COMPUTATIONAL COMPLEXITY OF ESTIMATION OF GENERALIZED SOLUTION SETS FOR INTERVAL LINEAR SYSTEMS*

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В работе исследуется вычислительная сложность задач распознавания и оценивания обобщенных множеств решений интервальных систем линейных алгебраических уравнений. Показано, что если матрица системы содержит "достаточно много" элементов с интервальной неопределенностью E-типа, то задачи распознавания и оценивания множества решений такой системы уравнений являются NP-трудными. Напротив, если в интервальной матрице системы присутствует "не очень много" E-неопределенных элементов и большинство элементов имеет A-тип неопределенности, то эти задачи являются полиномиально разрешимыми.

Introduction

We consider systems of linear interval equations of the form

$$\mathbf{A}x = \mathbf{b}, \tag{1}$$

where $\mathbf{A} = [\underline{A}, \overline{A}]$ is an interval $m \times n$-matrix, $\mathbf{b} = [\underline{b}, \overline{b}]$ is an interval m-vector, and $x \in \mathbb{R}^n$.

The interval matrix and the interval vector are traditionally understood [1] as the sets

$$\begin{aligned} \mathbf{A} &= \{ A \in \mathbb{R}^{m\times n} \mid \underline{\mathbf{A}} \le A \le \overline{\mathbf{A}} \}, \\ \mathbf{b} &= \{ b \in \mathbb{R}^{m} \mid \underline{\mathbf{b}} \le b \le \overline{\mathbf{b}} \} \end{aligned}$$

(by $\mathbb{R}^{m\times n}$ from now on we denote the set of $m \times n$-matrices). It is also assumed that $\underline{A} \le \overline{A}$, $\underline{b} \le \overline{b}$, and the inequalities between the matrices and the vectors are understood elementwise and coordinatewise, respectively.

As opposed to ordinary noninterval systems of equations, we can consider various *solution sets* to interval equations systems of the form (1). Historically, the so called *united solution set* [2, 3]

$$\Xi_{\mathrm{uni}}(\mathbf{A}, \mathbf{b}) = \{ x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b) \}, \tag{2}$$

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was the first and remains the most investigated one so far. As the time was going on, practical needs caused the introduction and investigation of another solution sets for the interval systems.

In [4, 5] set of inner solutions (later renamed as *tolerable solution set*)

$$\Xi_{\text{tol}}(\mathbf{A}, \mathbf{b}) = \{ x \in \mathbb{R}^n \mid (\forall A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b) \}, \tag{3}$$

was introduced and interpreted. In [6, 7] *controllable solution set* came into being as a result of solving the interval version of the automatic control problem. The controllable solution set is defined to be

$$\Xi_{\text{ctr}}(\mathbf{A}, \mathbf{b}) = \{ x \in \mathbb{R}^n \mid (\forall b \in \mathbf{b})(\exists A \in \mathbf{A})(Ax = b) \}. \tag{4}$$

It is obvious from the above examples that the difference between the solution sets is that:

(i) different quantifiers are attributed to different elements from $\mathbf{A}$ and $\mathbf{b}$, as well as;

(ii) the orders of the quantifiers can vary.

When addressing to these differences, i.e. when attributing either existential or universal quantifier to each of the elements from $\mathbf{A}$ and $\mathbf{b}$, and choosing some order for these quantifiers, we obtain a large number of different concepts (forms of understanding) of the solution sets to (1). For the first time such solution sets seems to have been appeared in [8] in connection with the game theory problems.

This paper will avoid the consideration of the most general definition of the solution sets, which one can consider on this way. Our treating will be restricted to the case investigated by S.P. Shary [9, 10]. Namely, we assume that a logical quantifier is attributed to each element from $\mathbf{A}$ and $\mathbf{b}$ and, furthermore, all the universal quantifiers precede all the existential quantifiers.

Following the papers [9, 10], let us give the precise definitions. We suppose that an $m \times n$-matrix $\Lambda = (\lambda_{ij}), \lambda_{ij} \in \{-1, 1\}$, $i = \overline{1, m}$, $j = \overline{1, n}$ and an m-vector $\beta = (\beta_1, \ldots, \beta_n)^\top$, $\beta_i \in \{-1, 1\}$, $i = \overline{1, m}$ are given along with the interval $m \times n$-matrix $\mathbf{A}$ and the interval m-vector $\mathbf{b}$. The matrix $\mathbf{A} = (\mathbf{a}_{ij})$ is decomposed into two matrices $\mathbf{A}^\exists = (\mathbf{a}^\exists_{ij})$ and $\mathbf{A}^\forall = (\mathbf{a}^\forall_{ij})$ so that

$$\mathbf{a}^\exists_{ij} = \begin{cases} \mathbf{a}_{ij}, & \text{if } \lambda_{ij} = 1, \\ 0, & \text{if } \lambda_{ij} = -1, \end{cases} \qquad \mathbf{a}^\forall_{ij} = \begin{cases} 0, & \text{if } \lambda_{ij} = 1, \\ \mathbf{a}_{ij}, & \text{if } \lambda_{ij} = -1. \end{cases}$$

Similarly, let us decompose the vector $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_m)^\top$ into two vectors

$$\mathbf{b}^\exists = (\mathbf{b}_1, \ldots, \mathbf{b}_m)^\top, \quad \mathbf{b}^\forall = (\mathbf{b}_1, \ldots, \mathbf{b}_m)^\top$$

such that

$$\mathbf{b}^\exists_i = \begin{cases} \mathbf{b}_i, & \text{if } \beta_i = 1, \\ 0, & \text{if } \beta_i = -1, \end{cases} \qquad \mathbf{b}^\forall_i = \begin{cases} 0, & \text{if } \beta_i = 1, \\ \mathbf{b}_i, & \text{if } \beta_i = -1. \end{cases}$$

It is furthermore obvious that $\mathbf{A} = \mathbf{A}^\forall + \mathbf{A}^\exists$, $\mathbf{b} = \mathbf{b}^\forall + \mathbf{b}^\exists$.

Definition 1 (S. P. Shary [9]). For given quantifier matrix Λ and quantifier vector β, the generalized AE-solution set of the type $\Lambda\beta$ is

$$\Xi_{\Lambda,\beta}(\mathbf{A}, \mathbf{b}) = \big\{ x \in \mathbb{R}^n \mid$$

$$(\forall A' \in \mathbf{A}^\forall)(\forall b' \in \mathbf{b}^\forall)(\exists A'' \in \mathbf{A}^\exists)(\exists b'' \in \mathbf{b}^\exists)((A' + A'')\, x = b' + b'') \big\}. \tag{5}$$

The main purpose of our paper is to inquire into the algorithmic complexity (in the sense of [11]) of two problems relating to these sets:

Problem 1. To find out (to determine) whether the set (5) is empty or not.

Problem 2. Compute the interval estimate for the set (5) provided that it is nonempty.

In the sequel, we assume that the reader is familiar with the principal concepts of the theory of computational complexity, such as: solvability within the polynomial time, NP-hardness, NP-completeness, polynomial reducibility of one problem to other one (see [11]). It is worth noting that the first result on NP-complexity of an interval linear algebraic problem, — that concerning the determination of singularity of an interval $n \times n$-matrix $\mathbf{A}$, — seems to have been obtained in the work [12].

Problems 1 and 2 for specific solution sets of the form (2) – (4) are known to be investigated previously (in different statements) by several authors. In [13] NP-completeness of problem 2 was proved for the united solution set Ξ_{uni} in the general case of an interval $m \times n$-matrices.

In the works [14, 15] problem 1 is shown to be NP-complete for the solution sets Ξ_{uni} and Ξ_{ctr}. It was also noted in [14, 15] that problem 1 for the solution set Ξ_{tol} is solvable for the polynomial time, the latter immediately following from the results of [16] as well as from the description of Ξ_{tol} obtained earlier in [5]. NP-completeness of problems 1 and 2 for Ξ_{uni} has been shown in [17] for interval $(2n+1) \times n$-matrices and, in essence, provided that the additional condition of finiteness of Ξ_{uni} is satisfied. Meanwhile, [18] suggested the proof of NP-completeness of problems 1 and 2 for Ξ_{uni} for the positive interval $(n+2) \times n$-matrices under the condition of finiteness of Ξ_{uni} (Ξ_{uni} has no more than 2^n elements), and [19] — a finite (in dimension) result on NP-completeness of problems 1 and 2 for Ξ_{uni} for positive interval $(n+1) \times n$-matrices under the condition of finiteness of Ξ_{uni}.

Finally, it was shown in [20] that, considering all the abovementioned restrictions, it is possible to assume that elements of the matrix $\mathbf{A}$ may be represented by only [0,0], [1,1] or [0,1], i.e. that these problems are NP-complete in the strong sense. On the other hand, in [21–23], NP-completeness of problem 2 for interval strongly regular $n \times n$-matrices $\mathbf{A}$ was shown for the united solution set $\Xi_{\text{uni}}(\mathbf{A}, \mathbf{b})$.

The large number of another NP-complete (and NP-hard) problems that naturally arise in connection with the interval computations can be found in [24].

1. Characterization of generalized solution sets

This section derives an Oettli — Prager-type description of the generalized solution sets, which will be needed further in our considerations. In doing this, we rely upon the characterization of the generalized solution sets to interval linear systems suggested by S.P. Shary in [9].

For the interval $m \times n$-matrix $\mathbf{A}$ and n-vector $x \in \mathbb{R}^n$, the product $\mathbf{A}x$ is defined as usual [1]:

$$\mathbf{A}x = \{ Ax \mid A \in \mathbf{A} \}.$$

We will also use traditional definitions of the interval operations and relations from [1]. Then the following statement holds.

Theorem 1 (S. P. Shary [9]). *For any given quantifiers* Λ *and* β *of the same size as* $\mathbf{A}$ *and* $\mathbf{b}$ *respectively the following equality holds*:

$$\Xi_{\Lambda,\beta}(\mathbf{A}, \mathbf{b}) = \{ x \in \mathbb{R}^n \mid \mathbf{A}^{\forall} x - \mathbf{b}^{\forall} \subseteq \mathbf{b}^{\exists} - \mathbf{A}^{\exists} x \}. \tag{6}$$

The proof of the equality (6) follows directly from the definitions and, hence, is omitted.

In the rest of the paper, for two $m \times n$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, by $A \circ B$ we will denote their Hadamard product [25] $A \circ B = (a_{ij}b_{ij})$. Using theorem 1 along with the well-known Oettli — Prager theorem (see e.g. [1, 3, 26]), it is possible to obtain the following statement.

Theorem 2 (J. Rohn [27]). *For any given* Λ *and* β*, the equality*

$$\Xi_{\Lambda,\beta}(\mathbf{A},\mathbf{b}) = \{\, x \in \mathbb{R}^n \mid |A_c x - b_c| \le (\Lambda \circ \Delta)|x| + \beta \circ \delta \,\},$$

holds, where $A_c = (\underline{\mathbf{A}} + \overline{\mathbf{A}})/2$, $\Delta = (\overline{\mathbf{A}} - \underline{\mathbf{A}})/2$, $b_c = (\underline{\mathbf{b}}1 + \overline{\mathbf{b}})/2$, $\delta = (\overline{\mathbf{b}} - \underline{\mathbf{b}})/2$.

Proof. (J.Rohn [27]). From theorem 1, we have that $x \in \Xi_{\Lambda,\beta}$ is equivalent to the condition

$$\mathbf{A}^{\forall}x - \mathbf{b}^{\forall} \subseteq \mathbf{b}^{\exists} - \mathbf{A}^{\exists}x.$$

Further, according to Oettli — Prager theorem [26]

$$\mathbf{A}^{\forall}x - \mathbf{b}^{\forall} = [A_c^{\forall}x - \Delta^{\forall}|x| - b_c^{\forall} - \delta^{\forall}, A_c^{\forall}x + \Delta^{\forall}|x| - b_c^{\forall} + \delta^{\forall}],$$

and

$$\mathbf{b}^{\exists} - \mathbf{A}^{\exists}x = [-A_c^{\exists}x - \Delta^{\exists}|x| + b_c^{\exists} - \delta^{\exists}, -A_c^{\exists}x + \Delta^{\exists}|x| + b_c^{\exists} + \delta^{\exists}],$$

where

$$\begin{aligned} A_c^{\exists} &= (\underline{\mathbf{A}^{\exists}} + \overline{\mathbf{A}^{\exists}})/2, & \Delta^{\exists} &= (\overline{\mathbf{A}^{\exists}} - \underline{\mathbf{A}^{\exists}})/2, \\ b_c^{\exists} &= (\underline{\mathbf{b}^{\exists}} + \overline{\mathbf{b}^{\exists}})/2, & \delta^{\exists} &= (\overline{\mathbf{b}^{\exists}} - \underline{\mathbf{b}^{\exists}})/2, \\ A_c^{\forall} &= (\underline{\mathbf{A}^{\forall}} + \overline{\mathbf{A}^{\forall}})/2, & \Delta^{\forall} &= (\overline{\mathbf{A}^{\forall}} - \underline{\mathbf{A}^{\forall}})/2, \\ b_c^{\forall} &= (\underline{\mathbf{b}^{\forall}} + \overline{\mathbf{b}^{\forall}})/2, & \delta^{\forall} &= (\overline{\mathbf{b}^{\forall}} - \underline{\mathbf{b}^{\forall}})/2. \end{aligned}$$

Hence, the above inclusion is equivalent to

$$-(\Delta^{\exists} - \Delta^{\forall})|x| - (\delta^{\exists} - \delta^{\forall}) \le (A_c^{\exists} + A_c^{\forall})x - (b_c^{\exists} + b_c^{\forall}) \le (\Delta^{\exists} - \Delta^{\forall})|x| + (\delta^{\exists} - \delta^{\forall}),$$

which gives

$$|(A_c^{\exists} + A_c^{\forall})x - (b_c^{\exists} + b_c^{\forall})| \le (\Delta^{\exists} - \Delta^{\forall})|x| + (\delta^{\exists} - \delta^{\forall}).$$

It only remains to note that due to the definition of the matrices $\mathbf{A}^{\exists}$, $\mathbf{A}^{\forall}$ and the vectors $\mathbf{b}^{\exists}$, $\mathbf{b}^{\forall}$ the equalities

$$A_c = A_c^{\exists} + A_c^{\forall}, \qquad b_c = b_c^{\exists} + b_c^{\forall}, \qquad \Lambda \circ \Delta = \Delta^{\exists} - \Delta^{\forall}, \qquad \beta \circ \delta = \delta^{\exists} - \delta^{\forall}$$

holds. Q.E.D.

2. Computational Complexity

In order to correctly state the interested us problems, we shall assume that for each m and n there are a fixed $m \times n$-matrix $\Lambda(m,n) = (\lambda_{ij}(m,n))$ such that $\lambda_{ij}(m,n) \in \{-1,1\}$, $i = \overline{1,m}$, $j = \overline{1,n}$ and an m-vector $\beta(m) = (\beta_1(m),\ldots,\beta_m(m))^T$ such that $\beta_i(m) \in \{-1,1\}$, $i = \overline{1,m}$.

In other words, the two functions Λ and β are given, such that the function Λ determines the correspondence between $m \times n$-matrices of $\{-1,1\}$ and the pairs of natural numbers (m,n),

$(m \geq 1, n \geq 1)$, while the function β sets a correspondence between the m-vectors of $\{-1, 1\}$ and the natural numbers m.

From now on, we will also use the following notation. For the real number λ, we denote the positive part of λ by $\lambda^+ = \max\{0, \lambda\}$, and the negative part by λ by $\lambda^- = \max\{0, -\lambda\}$ respectively. The positive parts Λ^+, β^+ and the negative parts Λ^-, β^- for the $m \times n$-matrices Λ and the m-vector β will be understood elementwise and componentwise respectively.

Hence, for any interval system of the form (1) having m equations for n variables, it is possible to define the set $\Xi_{\Lambda(m,n),\beta(m)}(\mathbf{A}, \mathbf{b})$. Furthermore, let assume that the matrix $\Lambda(m, n)$ and the vector $\beta(m)$ are "easily computable" in the following sense.

Definition 2. Let us speak that the functions Λ and β are *easily computable* if there exists a pseudo-polynomial time algorithm computing the matrix $\Lambda(m, n)$ and the vector $\beta(m)$, i.e. the algorithm whose processing time is limited by the polynomial of m and n.

Also, we will say that an interval matrix $\mathbf{A}$ is *integer* if the endpoints of its entries are integer numbers.

Let us consider the following two problems:

Problem $N(\Lambda, \beta)$
(checking nonemptyness of the generalized solution sets $\Xi_{\Lambda,\beta}(\mathbf{A}, \mathbf{b})$)

> **Given.** An integer interval $m \times n$-matrix $\mathbf{A} = [A_c - \Delta,\ A_c + \Delta]$ and an integer interval m-vector $\mathbf{b} = [b_c - \delta,\ b_c + \delta]$.
> **Question.** Is it true that $\Xi_{\Lambda(m,n),\beta(m)}(\mathbf{A}, \mathbf{b}) \neq \emptyset$?

The second problem is that of outer estimation of the solution set $\Xi_{\Lambda,\beta}(\mathbf{A}, \mathbf{b})$.

Problem $E(\Lambda, \beta)$
(estimation a coordinate of the nonempty generalized solution sets $\Xi_{\Lambda,\beta}(\mathbf{A},\mathbf{b})$)

> **Given.** An integer interval $m \times n$-matrix $\mathbf{A}$, an integer interval m-vector $\mathbf{b}$ and a number k_0, $1 \leq k_0 \leq n$ such that $\Xi_{\Lambda(m,n),\beta(m)}(\mathbf{A}, \mathbf{b}) \neq \emptyset$.
> **Question.** Is it true that
>
> $$\max\{x_{k_0} \mid x \in \Xi_{\Lambda(m,n),\beta(m)}(\mathbf{A}, \mathbf{b})\} \geq 0 \ ? \qquad (7)$$

It will be obvious from the foregoing considerations that the computational complexity of these problems is substantially determined by the number of the existential quantifiers in the definition of $\Xi_{\Lambda,\beta}(\mathbf{A}, \mathbf{b})$, i.e. by the number of $(+1)$'s in the matrix $\Lambda(m, n)$ and in the vector $\beta(m)$. Roughly speaking, if the number of existential quantifiers is "large enough", that is, a sufficiently large number of the columns of the matrix Λ contain at least one $(+1)$, and a sufficiently large number of rows of the extended matrix $(\Lambda\beta)$ contain at least one $+1$, then both above formulated problems are NP-complete. If the total number of $(+1)$'s in the matrix Λ grows slowly in comparison with the number mn (specifically, it has order of $\log_2(mn)$), then these problems can be solved in the polynomial time.

To formulate what is meant by the term "sufficiently many existential quantifiers", we need giving additional clarification. When defining the term precisely, we will use usual notation for the submatrices of some matrix [25], i.e. if $\Lambda = (\lambda_{ij})$ is an $m \times n$-matrix and $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_l\}$, $1 \leq i_1 < i_2 \ldots < i_k \leq m$, $1 \leq j_1 < j_2 \ldots < j_l \leq n$, then by $\Lambda(I|J)$ we denote the $k \times l$-matrix located at the intersections of the rows with the numbers $i_1, \ldots, i_k$ and the columns with the numbers $j_1, \ldots, j_l$. Similarly, for the m-vector β and $I = \{i_1, \ldots, i_k\}$, $1 \le i_1 < \ldots < i_k \le m$, by $\beta(I)$ we denote the k-vector with the corresponding coordinates.

Definition 3. Let us say that the functions Λ and β are *computationally 1-saturate* (or, briefly, 1-saturate) if there exists an algorithm allowing the numbers m, n, k, l and the two submatrices Λ_0, Λ_1 of the matrix $\Lambda(m, n)$ of dimensions $k \times s$ and $s \times l$, respectively, to be found for any natural number s, so that the following conditions hold:

1) the running time of the algorithms is restricted by a polynomial of s (that is, similar to definition 1, the algorithm is quasi-polynomial with respect to s);

2) $m \ge k + l + s + 1, \;\; n \ge l + s$;

3) if $\Lambda_0 = \Lambda(m, n)(K \mid J)$, $\Lambda_1 = \Lambda(m, n)(I \mid L)$, then $K \bigcap I = J \bigcap L = \emptyset$, i.e. submatrices are located in different rows and different columns;

4) each of the columns in the submatrix Λ_0 contains at least one $(+1)$;

5) each of the rows in the submatrix $(\Lambda_1 \gamma)$ obtained by adding of the column $\gamma = \beta(m)(I)$ to the submatrix Λ contains at least one $(+1)$.

In other words, up to within the transposition of rows and columns, the extended matrix $(\Lambda\beta)$ has the form

$$(\Lambda(m, n)\beta(m)) = \begin{pmatrix} \Lambda_0 & * & * & * \\ * & \Lambda_1 & * & \gamma \\ * & * & * & * \end{pmatrix}, \tag{8}$$

where the submatrices Λ_0 and Λ_1 possess the properties 4 and 5 from the definition 3.

Comment. Denote by $U_\Lambda(m, n)$ the number of $(+1)$'s in the matrix $\Lambda(m, n)$. If the functions Λ, β are 1-saturate then from the condition 1 of the definition 3 and from the fact that the complexity of the matrix $\Lambda(m, n)$ is greater or equal to mn it follows that there exist such numbers $C > 1$, $M > 1$ that $mn \le Cs^M$.

Since $U_\Lambda(m, n) \ge s$ according to condition 4, we get the estimate

$$U_\Lambda(m, n) \ge \left(\frac{1}{C}\right)^{\frac{1}{M}} (mn)^{\frac{1}{M}},$$

i.e., in this case for some $M > 1$ the relation

$$\limsup_{m,n \to \infty} \frac{U_\Lambda(m, n)}{\sqrt[M]{mn}} > 0. \tag{9}$$

holds true.

Therefore, the condition (9) is at least necessary for the functions Λ, β to be 1-saturate. It imposes a restriction from below on the order of growth of $U_\Lambda(m, n)$.

Theorem 3. *If the functions Λ, β are easily computable and 1-saturate then the problem $N(\Lambda, \beta)$ and the problem $E(\Lambda, \beta)$ are NP-complete.*

Proof. The fact that these problems lie in the class NP follows from the description of $\Xi_{\Lambda,\beta}$ given in theorem 2. Indeed, it obviously follows from the description that the intersection of $\Xi_{\Lambda,\beta}$ with any orthant may be defined as a set of solutions of the system of $2m+n$ linear inequalities of n variables, whose solvability and the estimate on the extremums of the coordinates of its solutions can be determined with the use of the polynomial algorithm [16].

We demonstrate now that NP-complete problem Partition [11] is polynomially reducible to both our problems under study. The problem **Partition** is as follows:

Given. Positive integer numbers $d_1, \ldots, d_s$, $s > 1$.
Question. Is there a sequence of signs $\varepsilon_l, \ldots, \varepsilon_s \in \{-1, 1\}$ such that $\varepsilon_l d_1 + \ldots + \varepsilon_s d_s = 0$.

Let us reduce this problem to the problem $N(\Lambda, \beta)$. For the given positive integer numbers $d_1, \ldots, d_s$, we choose $m = m_s$, $n = n_s$, $k = k_s$, $l = l_s$ so that the conditions of definition 3 is satisfied. Furthermore, without loss in generality, let us assume that the extended matrix $(\Lambda\beta)$ has the form (8).

We denote

$$l_i = \sum_{j=1}^{s} \lambda_{ij}^+$$

for $i = \overline{1, k}$ and

$$l_i = \sum_{j=s+1}^{s+l} \lambda_{ij}^+$$

for $i = \overline{k+1, k+s}$ and consider the following system composed of m_s interval equations of n_s variables

$$\begin{cases} \sum\limits_{j=1}^{s} [-\lambda_{ij}^+, \lambda_{ij}^+] x_j = l_i & \text{for} \quad i = \overline{1, k}, \\ (l_i + \beta_i^+) x_{i-k} + \sum\limits_{j=s+1}^{s+l} [-\lambda_{ij}^+, \lambda_{ij}^+] x_j = \beta_i^+ & \text{for} \quad i = \overline{k+1, k+s}, \\ x_{i-k} = 1 & \text{for} \quad i = \overline{k+s+1, k+s+l}, \\ x_1 d_1 + \ldots + x_s d_s = 0 & \text{for} \quad i = k+s+l+1, \\ 0 = 0 & \text{for} \quad i > k+s+l+1. \end{cases} \tag{10}$$

From theorem 2, we get that the vector $x \in \mathbb{R}^{n_s}$ belongs to $\Xi_{\Lambda,\beta}$ of the system (10) if it satisfies the following system of inequalities

$$\begin{cases} l_i \le \sum\limits_{j=1}^{s} \lambda_{ij}^+ |x_j| & \text{for} \quad i = \overline{1, k}, \\ (l_i + \beta_i^+) |x_{i-k}| \le \sum\limits_{j=s+1}^{s+l} \lambda_{ij}^+ |x_j| + \beta_i^+ & \text{for} \quad i = \overline{k+1, k+s}, \\ |x_{i-k} - 1| \le 0 & \text{for} \quad i = \overline{k+s+1, k+s+l}, \\ |x_1 d_1 + \ldots + x_s d_s| \le 0 & \text{for} \quad i = k+s+l+1. \end{cases} \tag{11}$$

It is time to demonstrate now that the vector $x = (x_1, \ldots, x_{n_s})^\top \in \mathbb{R}^{n_s}$ satisfies the system (11) if and only if

$$\begin{cases} x_{s+1} = \ldots = x_{s+l} = 1, \\ x_1, \ldots, x_s \in \{-1, 1\}, \\ x_1 d_1 + \ldots + x_s d_s = 0. \end{cases} \tag{12}$$

The fact that, under the conditions (12), $x \in \mathbb{R}^{n_s}$ satisfies the system (11) can easily be verified by the straightforward substitution.

To prove the converse implication, let us imagine that for $x \in \mathbb{R}^{n_s}$ the inequality (11) is satisfied. Then the first and last of the equalities (12) obviously follow from the third and forth inequalities of (11) respectively. Next, from the first equality of (12) for $i = \overline{k+1, k+s}$ we obtain

$$\sum_{j=s+1}^{s+l} \lambda_{ij}^{+}|x_j| = \sum_{j=s+1}^{s+l} \lambda_{ij}^{+} = l_i,$$

and then from the second inequality of (11) we have that $(l_i + \beta_i^{+})|x_{i-k}| \leq (l_i + \beta_i^{+})$. And since due to the condition 5 of definition 3 it follows that $l_i + \beta_i^{+} \geq 1$ for $i = \overline{k+1, k+s}$ then

$$|x_i| \leq 1, \quad \text{for} \quad i = \overline{1, s}. \tag{13}$$

Let $k_j = \sum\limits_{i=1}^{k} \lambda_{ij}^{+}$ for $j = \overline{1, s}$. Notice that $\sum\limits_{j=1}^{s} k_j = \sum\limits_{i=1}^{k} l_i$ and, due to the condition 4 of definition 3, $k_j \geq 1$, $j = \overline{1, s}$.

Further, by adding together all the first inequalities from (11) we obtain that

$$\sum_{j=1}^{s} k_j = \sum_{i=1}^{k} l_i \leq \sum_{i=1}^{k}\sum_{j=1}^{s} \lambda_{ij}^{+}|x_j| = \sum_{j=1}^{s}\left(\sum_{i=1}^{k} \lambda_{ij}^{+}\right)|x_j| = \sum_{j=1}^{s} k_j|x_j|,$$

i.e., $\sum\limits_{j=1}^{s} k_j \leq \sum\limits_{j=1}^{s} k_j|x_j|$.

It follows from the latter inequality and the inequality (13) that $|x_j| = 1$ for all $j = \overline{1, s}$. Hence, the second conditions of (12) is valid.

Next, we can obviously conclude that, for the system (10), the solution set $\Xi_{\Lambda,\beta}$ is nonempty if and only if there exists a solution of the problem **Partition** for the given $d_1, \ldots, d_s$. Furthermore, due to the condition of the theorem, the system (10) is constructed by applying $d_1, \ldots, d_s$ times the algorithms which are polynomial with respect to the length of the input. Therefore, the problem **Partition** is polynomially reducible to the problem $N(\Lambda, \beta)$.

We consider now the problem $E(\Lambda, \beta)$. Let positive integer numbers $d_1, \ldots, d_s$ be given and $d_{s+1} = d_1 + \ldots + d_s$. We choose $m = m_{s+1}$, $n = n_{s+1}$, $k = k_{s+1}$, $l = l_{s+1}$ so that the conditions of definition 2 (in which $(s+1)$ is substituted for s are satisfied. As before, we can assume that the matrix $(\Lambda\beta)$ has the form (8), putting $l_i = \sum\limits_{j=1}^{s+1} \lambda_{ij}^{+}$ for $i = \overline{1, k}$ and $l_i = \sum\limits_{j=s+2}^{s+l} \lambda_{ij}^{+}$ for $i = \overline{k+1, k+s+1}$.

Let us consider the system composed of m interval inequalities of n variables, which is similar to the system (10):

$$\begin{cases} \sum\limits_{j=1}^{s+1} [-\lambda_{ij}^{+}, \lambda_{ij}^{+}]x_j = l_i & \text{for} \quad i = \overline{1, k}, \\ (l_i + \beta_i^{+})x_{i-k} + \sum\limits_{j=s+2}^{s+l+1} [-\lambda_{ij}^{+}, \lambda_{ij}^{+}]x_j = \beta_i^{+} & \text{for} \quad i = \overline{k+1, k+s+1}, \\ x_{i-k} = 1 & \text{for} \quad i = \overline{k+s+2, k+s+l+1}, \\ 2x_1d_1 + \ldots + 2x_sd_s + x_{s+1}d_{s+1} = d_{s+1} & \text{for} \quad i = k+s+l+2, \\ 0 = 0 & \text{for} \quad i > k+s+l+2. \end{cases} \tag{14}$$

Repeatedly applying theorem 2, we conclude that the vector $x \in \mathbb{R}^{n_{s+1}}$ belongs to the solutions set $\Xi_{\Lambda,\beta}$ of the system (14) if and only if it satisfies the following system of inequalities

$$\begin{cases} l_i \le \sum\limits_{j=1}^{s+1} \lambda_{ij}^{+}|x_j| & \text{for} \quad i = \overline{1,k}, \\ (l_i + \beta_i^{+})|x_{i-k}| \le \sum\limits_{j=s+2}^{s+l+1} \lambda_{ij}^{+}|x_j| + \beta_i^{+} & \text{for} \quad i = \overline{k+1, k+s+1}, \\ |x_{i-k} - 1| \le 0 & \text{for} \quad i = \overline{k+s+2, k+s+l+1}, \\ |2x_1 d_1 + \ldots + 2x_s d_s + x_{s+1} d_{s+1} - d_{s+1}| \le 0 & \text{for} \quad i = k+s+l+1. \end{cases} \tag{15}$$

Similar to the above, we can draw that the system (15) is equivalent to the following conditions:

$$\begin{cases} x_{s+2} = \ldots = x_{s+l+1} = 1, \\ x_1, \ldots, x_{s+1} \in \{-1, 1\}, \\ 2x_1 d_1 + \ldots + 2x_s d_s + x_{s+1} d_{s+1} = d_{s+1}. \end{cases} \tag{16}$$

Note that, for the system (14), the set $\Xi_{\Lambda,\beta}$ is nonempty, since the vector $x^0 = (x_1^0, \ldots, x_{n_{s+1}}^0)$ such that $x_1^0 = \ldots = x_s^0 = 1$, $x_{s+1}^0 = -1$, $x_{s+2}^0 = \ldots = x_{s+l+1}^0 = 1$, $x_i = 0$ for $i > s + l + 1$ obviously satisfies the conditions (16) and, hence, belongs to $\Xi_{\Lambda,\beta}$.

Let in (7) $k_0 = s + 1$. Taking into account the second conditions of (16), we obtain that, for the system (14), the inequality (7) holds if and only if a vector $x \in \Xi_{\Lambda,\beta}$ can be found, such that $x_{s+1} = 1$. But from the third of the equalities (16) it follows that for $x \in \Xi_{\Lambda,\beta}$ the condition $x_{s+1} = 1$ is equivalent to the equality $x_1 d_1 + \ldots + x_s d_s = 0$. Therefore, the problem $E(\Lambda, \beta)$ can be solved positively for the system (14) if and only if the problem **Partition** is solvable for these $d_1, \ldots, d_s$. Since, furthermore, the system (19) is constructed with the use of $d_1, \ldots, d_s$ times of polynomial algorithms, we have that the problem Partition is polynomially reducible to the problem $E(\Lambda, \beta)$. Q.E.D.

Note that interval $m \times n$-matrices $\mathbf{A}$ and the interval m-vector $\mathbf{b}$ used in the proof of theorem 3 meet the additional requirement

$$\Lambda^- \circ \Delta = \Theta_{m,n}, \qquad \beta^- \circ \delta = \Theta_m,$$

where $\Theta_{m,n}$ is zero matrix, Θ_m is zero vector. Hence, the equality $\Xi_{\Lambda,\beta}(\mathbf{A}, \mathbf{b}) = \Xi_{\text{uni}}(\mathbf{A}, \mathbf{b})$ holds true for them, and, consequently, it is possible to use the technique from [18] to reduce both the problem $N(\Lambda, \beta)$ and the problem $E(\Lambda, \beta)$ to the same problems but with positive interval matrices.

More precisely, let us call the system (1) *strongly positive* if $\underline{A} > \Theta_{m,n}$, $\underline{b} > \Theta_m$. Then the following statement holds.

Corollary. If the functions Λ, β are easily computable and 1-saturate then the problem $N(\Lambda, \beta)$ and the problem $E(\Lambda, \beta)$ for strongly positive interval systems are NP-complete.

Let us now show that if the number of (+1)'s in the matrix $\Lambda(m, n)$ is "not too large", then the problem $N(\Lambda, \beta)$ and the problem $E(\Lambda, \beta)$ are polynomially solvable.

Theorem 4. *If the functions Λ, β are easily computable and if the condition*

$$\limsup_{m,n \to \infty} \frac{U_\Lambda(m, n)}{\log_2(mn)} \le C$$

holds for some fixed integer C, then there exist polynomial time algorithms that solve the problem $N(\Lambda, \beta)$ and the problem $E(\Lambda, \beta)$.

Proof. In accordance with theorem 2, the problem $N(\Lambda, N)$ is equivalent to the problem of checking the solvability of the system

$$| A_c x - b_c| \leq (\Lambda \circ \Delta)\, |x| + \beta \circ \delta.$$

By decomposing Λ into the positive and negative parts $\Lambda = \Lambda^+ - \Lambda^-$, the latter inequality can be rewritten in the form

$$| A_c x - b_c| + (\Lambda^- \circ \Delta)\, |x| \leq (\Lambda^+ \circ \Delta)|x| + \beta \circ \delta. \tag{17}$$

Note that, since $U_\Lambda(m, n) \leq C \log_2(mn)$, the right-hand side of this inequality has no more than $C \log_2(mn)$ coordinates of the vector x with nonzero coefficients. Since Λ is easily computable, we can find these coordinates in polynomial time, and assume that the right-hand side of (17) contains only $|x_1|, \dots, |x_k|$ up to the renumeration of the components, where $k \leq C \log_2(mn)$.

Next, it can be readily shown that the system (17) is equivalent to the following one

$$\begin{cases} A_c x - b_c = u_1 - u_2, \\ x = v_1 - v_2, \\ u \geq 0,\ u_2 \geq 0,\ v_1 \geq 0,\ v_2 \geq 0, \\ u_1 + u_2 + (\Lambda^- \circ \Delta)(v_1 + v_2) \ \leq (\Lambda^+ \circ \Delta)|x| + \beta \circ \delta. \end{cases} \tag{18}$$

Now, if we fix the signs of the first k coordinates of the vector x in the system (18), then it transforms into a system of linear equations and inequalities, while its solvability can be determined for polynomial time [16]. Therefore, in order to verify the solvability of (17) we need only to write out all the possible distributions of the signs for the first k coordinates of the vector x (2^k totally), and, for each one of them, to investigate the system of linear equations and inequalities. Since the number of such systems $2^k \leq 2^{C \log_2 mn} = (mn)^C$, we can solve also the question of solvability (17) for polynomial time. Hence, the problem $N(\Lambda, \beta)$ is polynomially solvable.

For the problems $E(\Lambda, \beta)$, the proof is similar. Q.E.D.

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