, the fixed interest rate i_{fix} can be computed by the following formula (see [6, 10]; compare also [11, 12]):

$$i_{\text{fix}} = \frac{1 - d_n}{\sum\limits_{k=1}^{n} d_k}, \qquad d_k = \mathrm{e}^{-it_k}, \quad k = 1, \ldots, n. \tag{4}$$

3. Swaptions

A swaption is a combination of the following two financial instruments: *Interest Rate Swap* and *Option.*

Definition 3. *A* Swaption[9] (Swap Option) *reserves the right for its holder to purchase a swap at a prescribed time and interest rate in the future (European Option).*

The holder of such a call option has the right, but not the obligation to pay fixed in exchange for variable interest rate. Therefore, this option is also known as "Payer Swaption". The holder of the equivalent put option has the right, but not the obligation to receive interest at a fixed rate (Receiver Swaption) and pay variable.

A swaption is an option on a forward interest rate. Like interest rate swaps, swaptions are used to mitigate the effects of unfavorable interest rate fluctuations at a future date. The

[5]R. Rauleder (1993) [11] page 11.

[6]see R. Rauleder (1993) [11] page 13: in the case of the variable cashflow interval being smaller than the fixed, fixed and variable cashflows will fall apart at certain dates.

[7]see B. Luderer; O. Zuchanke, [10], for a rigorous treatment of Swap-Pricing.

[8]see R. Rauleder [11], page 97; see R. Kohn, [9] .

[9]see P. Reißner (1991), [12], page 23.

premium paid by the holder of a swaption can more or less be considered as insurance against interest rate movements. In this way, businesses are able to guarantee limits in interest rates.

For instance, a five year swaption expiring in six months is the same as an option to contract a swap in six months time, and the swap will be valid for five years. To further buttress the point, an example is in order.

Consider the case of a firm that will start servicing its debt six months from now. The debt is serviceable within five years, at a floating interest rate payable every six months.

This firm can protect itself against rising interest rates by purchasing a payer swaption. By paying a premium, the firm obtains the right to recive variable payments (mostly LIBOR) to pay a predetermined fixed interest rate i_{fix} for a five year period. The swap begins six months from now (expiry date of the Swaption).

There will be two possible outcomes at the expiry date. If, on the one hand, the market swap rate is higher than i_{fix} the option is exercised and its holder is able to satisfy his variable interest rate commitment at a rate below the market interest rate. The firm thus gains. If, on the other hand, the market swap rate is below the strike rate the swaption is not exercised and the firm turns to lower interest rates in the market.

4. Swaption Pricing With Black Model

Notation

T	–	expiry date of swaption
σ_F	–	volatility of forward swap rate
i_S	–	strike rate
i_F	–	market swap rate (at par) at time T, forward swap rate
N	–	notional capital

Equal reset dates on both sides are assumed in the rest of what follows without loss of generality.

The input parameter σ is obtained from market data[10].
Let $t_1 < \cdots < t_n$ represent the coupon dates for the swap and $t_0 = T$.

In deriving a pricing formula, we look at the swap underlying the swaption. The swap begins on the expiration date (T) of the swaption (this coincides with the first cashflow) and ends at time t_n. The swap comprises payments at floating interest rate and payments at fixed interest rate, made at each of the n reset dates.

The individual variable interest rate payments are based on a benchmark (mostly LIBOR), valid at time t_k, a notional principal amount N and the length of the interest period $(t_i - t_{i-1})$. The fixed payments depend on the fixed interest rate i_F, as well as on the same notional principal N and the same time interval.

4.1. Pricing European Swaptions

The cash flow made to the buyer of a payer swaption at time T amounts to

[10] see [12], page 48 : historic or implicit volatility.

$$\sum_{i=1}^{n} N \cdot \mathrm{e}^{-(t_i-T)} \cdot (i_F - i_S) \cdot (t_i - t_{i-1}), \tag{5}$$

if $i_F > i_S$ and 0 otherwise. Its value today therefore equals

$$\begin{aligned} &\mathrm{e}^{-iT} \cdot \sum_{i=1}^{n} N \cdot \mathrm{e}^{-i(t_i-T)} \cdot (i_F - i_S) \cdot (t_i - t_{i-1}) = \\ &= N \cdot \sum_{i=1}^{n} \mathrm{e}^{-it_i} \cdot (i_F - i_S) \cdot (t_i - t_{i-1}). \end{aligned}$$

Now, the price of a european payer swaption is determined using the Black model. The Ith term

$$N(t_i - t_{i-1})(i_F - i_S)+$$

of the cash flow corresponds to the price of a european call option with expiry t_i. According to the Black model (see Section 1), the price of this option at time 0 is

$$N \cdot \mathrm{e}^{-it_i} \cdot (t_i - t_{i-1}) \left[i_F \Phi(d_1) - i_S \Phi(d_2) \right],$$

whereby the following holds by analogy to (3)

$$d_1 = \frac{\ln \dfrac{i_F}{i_S} + \dfrac{1}{2}\sigma_F^2 T}{\sigma_F \sqrt{T}}, \qquad d_2 = \frac{\ln \dfrac{i_F}{i_S} - \dfrac{1}{2}\sigma_F^2 T}{\sigma_F \sqrt{T}} = d_1 - \sigma_F \sqrt{T}. \tag{6}$$

The forward swap rate i_F is computed from (4), by replacing d_k with $\mathrm{e}^{-i(t_i-T)}$.
The value of the payer swaption P_{PS} itself is obtained by summing up all individual call options (see equation (1)):

$$P_{PS} = NA \left[i_F \Phi(d_1) - i_S \Phi(d_2) \right],$$

whereby

$$A = \sum_{i=1}^{n} \mathrm{e}^{-it_i} (t_i - t_{i-1}). \tag{7}$$

By analogy, in the case of a european receiver swaption, we obtain through (2):

$$P_{RS} = NA \left[i_S \Phi(-d_2) - i_F \Phi(-d_1) \right],$$

with A as in (7), and d_1, d_2 from (6); see also [1, 6, 9].

5. Risk Parameters and Hedging

Partial derivatives lay the practical foundation for the application of special trading and hedging techniques on options and derivative securities. They quantify the influence (risk) of changes in market factors on the option price. In this respect and in many instances, partial derivatives are as important as the theoretically determined price of the option, as they tell the user in a short and accurate manner which direction to go in the current investment (assets, liabilities): buy, sell or maintain. In the sequel, the most important risk parameters the ("Greeks") are presented with respect to swaptions. This is followed by some trading strategies in brief. Since the "Greeks" may apply to different types of instruments (to swaptions in particular also), we shall in the sequel denote the price of a general derivative security with D and the price of its underlying security with B. Concretely, in the case of swaptions (under consideration here) this means: D stands for the price P_{PS} of the Payer swaption and B stands for the corresponding swap price i_F.

5.1. Risk Parameters

5.1.1. Delta

Given a specific yield curve (interest rate structure) plus swap as underlying, the price of a swaption depends on the expiry date T and the strike price S (see, e. g., [11]). Therefore the impact of a shift in the swap price (strike rate i_S kept constant) on the swaption price is dependent on the variables T and i_F .

Generally, the parameter *delta* describes the rate of change of the price of the derivative security with respect to the asset (price of the underlying):

$$\Delta = \frac{\partial D}{\partial B}\,.$$

In the case under consideration, our underlying is the fixed forward swap rate i_F. By analogy with the delta of an equity call (see, e. g., [6, 8]), the following holds for the delta Δ_{PS} of a european payer swaption

$$\Delta_{PS} = \frac{\partial P_{PS}}{\partial i_F} = NA\Phi(d_1),$$

whereby A as defined in equation (7).

By analogy, we obtain for the delta of a european receiver swaption the following value

$$\Delta_{RS} = NA\left(\Phi(d_1) - 1\right)\,.$$

5.1.2. Gamma

The *gamma* of a derivative product (e.g., swaption or portfolio thereof) is the second derivative of the price of the derivative with respect to the underlying:

$$\Gamma = \frac{\partial^2 D}{\partial B^2} = \frac{\partial \Delta}{\partial B}\,.$$

Like in the case of delta, a closed form formula can be derived for gamma through differentiation (see [1, 6]):

$$\Gamma_{PS} = \frac{NA\varphi(d_1)}{\sqrt{T} i_F \sigma_F}\,,$$

whereby $\varphi(d_1) = \Phi'(d_1)$ stands for the probability density function of the gaussian distribution. According to the definition, gamma is the sensitivity of the delta to the underlying security. Therefore, it measures how much and how often Gamma must be rehedged (see Section 5.2 below), in order to maintain a delta-neutral portfolio.

5.1.3. Theta

The *theta* of a portfolio of derivative products is the rate of change of the portfolio price with respect to time to maturity T (time left for option to expire):

$$\Theta = \frac{\partial D}{\partial T}\,.$$

In the special case of a payer swaption one obtains

$$\Theta = \frac{\partial P_{PS}}{\partial T} = -i_F P_{PS} + \frac{NAi_F\sigma_F\varphi(d_1)}{2\sqrt{T}}\,.$$

5.1.4. Vega

Vega[11] is the sensitivity of the derivative (or portfolio of financial derivative products) price to volatility σ:

$$V = \frac{\partial D}{\partial \sigma}\,.$$

In the special case of swaptions one obtains the following equation through differentiation:

$$V_{PS} = V_{RS} = NAi_F\sqrt{T}\varphi(d_1)\,.$$

The implicit quantity σ_F is the estimate for the expected forward swap rate. A vega-hedged portfolio is protected against fluctuations of volatility.

5.2. Trading Strategies

5.2.1. Delta-Hedging

The so called *delta-hedging* is a dynamic hedging strategy. Here, it is sought, price changes of the swap to be compensated with price changes of the swaption. This is achieved by setting up a portfolio by holding (or shorting) the derivative (swaption) and shorting (or holding) a quantity Δ of the underlying (swap); this is referred to as *hedge portfolio*.

In this way, within the portfolio, price increases of the swap are compensated by price drops of the swaption and vice-versa. Risks caused by fluctuations of the underlying security are practically eliminated. As can be verified, this portfolio has a delta of zero (let P_{Port} be the price of the portfolio):

$$\Delta_{\text{Port}} = \frac{\partial P_{\text{Port}}}{\partial B} = \Delta \cdot \frac{\partial B}{\partial B} - \frac{\partial D}{\partial B} = \Delta \cdot 1 - \Delta = 0.$$

Therefore, by way of delta-hedging, one can eliminate (at least theoretically and to a great extent practically) the risk. The proportion of the underlying security in the portfolio must be continuously changed since the quantity Δ depends on both the price of the underlying and the remaining period to maturity of the swaption. This process is called *dynamic hedging* (or *rebalancing*) of the portfolio. Therefore (theoretically), one continuously has to buy and sell swaps. However, in the case of a discrete model, rebalancing of delta is done at discrete time intervals Δt.

[11] It is not denoted by a Greek character, since it is not a Greek letter although it belongs to the "Greeks".

5.2.2. Delta-Gamma-Hedging

A little value for gamma indicates that by definition, the rate of change of delta is little. This means rebalancing of the hedge-portfolio may be carried out in larger intervals of time. Conversely, larger gamma values are an indication that delta is very sensitive with respect to shifts in the underlying, resulting in the increase in risk inherent in a shift in portfolio value.

Because of the cost of frequent hedging, it is natural to try to minimize the need to rebalance the portfolio too frequently. The corresponding hedging procedure is called a *gamma-neutral* strategy (see [14]). To achieve this objective, we have to buy and sell more swaptions, not just the swap. By simple differentiation, you can check that a position in the underlying asset has zero gamma:

$$\frac{\partial^2 B}{\partial B^2} = 0, \qquad \text{particularly,} \qquad \frac{\partial^2 i_F}{\partial i_F^2} = 0\,.$$

Thus, we cannot change the gamma of our position by adding the underlying. However, we can add another swaption in quantity, which will make the portfolio gamma-neutral. By holding two different swaptions we can make the portfolio both delta- and gamma-neutral. Note that a delta-neutral portfolio $\Delta_{\text{Port}} = 0$ has gamma equal to Γ and a traded swaption has gamma equal to Γ_0. If the number of traded swaptions added to the portfolio is w_0, the gamma of the portfolio is

$$\widetilde{\Gamma}_{\text{Port}} = \Gamma + w_0\Gamma_0\,.$$

Hence, the portfolio becomes gamma-neutral, if our position in the traded swaption is equal to $w_0 = -\Gamma/\Gamma_0$. Of course, as we add the traded swaption, the delta of the portfolio changes. So the position in the underlying (swap) then has to be changed to maintain delta-neutrality. Due to

$$\widetilde{\Delta}_{\text{Port}} = 0 + w_0\Delta_0 = 0 - \left(\frac{\Gamma}{\Gamma_0}\right)\Delta_0,$$

the quantity $\widetilde{\Delta}_{\text{Port}}$ of the underlying (swap) has to be added for hedging.

5.2.3. Practical Approach

In practice, portfolio rebalancing to achieve delta-, gamma-, vega-neutrality etc. is not a continuous process. If it were, transaction costs would render it extremely expensive. Instead, the individual risks are analyzed to find out if they are worth taking or not. The aforementioned risk parameters play the role of quantifying various aspects of portfolio risk. If the risk is acceptable, no action is taken. Otherwise rebalancing is carried out as outlined above.

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