ON THE FOURIER TRANSFORM OF THE DISTRIBUTIONAL KERNEL $K_{\alpha,\beta,\gamma,\nu}$ RELATED TO THE OPERATOR $\oplus^k$

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Рассмотрено преобразование Фурье ядра $K_{\alpha,\beta,\gamma,\nu}$, где α,β,γ,ν — комплексные параметры. Исследовано преобразование Фурье свертки $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'}$, где $\alpha,\beta,\gamma,\nu,\alpha',\beta',\gamma',\nu'$ — комплексные параметры.

1. Introduction

The operator $\oplus^k$ can be factorized into the form

$$\oplus^k=\left[\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k\left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}+i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k\left[\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}-i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k, \quad (1.1)$$

where $p+q=n$ is the dimension of the space $\mathbb{C}^n$, $i=\sqrt{-1}$ and k is a nonnegative integer.

The operator $\left(\sum\limits_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2-\left(\sum\limits_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2$ is first introduced by A. Kananthai [1] and named the Dimond operator denoted by

$$\diamondsuit=\left(\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}\right)^2-\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2. \quad (1.2)$$

Let us denote the operators L_1 and L_2 by

$$L_1=\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}+i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}, \quad (1.3)$$

$$L_2=\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}-i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}. \quad (1.4)$$

Thus (1.1) can be written by

$$\oplus^k=\diamondsuit^k L_1^k L_2^k. \quad (1.5)$$

Now consider the convolutions $R_\alpha^H(u) * R_\beta^\ell(v) * S_\gamma(w) * T_\nu(z)$ where $R_\alpha^H, R_\beta^\ell, S_\gamma$ and T_ν are defined by (2.2), (2.4), (2.6) and (2.7) respectively.

We defined the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ by

$$K_{\alpha,\beta,\gamma,\nu} = R_\alpha^H * R_\beta^\ell * S_\gamma * T_\nu. \tag{1.6}$$

Since the function $R_\alpha^H(u), R_\beta^\ell(v), S_\gamma(w)$ and $T_\nu(z)$ are all tempered distribution see [1, p. 30, 31] and [6, p. 154, 155], then the convolutions on the right hand side of (1.6) exists and is a tempered distribution. Thus $K_{\alpha,\beta,\gamma,\nu}$ is well defined and also a tempered distribution.

In this paper, at first we study the Fourier transform $\Im K_{\alpha,\beta,\gamma,\nu}$ or $\widehat{K_{\alpha,\beta,\gamma,\nu}}$ where $K_{\alpha,\beta,\gamma,\nu}$ is defined by (1.6).

After that we put $\alpha = \beta = \gamma = \nu = 2k$, then we obtain $\widehat{K_{2k,2k,2k,2k}}$ related to the elementary solution of the operator $\oplus^k$.

We also study the Fourier transform of the convolution $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'}$.

2. Preliminaries

Definition 2.1. *Let $x = (x_1, x_2, ..., x_n)$ be a point in the space $\mathbb{C}^n$ of the n-dimensional complex space and write*

$$u = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2, \tag{2.1}$$

where $p + q = n$ is the dimension of $\mathbb{C}^n$.

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the set of an interior of the forward cone and $\overline{\Gamma_+}$ denotes it closure and $\mathbb{R}^n$ is the n-dimensional Euclidean space.

For any complex number α, define

$$R_\alpha^H(u) = \begin{cases} \dfrac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{for} \quad x \in \Gamma_+, \\ 0 \quad \text{for} & x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}.$$

The function R_α^H is called the ultra-hyperbolic Kernel of Marcel Riesz and was introduced by Y. Nozaki [5, p. 72].

It is well known that R_α^H is an ordinary function if $Re(\alpha) \geq n$ and is a distribution of α if $Re(\alpha) < n$. Let supp $R_\alpha^H(u)$ denote the support of $R_\alpha^H(u)$ and suppose supp $R_\alpha^H(u) \subset \bar{\Gamma}_+$, that is supp $R_\alpha^H(u)$ is compact.

Definition 2.2. *Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and write*

$$v = x_1^2 + x_2^2 + ... + x_n^2. \tag{2.3}$$

For any complex number β, define

$$R^{\ell}_{\beta}(v) = 2^{-\beta}\pi^{\frac{-n}{2}}\Gamma\left(\frac{n-\beta}{2}\right)\frac{v^{\frac{\beta-n}{2}}}{\Gamma\left(\frac{\beta}{2}\right)}. \tag{2.4}$$

The function $R^{\ell}_{\beta}(v)$ is called the elliptic Kernel of Marcel Riesz and is ordinary function for $Re(\beta) \geq n$ and is a distribution of β for $Re(\beta) < n$.

Definition 2.3. *Let $x = (x_1, x_2, ..., x_n)$ be a point of the space $\mathbb{C}^n$ of the n-dimensional complex space and write*

$$w = x_1^2 + x_2^2 + ... + x_p^2 - i\left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2\right), \tag{2.5}$$

where $p + q = n$ is the dimension of $\mathbb{C}^n$ and $i = \sqrt{-1}$.

For any complex number γ, define the function

$$S_{\gamma}(w) = 2^{-\gamma}\pi^{\frac{-n}{2}}\Gamma\left(\frac{n-\gamma}{2}\right)\frac{w^{\frac{\gamma-n}{2}}}{\Gamma\left(\frac{\gamma}{2}\right)}. \tag{2.6}$$

The function $S_{\gamma}(w)$is an ordinary function if $Re(\gamma) \geq n$ and is a distribution of γ for $Re(\gamma) < n$.

Definition 2.4. *For any complex number ν, define the function*

$$T_{\nu}(z) = 2^{-\nu}\pi^{\frac{-n}{2}}\Gamma\left(\frac{n-\nu}{2}\right)\frac{z^{\frac{\nu-n}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}, \tag{2.7}$$

where

$$z = x_1^2 + x_2^2 + ... + x_p^2 + i\left(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2\right), \tag{2.8}$$

$x = (x_1, x_2, ..., x_n) \in \mathbb{C}^n$, $p + q = n$ is the dimension of $\mathbb{C}^n$ and $i = \sqrt{-1}$.

We have $T_{\nu}(z)$is an ordinary function if $Re(\nu) \geq n$ and is a distribution of ν for $Re(\nu) < n$.

Definition 2.5. *Let $f(x)$ be continuous function on $\mathbb{R}^n$ where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. The Fourier transform of $f(x)$ denoted by $\Im f$ or $\hat{f}(\xi)$ and is defined by*

$$\Im f(x) = \hat{f}(\xi) = \int\limits_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)dx, \tag{2.9}$$

where $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$ and $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$.

Definition 2.6. *Let $\mu(x)$ be a tempered distribution with compact support. The Fourier transform of $\mu(x)$ is defined by*

$$\hat{\mu}(\xi) =< \mu(x), e^{-i(\xi,x)} > . \tag{2.10}$$

Lemma 2.1. *The functions R^H_{α}, R^{ℓ}_{β}, S_{γ} and T_{ν} defined by (2.2), (2.4), (2.6) and (2.7) respectively, are all tempered distributions.*

Proof see [1, p. 30, 31] and [6, p. 154, 155].

Lemma 2.2. *The function $(-1)^k K_{2k,2k,2k,2k}(x)$ is an elementary solution of the operator $\oplus^k$, that is $\oplus^k(-1)^k K_{2k,2k,2k,2k}(x) = \delta$ where $\oplus^k$ is defined by (1.1), $K_{2k,2k,2k,2k}(x)$ is defined by (1.6) with $\alpha = \beta = \gamma = \nu = 2k$ and δ is the Dirac-delta distribution.*

Proof see [4, p. 66].

Lemma 2.3. *1. The Fourier transform of the convolution $R^H_\alpha(u) * R^\ell_\beta(v)$ is given by the formula*

$$\Im\left(R^H_\alpha(u)*R^\ell_\beta(v)\right)=\frac{(i)^q 2^{\alpha+\beta}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\pi^n}{K_n(\alpha)H_n(\beta)\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-\beta}{2}\right)}\left(\sqrt{\sum_{r=1}^{p}\xi_r^2-\sum_{j=p+1}^{p+q}\xi_j^2}\right)^{-\alpha}\left(\sqrt{\sum_{r=1}^{n}\xi_r^2}\right)^{-\beta}, \tag{2.11}$$

where $R^H_\alpha(u)$ and $R^\ell_\beta(v)$ are defined by (2.2) and (2.4) respectively,

$$H_n(\beta)=\frac{\Gamma\left(\frac{\beta}{2}\right)2^\beta\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n-\beta}{2}\right)} \quad \text{and} \quad i=\sqrt{-1}.$$

In particular, if $\alpha=\beta=2k$ then (2.11) becomes

$$\Im(R^H_{2k}(u)*R^\ell_{2k}(v))=\frac{(-1)^k}{\left[\left(\sum_{r=1}^{p}\xi_r^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]^k}, \tag{2.12}$$

where k is nonnegative integer and $(-1)^k R^H_{2k}(u) * R^\ell_{2k}(v)$ is an elementary solution of the operator $\diamondsuit^k$ iterated k-times defined by (1.2).

Moreover $|\Im(R^H_{2k}(u)*R^\ell_{2k}(v))|\leq M$, where M is constant, that is $\Im$ is bounded, that implies $\Im$ is continuous on the space S' of the tempered distribution.

2. The Fourier transform of the convolution $S_\gamma(w) * T_\nu(z)$ is given by the formula

$$\Im(S_\gamma(w)*T_\nu(z))=\frac{1}{H_n(\gamma)H_n(\nu)}\frac{2^{\gamma+\nu}\pi^n\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)\Gamma\left(\frac{n-\nu}{2}\right)}\times$$

$$\times\left(\sqrt{\sum_{r=1}^{p}\xi_r^2+i\sum_{j=p+1}^{p+q}\xi_j^2}\right)^{-\gamma}\left(\sqrt{\sum_{r=1}^{p}\xi_r^2-i\sum_{j=p+1}^{p+q}\xi_j^2}\right)^{-\nu}, \tag{2.13}$$

where $S_\gamma(w)$ and $T_\nu(z)$ are defined by (2.6) and (2.7) respectively,

$$H_n(\gamma)=\frac{\Gamma\left(\frac{\gamma}{2}\right)2^\gamma\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n-\gamma}{2}\right)} \quad \text{and} \quad H_n(\nu)=\frac{\Gamma\left(\frac{\nu}{2}\right)2^\nu\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n-\nu}{2}\right)}.$$

In particular, if $\gamma=\nu=2k$ then (2.13) becomes

$$\Im(S_\gamma(w)*T_\nu(z))=\frac{1}{\left[\left(\xi_1^2+\xi_2^2+...+\xi_p^2\right)^2+\left(\xi_{p+1}^2+\xi_{p+2}^2+...+\xi_{p+q}^2\right)^2\right]^k}, \tag{2.14}$$

where k is a nonnegative integer and $(-1)^k(-i)^{\frac{q}{2}}S_{2k}(w)$ and $(-1)^k(-i)^{\frac{q}{2}}T_{2k}(z)$ are elementary solutions of the operators L_1 and L_2 defined by (1.3) and (1.4) respectively.

Proof: 1. To prove (2.11) and (2.12) see [2] and to show that $\Im$ is bounded, now

$$\left|\Im(R^H_{2k}(u)*R^\ell_{2k}(v))\right| = \left|\frac{(-1)^k}{\left[\left(\sum\limits_{r=1}^{p}\xi_r^2\right)^2 - \left(\sum\limits_{j=p+1}^{p+q}\xi_j^2\right)^2\right]^k}\right| \leq \frac{1}{\left|\left(\sum\limits_{r=1}^{p}\xi_r^2\right)^2 + \left(\sum\limits_{j=p+1}^{p+q}\xi_j^2\right)^2\right|^k} \leq M,$$

where $p+q=n$ for large $\xi_r \in \mathbb{R}$ $(r=1,2,...,n)$.

That implies that $\Im$ is continuous on the space S' of tempered distribution. For the case $(-1)^k R^H_{2k}(u)*R^\ell_{2k}(v)$ is an elementary solution of the operator $\diamondsuit^k$, see [1].

2. We have

$$S_\gamma(w) = \frac{w^{\frac{\gamma-n}{2}}}{H_n(\gamma)}, \quad \text{where} \quad H_n(\gamma) = \frac{\Gamma\left(\frac{\gamma}{2}\right)2^\gamma\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n-\gamma}{2}\right)}$$

and $w = x_1^2 + x_2^2 + ... + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2)$.

Now, changing the variable $x_1 = y_1$, $x_2 = y_2$, ..., $x_p = y_p$,

$$x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}}, \quad x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, \quad ..., \quad x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}.$$

Then we obtain $w = y_1^2 + y_2^2 + ... + y_p^2 + y_{p+1}^2 + y_{p+2}^2 + ... + y_{p+q}^2$.

Let $\rho^2 = y_1^2 + y_2^2 + ... + y_{p+q}^2$, $p+q=n$. Then

$$\Im S_\gamma(w) = \frac{1}{H_n(\gamma)}\int\limits_{\mathbb{R}^n} e^{-i(\xi,x)}w^{\frac{\gamma-n}{2}}dx = \frac{1}{H_n(\gamma)}\int\limits_{\mathbb{R}^n} e^{-i(\xi,x)}\rho^{\gamma-n}\frac{\partial(x_1,x_2,...,x_n)}{\partial(y_1,y_2,...,y_n)}dy_1dy_2...dy_n =$$

$$= \frac{1}{H_n(\gamma)(-i)^{\frac{q}{2}}}\int\limits_{\mathbb{R}^n} \rho^{\gamma-n}e^{-i(\xi_1y_1+\xi_2y_2+...+\xi_py_p+\frac{\xi_{p+1}}{\sqrt{-1}}y_{p+1}+...+\frac{\xi_{p+q}}{\sqrt{-1}}y_{p+q})}dy =$$

$$= \frac{1}{H_n(\gamma)(-i)^{\frac{q}{2}}}2^\gamma\pi^{\frac{n}{2}}\frac{\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)}\left(\sqrt{\xi_1^2+\xi_2^2+...+\xi_p^2+i(\xi_{p+1}^2+\xi_{p+2}^2+...+\xi_{p+q}^2)}\right)^{-\gamma} \tag{2.15}$$

by [5, p. 194]

Similarly, for $T_\nu(z) = \frac{z^{\frac{\nu-n}{2}}}{H_n(\nu)}$ we have $z = x_1^2 + x_2^2 + ... + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2)$. Putting $x_1 = y_1$, $x_2 = y_2$, ..., $x_p = y_p$, $x_{p+1} = \frac{y_{p+1}}{\sqrt{i}}$, ..., $x_{p+q} = \frac{y_{p+q}}{\sqrt{i}}$. Thus $z = y_1^2 + y_2^2 + ... + y_{p+q}^2$, $p+q=n$. Let $\rho^2 = y_1^2 + y_2^2 + ... + y_{p+q}^2$, $p+q=n$. Then

$$\Im T_\nu(z) = \frac{1}{H_n(\nu)}\int\limits_{\mathbb{R}^n} e^{-i(\xi,x)}z^{\frac{\nu-n}{2}}dx =$$

$$= \frac{1}{H_n(\nu)(i)^{\frac{q}{2}}}\int\limits_{\mathbb{R}^n} \rho^{\nu-n}e^{-i(\xi_1y_1+\xi_2y_2+...+\xi_py_p+\frac{\xi_{p+1}}{\sqrt{i}}y_{p+1}+...+\frac{\xi_{p+q}}{\sqrt{i}}y_{p+q})}dy =$$

$$= \frac{2^{\nu}\pi^{\frac{n}{2}}}{H_n(\nu)(i)^{\frac{q}{2}}} \frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{n-\nu}{2}\right)} \left(\sqrt{\xi_1^2+\xi_2^2+...+\xi_p^2 - i(\xi_{p+1}^2+\xi_{p+2}^2+...+\xi_{p+q}^2)}\right)^{-\nu}. \tag{2.16}$$

Since $S_\gamma(w)$ and $T_\nu(z)$ are tempered distributions, then $S_\gamma(w) * T_\nu(z)$ exists and $\Im(S_\gamma(w) * T_\nu(z)) = \Im(S_\gamma(w))\Im(T_\nu(z))$.

Thus

$$\Im(S_\gamma(w) * T_\nu(z)) = \Im(S_\gamma(w))\Im(T_\nu(z)) =$$

$$= \frac{2^{\gamma+\nu}\pi^n}{H_n(\gamma)H_n(\nu)} \frac{\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)} \frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{n-\nu}{2}\right)} \left(\sqrt{\sum_{r=1}^{p}\xi_r^2 + i\sum_{j=p+1}^{p+q}\xi_j^2}\right)^{-\gamma} \left(\sqrt{\sum_{r=1}^{p}\xi_r^2 - i\sum_{j=p+1}^{p+q}\xi_j^2}\right)^{-\nu} \tag{2.17}$$

by (2.15) and (2.16).

Now consider

$$\frac{2^{\gamma+\nu}\pi^n}{H_n(\gamma)H_n(\nu)} \frac{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{n-\gamma}{2}\right)\Gamma\left(\frac{n-\nu}{2}\right)}. \tag{2.18}$$

Putting $\gamma = \nu = 2k$, thus (2.18) becomes

$$\frac{2^{4k}\pi^n}{H_n(2k)H_n(2k)} \frac{\Gamma\left(\frac{2k}{2}\right)\Gamma\left(\frac{2k}{2}\right)}{\Gamma\left(\frac{n-2k}{2}\right)\Gamma\left(\frac{n-2k}{2}\right)} = \frac{2^{4k}\pi^n}{2^{4k}\pi^n} \frac{\Gamma\left(\frac{n-2k}{2}\right)\Gamma\left(\frac{n-2k}{2}\right)}{\Gamma(k)\Gamma(k)} \times$$

$$\times \frac{\Gamma(k)\Gamma(k)}{\Gamma\left(\frac{n-2k}{2}\right)\Gamma\left(\frac{n-2k}{2}\right)} = 1.$$

Thus, from (2.17)

$$\Im(S_{2k}(w) * T_{2k}(z)) = \frac{1}{\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]^k}. \tag{2.19}$$

3. Main results

Theorem 3.1. *The Fourier transform of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}(x)$ is given by the formula*

$$\Im K_{\alpha,\beta,\gamma,\nu}(x) = \left(\frac{(\pi)^{2n}(i)^q 2^{\alpha+\beta+\gamma+\nu}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{\nu}{2}\right)\left(\sqrt{\sum_{r=1}^{p}\xi_r^2 - \sum_{j=p+1}^{p+q}\xi_j^2}\right)^{-\alpha}}{K_n(\alpha)H_n(\beta)H_n(\gamma)H_n(\nu)}\right) \times$$

$$\times\left(\frac{\left(\sqrt{\sum_{r=1}^{n}\xi_r^2}\right)^{-\beta}\left(\sqrt{\sum_{r=1}^{p}\xi_r^2+i\sum_{j=p+1}^{p+q}\xi_j^2}\right)^{-\gamma}\left(\sqrt{\sum_{r=1}^{p}\xi_r^2-i\sum_{j=p+1}^{p+q}\xi_j^2}\right)^{-\nu}}{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-\beta}{2}\right)\Gamma\left(\frac{n-\gamma}{2}\right)\Gamma\left(\frac{n-\nu}{2}\right)}\right). \tag{3.1}$$

In particular, if $\alpha=\beta=\gamma=\nu=2k$ then (3.1) becomes

$$\Im K_{\alpha,\beta,\gamma,\nu}(x)=\frac{(-1)^k}{\left[\left(\xi_1^2+\xi_2^2+...+\xi_p^2\right)^4-\left(\xi_{p+1}^2+\xi_{p+2}^2+...+\xi_{p+q}^2\right)^4\right]^k}. \tag{3.2}$$

Moreover $(-1)^kK_{2k,2k,2k,2k}(x)$ is an elementary solution of the operator $\oplus^k$ defined by (1.1).

Proof. Now $K_{\alpha,\beta,\gamma,\nu}(x)=R_\alpha^H(u)*R_\beta^\ell(v)*S_\gamma(w)*T_\nu(z)$ by (1.6). Since $R_\alpha^H, R_\beta^\ell, S_\gamma(w)$ and $T_\nu(z)$ are all tempered distributions by Lemma 2.1, thus $\Im K_{\alpha,\beta,\gamma,\nu}(x)=\Im(R_\alpha^H(u)*R_\beta^\ell(v))\Im(S_\gamma(w)*T_\nu(z))$. By (2.11) and (2.17), we obtained (3.1) as required. For the case $\alpha=\beta=\gamma=\nu=2k$, by (2.12) and (2.19) we obtain (3.2) as required.

For $(-1)^kK_{2k,2k,2k,2k}(x)$ is an elementary solution of the operator $\oplus^k$ see [4, p. 66].

Theorem 3.2. *The Fourier transform of the convolution $K_{\alpha,\beta,\gamma,\nu}(x)*K_{\alpha',\beta',\gamma',\nu'}(x)$ is given by the formula*

$$\Im\left(K_{\alpha,\beta,\gamma,\nu}(x)*K_{\alpha',\beta',\gamma',\nu'}(x)\right)=\Im K_{\alpha,\beta,\gamma,\nu}(x)\Im K_{\alpha',\beta',\gamma',\nu'}(x), \tag{3.3}$$

where $K_{\alpha,\beta,\gamma,\nu}(x)$ is defined by (1.6), α, β, γ, ν, α', β', γ' and ν' are complex numbers.

Proof. Now $K_{\alpha,\beta,\gamma,\nu}(x)=R_\alpha^H(u)*R_\beta^\ell(v)*S_\gamma(w)*T_\nu(z)$ by (1.6). Since $K_{\alpha,\beta,\gamma,\nu}(x)$ is the convolutions of all tempered distributions, thus $K_{\alpha,\beta,\gamma,\nu}(x)$ is also a tempered distribution and the convolution $K_{\alpha,\beta,\gamma,\nu}(x)*K_{\alpha',\beta',\gamma',\nu'}(x)$ exists.

Since $K_{\alpha,\beta,\gamma,\nu}(x)$ is a tempered distribution, then the Fourier transform

$$\Im\left(K_{\alpha,\beta,\gamma,\nu}(x)*K_{\alpha',\beta',\gamma',\nu'}(x)\right)=\left(\Im K_{\alpha,\beta,\gamma,\nu}(x)\right)\left(\Im K_{\alpha',\beta',\gamma',\nu'}(x)\right),$$

where $\Im(K_{\alpha,\beta,\gamma,\nu}(x))$ is given by (3.1).

Corollary 3.1. *(The alternative proof of Theorem 3.1). The Fourier transform*

$$\Im K_{2k,2k,2k,2k}(x)=\frac{(-1)^k}{\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^4-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right]^k},$$

where k is a nonnegative integer and $K_{\alpha,\beta,\gamma,\nu}(x)$ is defined by (1.6).

Proof. From Theorem 3.1 with the particular case $\alpha=\beta=\gamma=\nu=2k$, we can find $\Im K_{2k,2k,2k,2k}(x)$ directly from the elementary solution of the operator $\oplus^k$ defined by (1.1). Since $(-1)^kK_{2k,2k,2k,2k}(x)$ is an elementary solution of the operator $\oplus^k$.

Thus $\oplus^k(-1)^kK_{2k,2k,2k,2k}(x)=\delta$ or $(\oplus^k(-1)^k\delta)*K_{2k,2k,2k,2k}(x)=\delta$.

By taking the Fourier transform both sides, we obtain

$$\Im(\oplus^k(-1)^k\delta)*\Im K_{2k,2k,2k,2k}(x)=\Im\delta=1. \tag{3.4}$$

Now consider $\Im(\oplus^k(-1)^k\delta)$. Since δ is tempered distribution with compact support. Thus $\Im(\oplus^k(-1)^k\delta) =< \oplus^k(-1)^k\delta, e^{-i(\xi,x)} >= < \diamondsuit^kL_1^kL_2^k(-1)^k\delta, e^{-i(\xi,x)} >$ by (2.10) where $\oplus^k=\diamondsuit^kL_1^kL_2^k$ by (1.5). Thus

$$<\diamondsuit^kL_1^kL_2^k(-1)^k\delta, e^{-i(\xi,x)}>=<\diamondsuit^kL_1\delta,(-1)^kL_2^ke^{-i(\xi,x)}>=$$

$$=<\diamondsuit^k L_1\delta,(-1)^k(-1)^k\left(\sum_{r=1}^{p}\xi_r^2-i\sum_{j=p+1}^{p+q}\xi_j^2\right)^k e^{-i(\xi,x)}>=<\diamondsuit^k\delta,\left(\sum_{r=1}^{p}\xi_r^2-i\sum_{j=p+1}^{p+q}\xi_j^2\right)^k L_1 e^{-i(\xi,x)}>=$$

$$=<\diamondsuit^k\delta,\left(\sum_{r=1}^{p}\xi_r^2-i\sum_{j=p+1}^{p+q}\xi_j^2\right)^k\left(\sum_{r=1}^{p}\xi_r^2+i\sum_{j=p+1}^{p+q}\xi_j^2\right)^k(-1)^k e^{-i(\xi,x)}>=$$

$$=<\delta,(-1)^k\left(\left(\sum_{r=1}^{p}\xi_r^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k\diamondsuit^k e^{-i(\xi,x)}>=$$

$$=<\delta,(-1)^k\left(\left(\sum_{r=1}^{p}\xi_r^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k v\times\left(\left(\sum_{r=1}^{p}\xi_r^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k e^{-i(\xi,x)}>$$

$$=<\delta,(-1)^k\left(\left(\sum_{r=1}^{p}\xi_r^2\right)^4-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k e^{-i(\xi,x)}>=(-1)^k\left(\left(\sum_{r=1}^{p}\xi_r^2\right)^4-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k.$$

Thus $\Im(\oplus^k(-1)^k\delta)=(-1)^k\left(\left(\sum\limits_{r=1}^{p}\xi_r^2\right)^4-\left(\sum\limits_{j=p+1}^{p+q}\xi_j^2\right)^4\right)^k.$

Thus by (3.4) we obtain

$$\Im K_{2k,2k,2k,2k}(x)=\frac{(-1)^k}{\left[\left(\sum\limits_{i=1}^{p}\xi_i^2\right)^4-\left(\sum\limits_{j=p+1}^{p+q}\xi_j^2\right)^4\right]^k}.$$

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