CHANNEL FLOWS AND STEADY VARIATIONAL INEQUALITIES OF THE NAVIER—STOKES TYPE*

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Исследуется стационарное течение вязкой несжимаемой жидкости в канале с условиями на выходе, отличными от условий Дирихле. Для того чтобы контролировать кинетическую энергию жидкости в канале, предполагается, что возможные обратные течения на выходе в некотором смысле ограничены. Течения, удовлетворяющие этому условию, заполняют выпуклое подмножество пространства определенных функций. На этом выпуклом множестве формулируется вариационное неравенство типа Навье—Стокса и доказывается существование слабого решения. Предположение, используемое для определения выпуклого подмножества, более ограничительно, чем предположение, из работы [3]. С другой стороны, условие в теореме существования менее строго, чем условие из [3]. Кроме того, изучается вопрос о том, в каком смысле слабое решение удовлетворяет уравнениям Навье—Стокса и смешанным граничным условиям, если это решение гладкое.

Introduction

Let Ω be a simply connected bounded domain with a Lipschitz boundary $\partial\Omega$ in $\mathbb{R}^3$. Ω can be considered as a channel filled up by a moving fluid, Γ_1 will denote the part of the boundary where the fluid is flowing into the channel or where the channel has fixed boundaries and Γ_2 will denote the part of the boundary where the fluid is supposed to leave the channel. Precisely, we will suppose that Γ_1, Γ_2 are open disjoint subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ and the $\partial\Omega - \Gamma_1 - \Gamma_2$ consists of a finite number of closed simple smooth curves whose each point belongs to $\partial\Gamma_1 \cap \partial\Gamma_2$ ($\partial\Gamma_1$, respectively $\partial\Gamma_2$, denotes $\overline{\Gamma_1} - \Gamma_1$, respectively $\overline{\Gamma_2} - \Gamma_2$). We will also suppose that $\overline{\Gamma_2}$ is a union of a finite number of disjoint simple smooth surfaces $S_1, \ldots, S_r$. Let us denote their boundaries by $C_1, \ldots, C_r$. It follows from the previous assumptions that $C_1, \ldots, C_r$ are closed simple smooth curves.

The motion of a viscous incompressible fluid in Ω, generally non-stationary, can be described by the Navier—Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla\mathbf{u} = \mathbf{f} - \nabla p + \nu \Delta \mathbf{u} \tag{1}$$

and the equation of continuity

$$\operatorname{div} \mathbf{u} = 0, \tag{2}$$

*The research was supported by the Grant Agency of the Czech Republic (grant No. 201/99/0267) and by the research plan of the Ministry of Education of the Czech Republic (MSM98/21000010).

The authors are responsible for possible misprints and the quality of translation.

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity, p is the pressure, $\mathbf{f} = (f_1, f_2, f_3)$ is an external body force and ν is the kinematic coefficient of viscosity. ν is a positive constant.

It is natural to prescribe a Dirichlet boundary condition for the velocity of the fluid on Γ_1. However, since the situation on the output of the channel depends on the behaviour of the fluid inside the channel and it is not known in advance, it is not reasonable to use a boundary condition of the same type on Γ_2. It is a matter of discussion which boundary condition should be used on Γ_2. One of the possibilities is the condition

$$-p\,\mathbf{n} + \nu\,\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = \mathbf{F}, \tag{3}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is an outer normal vector on Γ_2 and $\mathbf{F} = (F_1, F_2, F_3)$ is a prescribed vector function on Γ_2. It can be shown that a weak problem for equations (1), (2) with no condition on Γ_2 involves condition (3) implicitly. It means that if its solution is "smooth enough" then except for equations (1), (2), it must also satisfy condition (3). This is why condition (3) is often called the "do nothing condition".

Existence or uniqueness of solutions of system (1), (2) with condition (3) on the part of $\partial\Omega$ is known either locally in time (see e. g. P. Kučera, Z. Skalák [4]) or for "small data" (i. e. "small" initial velocity, "small" external force and "small" function F in condition (3) — see e. g. P. Kučera [5, 7]). The problem with condition (3) is difficult mainly because solutions of (1), (2), (3) need not satisfy an energy inequality. This is due to the fact that boundary condition (3) on Γ_2 does not exclude backward flows on Γ_2, bringing into Ω an uncontrollable amount of kinetic energy. The kinetic energy in Ω can be estimated by an additional condition on Γ_2 which estimates the backward flows. For example if $c_0 > 0$ then the following condition can be used:

$$\int_{\Gamma_2} \Big[\operatorname{dist}(\mathbf{u}(x), K_\alpha(x))\Big]^a dS_x \le c_0, \tag{4}$$

where $K_\alpha(x)$ denotes the cone of vectors in $\mathbb{R}^3$ whose angle with $\mathbf{n}(x)$ is less than or equal to α, $\alpha \in \left(0, \frac{\pi}{2}\right)$, and $\operatorname{dist}(\mathbf{u}(x), K_\alpha(x))$ means the distance between $\mathbf{u}(x)$ and $K_\alpha(x)$. However, condition (4) has the consequence: if we use it then we are searching for a solution not in a whole function space (which will be exactly specified later) but in its convex subset. This is why we do not use the Navier — Stokes equation (1) and instead of it, we describe the flow by means of a certain variational inequality which arises from equation (1). We have already used this approach in paper [2] where we studied the non-stationary case and we have proved the global in time existence of a weak solution without any restriction on the size of the input data.

In this paper, we study a steady motion of a viscous incompressible fluid in channel Ω with the Dirichlet boundary condition

$$\mathbf{u} = \mathbf{u}_* \tag{5}$$

on Γ_1 and with conditions (3) and (4) on Γ_2. Analogously as in [2], we formulate a variational inequality of the Navier — Stokes type, which is now steady, and we prove its weak solvability without any requirement on the size of the input data $\mathbf{f}$, $\mathbf{F}$ and $\mathbf{u}_*$. We also show that if the solution is smooth enough then it satisfies the Navier — Stokes equation and moreover, if it in some sense finds itself in the interior of the convex set which is defined by means of condition (4) then it also satisfies condition (3) on Γ_2.

1. Formulation of the problem and some properties of its solution

Let c_0 be a positive real number and $a \in (2,4)$. Since $2a/(a-1) < 4$, there exists a continuous operator of traces from V into $L^{2a/(a-1)}(\partial\Omega)$. Let $K_\alpha(x) = \left\{ y \in \mathbb{R}^3 : \dfrac{y\mathbf{n}(x)}{|y|} \geq \cos\alpha \right\}$, $\alpha \in \left(0, \dfrac{\pi}{2}\right)$, $x \in \Gamma_2$.

$\|\,.\,\|_r$ will denote the L^r-norm on domain Ω, $\|\,.\,\|_{r,\Gamma_2}$ the L^r-norm on Γ_2 and $\|\,.\,\|_{r,s}$ will denote the $W^{r,s}$-norm on Ω. The $W^{r,s}$-norm on $\partial\Omega$ will be denoted by $\|\,.\,\|_{r,s(\partial\Omega)}$. In order not to complicate the notation, we will denote traces on $\partial\Omega$ of functions which are defined a. e. in Ω by the same letters as the functions themselves.

Suppose that function $\mathbf{u}_*$ in boundary condition (5) is such that it can be extended from Γ_1 onto the whole boundary $\partial\Omega$ so that the extended function (it will be also denoted by $\mathbf{u}_*$) belongs to $W^{1/2,2}(\partial\Omega)^3$,

$$\int_{\partial\Omega} \mathbf{u}_* \cdot \mathbf{n}\, dS = 0$$

and on each simple smooth surface S_k, which is a component of $\overline{\Gamma_2}$, $\mathbf{u}_* = \mathbf{0}$ in a certain neighbourhood S_k' of the boundary C_k and $\mathbf{u}_*(x) = k(x)\,\mathbf{n}$ (for some $k(x) \geq 0$) in all other points $x \in S_k$.

Lemma 1. *There exists a function $\mathbf{V} \in W^{1,2}(\Omega)^3$ with the following properties:*

1. $\operatorname{div} \mathbf{V} = 0$ *a. e. in* Ω,
2. $\mathbf{V} = \mathbf{u}_*$ *a. e. on* Γ_1,
3. $\displaystyle\int_{\Gamma_2} \Big[\operatorname{dist}(\mathbf{V}(x), K_\alpha(x))\Big]^a dS_x = 0,$
4. $\displaystyle\int_{\Omega} (\mathbf{v}^1 \nabla \mathbf{v}^2)\mathbf{V}\, dx \leq (\nu/2)\, \|\nabla \mathbf{v}^1\|_2\, \|\nabla \mathbf{v}^2\|_2$ *for all* $\mathbf{v}^1$, $\mathbf{v}^2 \in W^{1,2}(\Omega)^3$ *whose traces are zero on* Γ_1.

Proof: The proof is in many steps similar to the proof of Lemma VIII.4.2 in [1] or Lemma II.1.8 in [9]. The main differences between our situation and the situation treated in [1] and in [9] are that we require $\mathbf{V}$ to be equal to $\mathbf{u}_*$ only on the part Γ_1 of $\partial\Omega$, our $\mathbf{v}^1$, $\mathbf{v}^2$ have the traces equal to zero only on Γ_1 and we require function $\mathbf{V}$ to satisfy the equality in item 3 of the lemma. We will show the construction of function $\mathbf{V}$ and we will especially pay attention to the equality in item 3. We follow the arguments given in [3]:

There exists a function $\mathbf{w}^1 \in W^{2,2}(\Omega)^3$ such that $\mathbf{u}_* = \operatorname{curl} \mathbf{w}^1$ in the sense of traces on $\partial\Omega$. Moreover,

$$\|\mathbf{w}^1\|_{2,2} \leq c_1\, \|\mathbf{u}_*\|_{1/2,2(\partial\Omega)}, \tag{6}$$

where $c_1 = c_1(\Omega)$. (See Lemma VIII.4.1 in [1].) It follows from the Stokes theorem that the line integral of $\mathbf{w}^1$ on each closed simple curve C' in S_k' (where $k \in \{1; \ldots; r\}$) whose interior is a subset of S_k' is equal to zero. (By interior we mean interior on surface S_k — i. e. that one of the two components of the set $S_k - C'$ which has a positive distance from the boundary C_k of surface S_k.) Furthermore, the line integral of $\mathbf{w}^1$ on each closed simple curve C'' in S_k' whose interior contains $S_k - S'_k$ and which is positively oriented when observed from the outer part of S_k is equal to a certain nonnegative number β_k — the flux of $\mathbf{u}_*$ through surface S_k. Suppose for simplicity that $\beta_k > 0$ for all $k \in \{1; \dots; r\}$.

If $x \in \overline{\Omega}$ then we define $\delta(x)$ as the infimum of lengths of all possible curves in $\overline{\Omega}$ whose initial point is x and their terminal point is on Γ_1. (I. e. $\delta(x)$ is a distance of x from Γ_1, measured only on trajectories never leaving $\overline{\Omega}$.)

Using the same approach as in the proof of Lemma III.6.2 and in the proof of Lemma III.6.3 in [1], it can be shown that there exists $c_3 > 0$ such that

$$\|\mathbf{u}/\delta\|_2 \le c_3 \|\mathbf{u}\|_{1,2} \tag{7}$$

for all $\mathbf{u} \in W^{1,2}(\Omega)^3$ whose trace on Γ_1 is zero.

Let ϵ be a positive parameter. We define the function

$$\xi_\epsilon(\lambda) = \begin{cases} 1 & \text{if } \lambda < \mathrm{e}^{-2/\epsilon}, \\ -1 - \epsilon \ln \lambda & \text{if } \mathrm{e}^{-2/\epsilon} \le \lambda < \mathrm{e}^{-1/\epsilon}, \\ 0 & \text{if } \mathrm{e}^{-1/\epsilon} \le \lambda. \end{cases}$$

Let R_ϵ be the mollifier with the kernel whose support has the diameter $\frac{1}{2}\,\mathrm{e}^{-2/\epsilon}$. We define

$$\psi_\epsilon(x) = R_\epsilon \xi_\epsilon(\delta(x)).$$

Then

i) $|\psi_\epsilon(x)| \le 1$ for all $x \in \overline{\Omega}$,

ii) $\psi_\epsilon(x) = 1$ if $\delta(x) \le \frac{1}{2}\,\mathrm{e}^{-2/\epsilon}$,

iii) $0 < \psi_\epsilon(x) < 1$ if $\frac{1}{2}\,\mathrm{e}^{-2/\epsilon} < \delta(x) < \mathrm{e}^{-1/\epsilon} + \frac{1}{2}\,\mathrm{e}^{-2/\epsilon}$,

iv) $\psi_\epsilon(x) = 0$ if $\delta(x) \ge \mathrm{e}^{-1/\epsilon} + \frac{1}{2}\,\mathrm{e}^{-2/\epsilon}$,

v) $|\nabla\psi_\epsilon(x)| \le \epsilon/\delta(x)$ if $\delta(x) \le \mathrm{e}^{-1/\epsilon} + \frac{1}{2}\,\mathrm{e}^{-2/\epsilon}$.

It can be shown that if ϵ is sufficiently small (what we will further assume) then the following assertions hold for each $k \in \{1; \dots; r\}$:

— The set $S''_{k,\epsilon} = \{x \in S_k;\ \nabla\psi_\epsilon(x) \ne \mathbf{0}\}$ is a subset of S'_k and $S''_{k,\epsilon}$ can be expressed as a union of mutually disjoint closed simple smooth curves $C^y_{k,\epsilon}$ (for $y \in (0,1)$), each of whose is an equipotential line of function ψ_ϵ: $C^y_{k,\epsilon} = \{x \in S_k;\ \psi_\epsilon(x) = y\}$.

— Each of the curves $C^y_{k,\epsilon}$ contains the set $S_k - S'_k$ in its interior.

— The vector $\mathbf{n} \times \nabla\psi_\epsilon$ is tangent to each of the curves $C^y_{k,\epsilon}$. Let us further assume that this vector defines the orientation of $C^y_{k,\epsilon}$.

— The system $\mathcal{S}$ of curves in $S''_{k,\epsilon}$ whose tangent vector is $\nabla\psi_\epsilon - (\nabla\psi_\epsilon \mathbf{n})\,\mathbf{n}$ (the tangent to S_k component of $\nabla\psi_\epsilon$) is perpendicular to the system of curves $C^y_{k,\epsilon}$ and these curves also cover the whole set $S''_{k,\epsilon}$.

Suppose further that $P(t)$; $t \in [0, \beta_k]$ is a parametrization of one of the curves $C^y_{k,\epsilon}$ — let it be e. g. the curve $C^{1/2}_{k,\epsilon}$. Due to the smoothness of the curve $C^{1/2}_{k,\epsilon}$, the parametrization can be chosen so that $\dot{P}_+(0) = \dot{P}_-(\beta_k)$. Let us denote by ${}^{\perp}C^t_{k,\epsilon}$ (for $t \in [0, \beta_k)$) that one of the curves from system $\mathcal{S}$ defined above, whose intersection with curve $C^{1/2}_{k,\epsilon}$ is the point $P(t)$.

Let us now define a real function ϕ_1 on $S''_{k,\epsilon}$ in this way: $\phi_1(x) = t$ in all points $x \in {}^{\perp}C^t_{k,\epsilon}$. Put $\mathbf{w}^2 = \nabla\phi_1$ in $S''_{k,\epsilon}$. In fact, since function ϕ_1 has a discontinuity on curve ${}^{\perp}C^0_{k,\epsilon}$, $\nabla\phi_1$ is not defined on ${}^{\perp}C^0_{k,\epsilon}$. However, it follows from the introduction of function ϕ_1 that $\nabla\phi_1$ can be continuously extended to ${}^{\perp}C^0_{k,\epsilon}$. Thus, we understand by $\nabla\phi_1$ the value of this extension on ${}^{\perp}C^0_{k,\epsilon}$. Function $\mathbf{w}^2$ has obviously these properties:

— The line integral of $\mathbf{w}^2$ on each closed simple curve C' in S''_k whose interior is a subset of S''_k is equal to zero and the line integral of $\mathbf{w}^2$ on each closed simple curve C'' in S''_k whose interior contains $S_k - S'_k$ and which is positively oriented when observed from the outer part of S_k is equal to β_k.

— $\mathbf{w}^2$ is tangent to the curves $C^y_{k,\epsilon}$ and so it has the same direction as $\mathbf{n} \times \nabla\psi_\epsilon$ in all points of $S''_{k,\epsilon}$.

Thus, the line integral of $\mathbf{w}^1 - \mathbf{w}^2$ does not depend on the path in $S''_{k,\epsilon}$. Hence we can define a scalar function ϕ_2 on $S''_{k,\epsilon}$ so that we choose a fixed point $x_0 \in S''_{k,\epsilon}$ and we put $\phi_2(x)$ equal to the value of the line integral of $\mathbf{w}^1 - \mathbf{w}^2$ on any curve in $S''_{k,\epsilon}$ which starts in x_0 and terminates in x. Function ϕ_2 can be extended from the union of all $S''_{k,\epsilon}$ ($k \in \{1; \ldots; r\}$) to a smooth function in $\overline{\Omega}$ such that its gradient is in $W^{2,2}(\Omega)^3$ and $\nabla\phi_2\mathbf{n} = 0$ on $S''_{k,\epsilon}$ ($k \in \{1; \ldots; r\}$). Then the function $\mathbf{w} = \mathbf{w}^1 - \nabla\phi_2$ coincides with $\mathbf{w}^2$ on $S''_{k,\epsilon}$ and $\operatorname{curl}\mathbf{w}$ is identical with $\operatorname{curl}\mathbf{w}^1$. It also follows from (6) that for some $c_2 > 0$

$$\|\mathbf{w}\|_{2,2} \le c_1 \|\mathbf{u}_*\|_{1/2,2(\partial\Omega)} + c_2. \tag{8}$$

It follows from the smoothness of curves C_k that c_2 can be chosen so that it is independent of ϵ.

We put $\mathbf{V}_\epsilon = \operatorname{curl}(\psi_\epsilon \mathbf{w})$. Then $V_\epsilon = \psi_\epsilon \operatorname{curl}\mathbf{w} + \nabla\psi_\epsilon \times \mathbf{w}$. It follows from our choice of ϵ that $\psi_\epsilon(x) = 0$ in all points $x \in S_k$ where $\operatorname{curl}\mathbf{w}(x) \neq \mathbf{0}$. Thus, $\mathbf{V}_\epsilon\mathbf{n} = (\nabla\psi_\epsilon \times \mathbf{w})\mathbf{n} = (\mathbf{n} \times \nabla\psi_\epsilon)\mathbf{w}$. This is obviously positive in all points of $S''_{k,\epsilon}$ and equal to zero in all points of $S_k - S''_{k,\epsilon}$ (for each $k \in \{1; \ldots; r\}$). Moreover, it follows from the smoothness of S_k that fo sufficiently small ϵ the angle between $\nabla\psi_\epsilon$ and $\mathbf{n}(x)$ on $S''_{k,\epsilon}$ is from $[\pi/2 - \alpha/2, \pi/2 + \alpha/2]$. Hence, the angle between $\mathbf{V}_\epsilon = \nabla\psi_\epsilon \times \mathbf{w}$ and $\mathbf{n}$ is from $[0, \alpha/2]$, and function $\mathbf{V}_\epsilon$ satisfies condition in item 3 of the lemma.

It remains to verify that if ϵ is small enough and we put $\mathbf{V} = \mathbf{V}_\epsilon$ then the inequality in item 4 is satisfied. However, the proof can be done in the same way as the proof of estimate (4.38) in [1], p. 32, and so we do not show it here. ■

We will further search for the velocity $\mathbf{u}$ in the form $\mathbf{u} = \mathbf{V} + \mathbf{v}$ where $\mathbf{V}$ is the function given by Lemma 1 and $\mathbf{v} \in W^{1,2}(\Omega)^3$ is a new unknown function such that $\mathbf{v}|_{\Gamma_1} = \mathbf{0}$. Let $\mathcal{V}$ be a set of infinitely differentiable divergence-free vector functions in $\overline{\Omega}$ which have a compact support in $\Omega \cup \Gamma_2$. Let $\mathcal{K}$ be a subset of $\mathcal{V}$ which contains only functions $\mathbf{h}$ satisfying the condition

$$\int_{\Gamma_2} \Big[\operatorname{dist}(\mathbf{V}(x) + \mathbf{h}(x), K_\alpha(x))\Big]^a dS_x \le c_0. \tag{9}$$

Denote by H (respectively V) the closure of $\mathcal{V}$ in $L^2(\Omega)^3$ (respectively in the norm $\|\nabla\,.\,\|_2$) and denote by K the closure of $\mathcal{K}$ in V. The dual space to V will be denoted by V' and the duality between elements of V' and V will be denoted by $\langle\,.\,,\,.\,\rangle$.

It follows from the existence of a continuous operator of traces from V into $L^a(\Gamma_2)^3$ that

$$K = \left\{ \mathbf{h} \in V : \int_{\Gamma_2} \Big[\operatorname{dist}(\mathbf{V}(x) + \mathbf{h}(x), K_\alpha(x))\Big]^a dS_x \le c_0 \right\}.$$

If we also use item 3 of Lemma 1, we can show that there exists $\epsilon_1 > 0$ so that K contains the ϵ_1-neighbourhood of zero in V. Moreover, K is a convex set in V; This can be proved by means of the Minkowski inequality.

Let us formally derive the Navier—Stokes variational inequality now. (See e.g. [2] or [8] for similar approaches.) We use the steady Navier—Stokes equation in the form

$$\mathbf{u}\nabla\mathbf{u} - \mathbf{f} + \nabla p - \nu\,\Delta\mathbf{u} = \mathbf{0}, \tag{10}$$

we multiply the left hand side by $\mathbf{q}-\mathbf{v}$ (where $\mathbf{q}$ is a test function from $\mathcal{K}$), we integrate over Ω and we require the result to be greater or equal to zero. We obtain:

$$\int_\Omega [\,\mathbf{u}\nabla\mathbf{u} - \mathbf{f} + \nabla p - \nu\,\Delta\mathbf{u}\,](\mathbf{q}-\mathbf{v})\,dx \;\ge\; 0.$$

If we express $\mathbf{u}$ in the form $\mathbf{V}+\mathbf{v}$, integrate by parts, use boundary condition (3) and write the duality $\langle\,\mathbf{f}\,,\,\mathbf{q}-\mathbf{v}\,\rangle$ instead of the scalar product of $\mathbf{f}$ and $\mathbf{q}-\mathbf{v}$ in H, we obtain the inequality

$$\int_\Omega [\,(\mathbf{V}+\mathbf{v})\nabla(\mathbf{V}+\mathbf{v})\,](\mathbf{q}-\mathbf{v})\,dx + \nu\int_\Omega \nabla(\mathbf{V}+\mathbf{v})\nabla(\mathbf{q}-\mathbf{v})\,dx -$$
$$- \langle\,\mathbf{f}\,,\,\mathbf{q}-\mathbf{v}\,\rangle - \int_{\Gamma_2}\mathbf{F}(\mathbf{q}-\mathbf{v})\,dS \;\ge\; 0. \tag{11}$$

We will solve the following problem:

Problem 1. *Let* $\mathbf{f}\in V'$ *and* $\mathbf{F}\in L^2(\Gamma_2)^3$. *We are looking for function* $\mathbf{v}\in K$ *such that inequality (11) holds for all functions* $\mathbf{q}\in\mathcal{K}$.

The next two theorems show that if a solution of Problem 1 is aposteriori smooth enough then it satisfies the Navier—Stokes equation (10) and the inequality holds only on Γ_2 (Theorem 1) or the solution satisfies boundary condition (3) on Γ_2 (Theorem 2).

Theorem 1. *Let* $\mathbf{f}\in L^2(\Omega)^3$, *let function* $\mathbf{u}_*$ *can be extended from* Γ_1 *to* $\partial\Omega$ *so that except for already mentioned properties, the extended function belongs to* $W^{3/2,2}(\partial\Omega)^3$ *and let* $\mathbf{v}$ *be a solution of Problem 1 such that* $\mathbf{v}\in W^{2,2}(\Omega)^3$. *Then there exists* $p\in W^{1,2}(\Omega)$ *such that* $\mathbf{u}\equiv\mathbf{V}+\mathbf{v}$ *and* p *satisfy the steady Navier—Stokes equation (10) in* Ω *in a strong sense and*

$$\int_{\Gamma_2}\left[\,\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\,\mathbf{n} - \mathbf{F}\right](\mathbf{q}'-\mathbf{v})\,dS \;\ge\; 0 \tag{12}$$

for all $\mathbf{q}'\in K$.

Proof: Since function $\mathbf{u}_*$ has a higher regularity than it was assumed in Lemma 1, function $\mathbf{V}$ given by Lemma 1 can also be found so that it has a higher regularity, namely that it belongs to $W^{2,2}(\Omega)^3$. Let $\mathbf{q}'\in W^{2,2}(\Omega)^3\cap K$ at first and $\theta\in(0,1)$. There exists a sequence $\{\mathbf{q}^n\}$ in $\mathcal{K}$ such that $\mathbf{q}^n\to\theta\mathbf{q}'+(1-\theta)\mathbf{v}$ in $W^{2,2}(\Omega)^3$. If we use $\mathbf{q}=\mathbf{q}^n$ in inequality (11) and assume that $n\to+\infty$, we obtain

$$\theta\int_\Omega [\,(\mathbf{V}+\mathbf{v})\nabla(\mathbf{V}+\mathbf{v})\,](\mathbf{q}'-\mathbf{v})\,dx + \nu\,\theta\int_\Omega \nabla(\mathbf{V}+\mathbf{v})\nabla(\mathbf{q}'-\mathbf{v})\,dx -$$
$$-\,\theta\int_\Omega \mathbf{f}(\mathbf{q}'-\mathbf{v})\,dx - \theta\int_{\Gamma_2}\mathbf{F}(\mathbf{q}'-\mathbf{v})\,dS \;\ge\; 0.$$

If we divide this inequality by θ, assume that $\theta \to 0+$ and write $\mathbf{u}$ instead of $\mathbf{w}+\mathbf{v}$, we get

$$\int_\Omega [\,\mathbf{u}\nabla\mathbf{u} - \mathbf{f}\,](\mathbf{q}'-\mathbf{v})\,dx + \nu\int_\Omega \nabla\mathbf{u}\nabla(\mathbf{q}'-\mathbf{v})\,dx - \int_{\Gamma_2} \mathbf{F}(\mathbf{q}'-\mathbf{v})\,dS \;\geq\; 0,$$

$$\int_\Omega [\,\mathbf{u}\nabla\mathbf{u} - \nu\,\Delta\mathbf{u} - \mathbf{f}\,](\mathbf{q}'-\mathbf{v})\,dx + \int_{\Gamma_2}\Big[\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \mathbf{F}\Big](\mathbf{q}'-\mathbf{v})\,dS \;\geq\; 0. \tag{13}$$

Let ϕ be any function from $W^{2,2}(\Omega)^3 \cap W_0^{1,2}(\Omega)^3$ such that $\operatorname{div}\phi = 0$. If we successively use $\mathbf{q}' = \mathbf{v}+\phi$ and $\mathbf{q}' = \mathbf{v}-\phi$ in (13), we obtain

$$\int_\Omega [\,\mathbf{u}\nabla\mathbf{u} - \nu\,\Delta\mathbf{u} - \mathbf{f}\,]\phi\,dx \;=\; 0.$$

This implies the existence of $p \in W^{1,2}(\Omega)$ such that $\|p\|_2 \leq c_3(\Omega)\|\nabla p\|_2$ and

$$\mathbf{u}\nabla\mathbf{u} - \nu\,\Delta\mathbf{u} - \mathbf{f} \;=\; -\nabla p \tag{14}$$

a. e. in Ω (see [1]). Let us now return to inequality (13). If we use (14), apply the integration by parts and use the fact that functions $\mathbf{q}'$ and $\mathbf{v}$ are divergence-free, we obtain (12). The validity of (12) for all $\mathbf{q}' \in K$ follows from the density of $W^{2,2}(\Omega)^3 \cap K$ in K. ■

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled and moreover, let there exist a neighbourhood U of zero in V so that $\mathbf{v}+\mathbf{h} \in K$ for all $\mathbf{h} \in U$. Then $\mathbf{u} \equiv \mathbf{V}+\mathbf{v}$ and p (given by Theorem 1) satisfy boundary condition (3) a. e. on Γ_2.*

Proof: Let $\mathbf{h} \in U$. The functions $\mathbf{q}^1 = \mathbf{v}+\mathbf{h}$ and $\mathbf{q}^2 = \mathbf{v}-\mathbf{h}$ belong to K. If we successively use them in (12) instead of $\mathbf{q}'$, we obtain:

$$\int_{\Gamma_2}\Big[\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\,\mathbf{n} - \mathbf{F}\Big]\mathbf{h}\,dS \;=\; 0.$$

Since this holds for all $\mathbf{h} \in U$, $\mathbf{u}$ and p satisfy condition (3) a. e. on Γ_2. ■

2. Approximations and their estimates

We will prove that if c_0 is an arbitrary positive constant then Problem 1 has at least one solution in Section 4. In Section 3, we will construct a sequence of approximations and derive some estimates. We will use the Galerkin method combined with the method of penalisation.

Let P be the projector of V onto K which assigns to each element of V the nearest element in K and put $\Psi(\mathbf{h}) = \mathbf{h} - P\mathbf{h}$ for $\mathbf{h} \in V$. It follows from the convexity of K that Ψ is a monotone operator in V. It will be used as a penalisation in the following. Let us prove that

$$(\,\Psi(\mathbf{h})\,,\,\mathbf{h}\,)_V \geq \|\Psi(\mathbf{h})\|_V^2, \quad \text{and} \quad (\,\Psi(\mathbf{h})\,,\,\mathbf{h}\,)_V \geq \epsilon_1\|\Psi(\mathbf{h})\|_V \tag{15}$$

(for all $\mathbf{h} \in V$) at first. (We remind that K contains the ϵ_1-neighbourhood of zero in V.) The first inequality in (15) is obvious:

$$(\Psi(\mathbf{h}),\mathbf{h})_V = (\mathbf{h}-P\mathbf{h},\mathbf{h})_V = (\mathbf{h}-P\mathbf{h},\mathbf{h}-P\mathbf{h})_V + (\mathbf{h}-P\mathbf{h},P\mathbf{h}-\mathbf{0})_V$$

and since $\mathbf{0} \in K$ and K is convex, $(\mathbf{h} - P\mathbf{h}, P\mathbf{h} - \mathbf{0})_V \geq 0$ and so we obtain the desired inequality. The second inequality is clearly satisfied if $\mathbf{h} \in K$. Thus, let $\mathbf{h} \notin K$. Put $\mathbf{p} = \epsilon_1(\mathbf{h} - P\mathbf{h})/\|\mathbf{h} - P\mathbf{h}\|_V$. Then $\mathbf{p} \in K$ and

$$(\Psi(\mathbf{h}), \mathbf{h})_V = (\mathbf{h} - P\mathbf{h}, \mathbf{h})_V = (\mathbf{h} - P\mathbf{h}, \mathbf{h} - \mathbf{p})_V + (\mathbf{h} - P\mathbf{h}, \mathbf{p})_V =$$

$$= (\mathbf{h} - P\mathbf{h}, \mathbf{h} - P\mathbf{h})_V + (\mathbf{h} - P\mathbf{h}, P\mathbf{h} - \mathbf{p})_V + \epsilon_1(\mathbf{h} - P\mathbf{h}, \mathbf{h} - P\mathbf{h})_V/\|\mathbf{h} - P\mathbf{h}\|_V \geq$$

$$\geq \epsilon_1\|\mathbf{h} - P\mathbf{h}\|_V = \epsilon_1 \|\Psi(\mathbf{h})\|_V .$$

Put $V^2 = V \cap W^{2,2}(\Omega)^3$. Then V^2 is a Hilbert space with the same scalar product as $W^{2,2}(\Omega)^3$. Let functions e^k $(k = 1, 2, \dots)$ form a basis of V^2 which is orthonormal in H. It follows from the density of V^2 in V and in H and from the continuity of imbeddings of V^2 into V and into H that $\{\mathrm{e}^1, \mathrm{e}^2, \dots\}$ is also a basis in V and in H. Functions e^k $(k = 1, 2, \dots)$ can be chosen so that they all belong to $\mathcal{V}$. Let $n \in \mathbb{N}$ be given. We are looking for $\theta_k^n \in \mathbb{R}$ $(k = 1, \dots, n)$ so that the function

$$\mathbf{v}^n = \sum_{k=1}^{n} \theta_k^n \; \mathrm{e}^k \tag{16}$$

satisfies for $k = 1, \dots, n$ the equations

$$\int_\Omega \Big[(P\mathbf{v}^n + \mathbf{V})\nabla(\mathbf{v}^n + \mathbf{V}) \Big] \mathrm{e}^k \, dx + \nu \int_\Omega \nabla(\mathbf{v}^n + \mathbf{V})\nabla \, \mathrm{e}^k \, dx -$$

$$- \langle \mathbf{f}, \mathrm{e}^k \rangle - \int_{\Gamma_2} \mathbf{F} \, \mathrm{e}^k \, dS + n \left(1 + \|\mathbf{v}^n\|^2_{q,\Gamma_2}\right) \left(\Psi(\mathbf{v}^n), \mathrm{e}^k \right)_V = 0, \tag{17}$$

where $q \in (1, 4)$ will be chosen later. Substituting here from (16), we get a system of algebraic equations for unknowns θ_k^n:

$$\int_\Omega \Big[\Big(P\sum_{l=1}^{n} \theta_l^n \mathrm{e}^l + \mathbf{V}\Big)\nabla\Big(\sum_{m=1}^{n} \theta_m^n \mathrm{e}^m + \mathbf{V}\Big)\Big]\mathrm{e}^k dx + \nu \int_\Omega \nabla\Big(\sum_{l=1}^{n} \theta_l^n \mathrm{e}^l + \mathbf{V}\Big)\nabla \mathrm{e}^k dx - \langle \mathbf{f}, \mathrm{e}^k\rangle -$$

$$- \int_{\Gamma_2} \mathbf{F} \, \mathrm{e}^k \, dS + n \left(1 + \left(\int_{\Gamma_2} \left|\sum_{l=1}^{n} \theta_l^n \mathbf{e}^l\right|^q dS\right)^{\frac{2}{q}}\right) \left(\Psi(\sum_{m=1}^{n} \theta_m^n \, \mathrm{e}^m), \mathrm{e}^k \right)_V = 0 \tag{18}$$

for $k = 1, \dots, n$. Let us denote by θ^n the n-tuple $[\theta_1^n, \dots, \theta_n^n]$, by $\mathcal{G}_k(\theta^n)$ the left-hand side of equation (18) and by $\mathcal{G}(\theta^n)$ the n-tuple $[\mathcal{G}_1(\theta^n), \dots, \mathcal{G}_n(\theta^n)]$. Then system (18) is equivalent to the equation

$$\mathcal{G}(\theta^n) = 0^n, \tag{19}$$

where 0^n is the zero element in $\mathbb{R}^n$. $\mathcal{G}$ is a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.

We will show that there exists $R > 0$ (independent of n) such that

$$\mathcal{G}(\theta^n)\theta^n > 0 \tag{20}$$

for all $\theta^n \in \mathbb{R}^n$ such that $|\theta^n| = R$:

Multiplying $\mathcal{G}_k(\theta^n)$ by θ_k^n, summing over k from 1 to n, using (16), and using item 4 of Lemma 1 we obtain

$$\sum_{k=1}^{n} \mathcal{G}_k(\theta^n)\theta_k^n = \mathcal{G}(\theta^n)\theta^n = \int_\Omega \Big[(P\mathbf{v}^n + \mathbf{V})\nabla(\mathbf{v}^n + \mathbf{V}) \Big] \mathbf{v}^n \, dx + \nu \int_\Omega \nabla(\mathbf{v}^n + \mathbf{V})\nabla\mathbf{v}^n \, dx -$$

$$- \langle \mathbf{f}, \mathbf{v}^n \rangle - \int_{\Gamma_2} \mathbf{F}\mathbf{v}^n \, dS + n \left(1 + \|\mathbf{v}^n\|_{q,\Gamma_2}^2\right) \left(\Psi(\mathbf{v}^n), \mathbf{v}^n\right)_V =$$

$$= \int_\Omega \Big[(P\mathbf{v}^n + \mathbf{V})\nabla(\mathbf{v}^n + \mathbf{V}) \Big] (\mathbf{v}^n + \mathbf{V}) \, dx - \int_\Omega \Big[(P\mathbf{v}^n + \mathbf{V})\nabla(\mathbf{v}^n + \mathbf{V}) \Big] \mathbf{V} \, dx +$$

$$+ \nu \int_\Omega \nabla(\mathbf{v}^n + \mathbf{V})\nabla\mathbf{v}^n \, dx - \langle \mathbf{f}, \mathbf{v}^n \rangle - \int_{\Gamma_2} \mathbf{F}\mathbf{v}^n \, dS + n \left(1 + \|\mathbf{v}^n\|_{q,\Gamma_2}^2\right) \left(\Psi(\mathbf{v}^n), \mathbf{v}^n\right)_V =$$

$$= \int_{\Gamma_2} \Big[(P\mathbf{v}^n + \mathbf{V})\mathbf{n} \Big] \frac{1}{2} |\mathbf{v}^n + \mathbf{V}|^2 \, dS + \int_{\Gamma_1} (\mathbf{V}\mathbf{n}) \frac{1}{2} |\mathbf{V}|^2 \, dS - \int_\Omega \Big[P\mathbf{v}^n \nabla\mathbf{v}^n \Big] \mathbf{V} \, dx -$$

$$- \int_\Omega \Big[P\mathbf{v}^n \nabla\mathbf{V} \Big] \mathbf{V} \, dx - \int_\Omega \Big[\mathbf{V}\nabla(\mathbf{v}^n + \mathbf{V}) \Big] \mathbf{V} \, dx +$$

$$+ \nu \|\nabla\mathbf{v}^n\|_2^2 + \nu \int_\Omega \nabla\mathbf{V}\nabla\mathbf{v}^n \, dx - \langle \mathbf{f}, \mathbf{v}^n \rangle - \int_{\Gamma_2} \mathbf{F}\mathbf{v}^n \, dS + n \left(1 + \|\mathbf{v}^n\|_{q,\Gamma_2}^2\right) \left(\Psi(\mathbf{v}^n), \mathbf{v}^n\right)_V \geq$$

$$\geq \frac{1}{2} \int_{\Gamma_2^-} \Big((P\mathbf{v}^n + \mathbf{V})\mathbf{n} \Big) |\mathbf{v}^n + \mathbf{V}|^2 \, dS + \int_{\Gamma_1} (\mathbf{u}_*\mathbf{n}) \frac{1}{2} |\mathbf{u}_*|^2 \, dS - \frac{1}{2}\nu \|\nabla P\mathbf{v}^n\|_2 \|\nabla\mathbf{v}^n\|_2 -$$

$$- c_4 \|\nabla\mathbf{v}^n\|_2 - c_5 + \nu \|\nabla\mathbf{v}^n\|_2^2 + n \left(1 + \|\mathbf{v}^n\|_{q,\Gamma_2}^2\right) \left(\Psi(\mathbf{v}^n), \mathbf{v}^n\right)_V,$$

where Γ_2^- is the part of Γ_2 with $(P\mathbf{v}^n + \mathbf{V})\mathbf{n} \leq 0$.

The term $\frac{1}{2} \int_{\Gamma_2^-} [(P\mathbf{v}^n + \mathbf{V})\,\mathbf{n}] \, |\mathbf{v}^n + \mathbf{V}|^2 \, dS$ can be estimated by using Hölder's inequality. There exists $\varepsilon \in (0, 2/3)$, $r \in [1, a]$, and $q \in (1, 4)$ ($r = r(\varepsilon) > 4a\varepsilon/(2a - 4 + a\varepsilon)$), such that

$$\frac{1}{2} \int_{\Gamma_2^-} [(P\mathbf{v}^n + \mathbf{V})\,\mathbf{n}] \, |\mathbf{v}^n + \mathbf{V}|^2 \, dS \geq$$

$$\geq -\frac{1}{2} \left(\int_{\Gamma_2^-} |(P\mathbf{v}^n + \mathbf{V})\,\mathbf{n}|^a \, dS \right)^{\frac{1}{a}} \left[\left(\int_{\Gamma_2^-} |\mathbf{v}^n + \mathbf{V}|^q \, dS \right)^{\frac{1}{q}} \right]^{(2-\varepsilon)} \left[\left(\int_{\Gamma_2^-} |\mathbf{v}^n + \mathbf{V}|^r \, dS \right)^{\frac{1}{r}} \right]^{\varepsilon} \geq$$

$$\geq -c_0^{\frac{1}{a}} c_6 \|\mathbf{v}^n + \mathbf{V}\|_{q,\Gamma_2^-}^{2-\varepsilon} \|\mathbf{v}^n + \mathbf{V}\|_{r,\Gamma_2^-}^{\varepsilon} \geq$$

$$\geq -c_0^{\frac{1}{a}} c_6 \|\mathbf{v}^n + \mathbf{V}\|_{q,\Gamma_2^-}^{2-\varepsilon} \left(\|\mathbf{v}^n - P\mathbf{v}^n\|_{r,\Gamma_2^-} + \|P\mathbf{v}^n + \mathbf{V}\|_{r,\Gamma_2^-} \right)^{\varepsilon} \geq$$

$$\geq -c_0^{\frac{1}{a}} c_6 \|\mathbf{v}^n + \mathbf{V}\|_{q,\Gamma_2^-}^{2-\varepsilon} \left(\|\mathbf{v}^n - P\mathbf{v}^n\|_{r,\Gamma_2^-}^{\varepsilon} + \|P\mathbf{v}^n + \mathbf{V}\|_{r,\Gamma_2^-}^{\varepsilon} \right) \geq$$

$$\geq -c_0^{\frac{1}{a}} c_6 \left\|\mathbf{v}^n + \mathbf{V}\right\|_{q,\Gamma_2^-}^{2-\varepsilon} \left[\left\|\Psi\left(\mathbf{v}^n\right)\right\|_{r,\Gamma_2^-}^{\varepsilon} + \left(\frac{c_7}{\cos\alpha} \left(\int\limits_{\Gamma_2^-} \operatorname{dist}\left(P\mathbf{v}^n + \mathbf{V}, K_\alpha\right)^a dS \right)^{\frac{1}{a}} \right)^{\varepsilon} \right] \geq$$

(because, if $\mathbf{w}\mathbf{n} \leq 0$ then $|\mathbf{w}| \leq (1/\cos\alpha) \operatorname{dist}(\mathbf{w}, K_\alpha)$)

$$\geq -c_0^{\frac{1}{a}} c_6 \left\|\mathbf{v}^n + \mathbf{V}\right\|_{q,\Gamma_2^-}^{2-\varepsilon} \left[\left\|\Psi\left(\mathbf{v}^n\right)\right\|_{r,\Gamma_2^-}^{\varepsilon} + \left(\frac{c_7}{\cos\alpha} c_0^{\frac{1}{a}} \right)^{\varepsilon} \right] \geq$$

$$\geq -C_1 \left(\left\|\mathbf{v}^n\right\|_{q,\Gamma_2^-}^{2-\varepsilon} + 1 \right) \left[\left(\left\|\Psi\left(\mathbf{v}^n\right)\right\|_{r,\Gamma_2^-}^{\varepsilon} + C_2 \right) \right].$$

Constants C_1, C_2 depend on $\Omega,\ \Gamma_1,\ \Gamma_2,\ \mathbf{u}_*,\ \mathbf{V},\ a,\ \alpha,\ c_0,\ \varepsilon,$ and q. Using now the estimate of the term $\frac{1}{2}\int\limits_{\Gamma_2^-} \left[\left(P\mathbf{v}^n + \mathbf{V}\right)\mathbf{n}\right] \left|\mathbf{v}^n + \mathbf{V}\right|^2 dS$ for evaluation of $(\mathcal{G}\theta^n\theta^n)$ we get

$$\mathcal{G}\left(\theta^n\right)\theta^n \geq$$

$$\geq \begin{cases} \left\{ \left\|\nabla\mathbf{v}^n\right\|_2 \left[\left\|\nabla\mathbf{v}^n\right\|_2^{1-\varepsilon} \left(\frac{\nu}{2} \left\|\nabla\mathbf{v}^n\right\|_2^{\varepsilon} - C_1 c_9 \left(1 + C_2\right) \right) - c_4 \right] - c_8 \right\} + \\ +n \left(\left\|\mathbf{v}^n\right\|_{q,\Gamma_2}^2 + 1 \right) \left(\Psi\left(\mathbf{v}^n\right), \mathbf{v}^n\right)_V, \quad \left\|\Psi\left(\mathbf{v}^n\right)\right\|_{q,\Gamma_2} \leq 1, \\ \left\|\nabla\mathbf{v}^n\right\|_2 \left(\frac{\nu}{2} \left\|\nabla\mathbf{v}^n\right\|_2 - c_4 \right) - c_8 + (n - n_0) \left(\left\|\mathbf{v}^n\right\|_{q,\Gamma_2}^2 + 1 \right) \left(\Psi\left(\mathbf{v}^n\right), \mathbf{v}^n\right)_V, \quad \left\|\Psi\left(\mathbf{v}^n\right)\right\|_{q,\Gamma_2} \geq 1, \end{cases}$$

where n_0 is such that $n_0\epsilon_1 \geq C_1(1+C_2)c_9$, with ϵ_1 from relation (15). $c_4, \ldots, c_9$ are appropriate constants which depend only on $\Omega,\ \Gamma_1,\ \Gamma_2,\ \mathbf{u}_*,\ \mathbf{V},\ a,\ \alpha,\ c_0,\ \varepsilon,$ and q.

Let us define constant R_0 with the following property:

$$\left\|\nabla\mathbf{v}^n\right\|_2 \geq R_0 \ \Rightarrow\ \mathcal{G}\left(\theta^n\right)\theta^n > 0. \tag{21}$$

It follows from (21) that R_0 can be taken e. g.

$$R_0 = \max\left\{ \left[\frac{2}{\nu} C_1 \left(C_2 + 1\right) c_9 + 1 \right]^{\frac{1}{\varepsilon}}, \left(c_4 + 1\right)^{\frac{1}{1-\varepsilon}}, c_8 + 1, \frac{2}{\nu}\left(c_4 + 1\right) \right\}.$$

Suppose that

$$\left|\theta^n\right| = \left\|\mathbf{v}^n\right\|_2 = R > R_0 c_{10}^{-1}, \tag{22}$$

where c_{10} is the constant from the inequality $\left\|\nabla\mathbf{v}^n\right\|_2 \geq c_{10} \left\|\mathbf{v}^n\right\|_2$. Then

$$R_0 \ < \ c_{10}R \ = \ c_{10}\|\mathbf{v}^n\|_2 \ \leq \|\nabla\mathbf{v}\|_2,$$

and so from (21) we have

$$\mathcal{G}\left(\theta^n\right)\theta^n > 0.$$

Hence equation (19) has a solution θ^n in the ball $B_R(0^n)$ in $\mathbb{R}^n$ (see [9], Lemma II.1.4). The radius R of this ball does not depend on n.

The fact that solution θ^n of equation (19) satisfies $|\theta^n| < R$ means that function $\mathbf{v}^n$ given by (16) satisfies the estimate $\|\mathbf{v}^n\|_2 < R$. However, we will also need estimates of $\|\nabla\mathbf{v}^n\|_2$

and the penalisation term. If we assume a solution θ^n of equation (19) and its corresponding $\mathbf{v}^n$ we get from (21) $\|\nabla \mathbf{v}^n\| \le R_0$ because $\mathcal{G}(\theta^n)\theta^n = 0$.

Taking into account these two facts together with estimate (21) we get an estimate of the penalisation term. So we have

$$\|\nabla \mathbf{v}^n\| \le R_0, \quad (n-n_0)\left(\|\mathbf{v}^n\|^2_{q,\Gamma_2}+1\right)\left(\Psi\left(\mathbf{v}^n\right), \mathbf{v}^n\right)_V \le c_{11} \tag{23}$$

for $n > n_0$ and a positive constant c_{11} which does not depend on n.

3. The limit procedure for $n \to +\infty$

It follows from (23) that there exists a function $\mathbf{v} \in V$ and a subsequence of $\{\mathbf{v}^n\}$ (which will be also denoted by $\{\mathbf{v}^n\}$ in order to keep a simple notation) such that

$$\mathbf{v}^n \ \to \ \mathbf{v} \quad \text{weakly in } V. \tag{24}$$

The operator of traces from V into $L^q(\partial\Omega)^3$ is compact for $q \in (1,4)$. This implies that

$$\mathbf{v}^n \ \to \ \mathbf{v} \quad \text{strongly in } L^q(\partial\Omega)^3 \tag{25}$$

for $q \in (1,4)$. It follows from (15) and (23) that

$$\Psi(\mathbf{v}^n) \ \to \ 0 \quad \text{strongly in } V. \tag{26}$$

Due to the monotonicity of operator Ψ, we also have

$$\Big(\,\Psi(\mathbf{v}^n) - \Psi(\mathbf{z})\,,\,\mathbf{v}^n - \mathbf{z}\,\Big)_V \ \ge\ 0 \tag{27}$$

for all $n \in \mathbb{N}$ and $\mathbf{z} \in V$. Thus, using the boundedness of the sequence $\{\mathbf{v}^n - \mathbf{z}\}$ in V and (26), we get $\lim\limits_{n\to+\infty}\Big(\Psi(\mathbf{v}^n),\, \mathbf{v}^n - \mathbf{z}\Big)_V = 0$. Condition (24) implies that $\lim\limits_{n\to+\infty}\Big(\Psi(\mathbf{z}),\, \mathbf{v}^n\Big)_V = (\Psi(\mathbf{z}),\, \mathbf{v})_V$. If we pass to the limit for $n \to +\infty$ in (27), we obtain $-\Big(\Psi(\mathbf{z}),\, \mathbf{v}-\mathbf{z}\Big)_V \ge 0$. Put $\mathbf{z} = \mathbf{v} - \epsilon\,\Psi(\mathbf{v})$ where $\epsilon > 0$. Dividing the inequality by ϵ and passing to the limit for $\epsilon \to 0+$, we get: $-\Big(\Psi(\mathbf{v}),\, \Psi(\mathbf{v})\Big)_V \ge 0$ which means that $\Psi(\mathbf{v}) = \mathbf{0}$. Hence $\mathbf{v} \in K$ and $P\mathbf{v} = \mathbf{v}$.

Let $\mathcal{M}$ be a set of functions from $\mathcal{K}$ which are linear combinations of a finite number of functions $\mathrm{e}^1,\ \mathrm{e}^2, \dots$

Let us assume that $\mathbf{q} \in \mathcal{M}$ at first. Then there exists $m \in \mathbb{N}$ (depending on $\mathbf{q}$) and real numbers β_k $(k = 1, \dots, m)$ such that $\mathbf{q} = \sum\limits_{k=1}^{m} \beta_k\, \mathrm{e}^k$. Let us choose $n \in \mathbb{N}$ so that $n > m$. Let us multiply (17) by $(-\theta^n_k + \beta_k)$ if $k \le m$ and by $(-\theta^n_k)$ if $m < k \le n$. Let us sum for $k = 1, \dots, n$ the obtained equations. We get

$$\nu \int\limits_{\Omega} \nabla(\mathbf{v}^n + \mathbf{V})\nabla(\mathbf{q} - \mathbf{v}^n)\,dx - \int\limits_{\Gamma_2} \mathbf{F}(\mathbf{q} - \mathbf{v}^n)\,dS + n\left(\|\mathbf{v}^n\|^2_{q,\Gamma_2} + 1\right)\Big(\Psi(\mathbf{v}^n), \mathbf{q} - \mathbf{v}^n\Big)_V = 0. \tag{28}$$

It follows from the monotonicity of operator Ψ and from the fact that $\mathbf{q} \in K$ (which means that $\Psi(\mathbf{q}) = \mathbf{0}$) that the last term on the left hand side of (28) is nonpositive. If we omit this term, we get

$$\int\limits_{\Omega}\Big[\,(P\mathbf{v}^n+\mathbf{V})\nabla(\mathbf{v}^n+\mathbf{V})\,\Big](\mathbf{q}-\mathbf{v}^n)\,dx\;-\;\langle\,\mathbf{f}\,,\,\mathbf{q}-\mathbf{v}^n\rangle\;+$$

$$+\,\nu\int\limits_{\Omega}\nabla(\mathbf{v}^n+\mathbf{V})\nabla(\mathbf{q}-\mathbf{v}^n)\,dx\;-\;\int\limits_{\Gamma_2}\mathbf{F}(\mathbf{q}-\mathbf{v}^n)\,dS\;\geq\;0. \tag{29}$$

We are going to pass to the limit for $n\to+\infty$ in (29) now. We will use all convergences (24) – (26). Since some steps are standard, we do not show all the details here. Let us deal for example with the two nonlinear terms:

a)
$$\lim_{n\to+\infty}\;\inf\;\left(-\nu\int\limits_{\Omega}\nabla\mathbf{v}^n\nabla\mathbf{v}^n\,dx\right)\;\leq\;-\nu\int\limits_{\Omega}\nabla\mathbf{v}\nabla\mathbf{v}\,dx,$$

b)
$$\int\limits_{\Omega}\Big[(P(\mathbf{v}^n)+\mathbf{V})\nabla(\mathbf{v}^n+\mathbf{V})\Big](\mathbf{q}-\mathbf{v}^n)\,dx\;=\;\int\limits_{\Omega}\Big[(P\mathbf{v}^n+\mathbf{V})\nabla(\mathbf{v}^n+\mathbf{V})\Big](\mathbf{V}+\mathbf{q})\,dx\,-$$

$$-\,\frac{1}{2}\int\limits_{\Gamma_1}(\mathbf{V}\mathbf{n})\,|\mathbf{V}|^2\,dS\;-\;\frac{1}{2}\int\limits_{\Gamma_2}\Big[(P\mathbf{v}^n+\mathbf{V})\mathbf{n}\Big]\,|\mathbf{v}^n+\mathbf{V}|^2\,dS.$$

Due to (25), we have

$$\int\limits_{\Gamma_2}\Big[(P\mathbf{v}^n+\mathbf{V})\mathbf{n}\Big]\,|\mathbf{v}^n+\mathbf{V}|^2\,dS\;\to\;\int\limits_{\Gamma_2}\Big[(\mathbf{v}+\mathbf{V})\mathbf{n}\Big]\,|\mathbf{v}+\mathbf{V}|^2\,dS.$$

If we use (26), the strong convergence of $\mathbf{v}^n$ to $\mathbf{v}$ in $L^4(\Omega)^3$ (following from (24)) and the decomposition $P\mathbf{v}^n-\mathbf{v}=(P\mathbf{v}^n-\mathbf{v}^n)+(\mathbf{v}^n-\mathbf{v})=-\Psi(\mathbf{v}^n)+(\mathbf{v}^n-\mathbf{v})$, we obtain that $P\mathbf{v}^n\to\mathbf{v}$ in $L^4(\Omega)^3$). Hence

$$\lim_{n\to+\infty}\int\limits_{\Omega}\Big[(P(\mathbf{v}^n)+\mathbf{V})\nabla(\mathbf{v}^n+\mathbf{V})\Big](\mathbf{V}+\mathbf{q})\,dx\;=\;\int\limits_{\Omega}\Big[(\mathbf{v}+\mathbf{V})\nabla(\mathbf{v}+\mathbf{V})\Big](\mathbf{V}+\mathbf{q})\,dx.$$

Thus, we have

$$\lim_{n\to+\infty}\int\limits_{\Omega}\Big[(P(\mathbf{v}^n)+\mathbf{V})\nabla(\mathbf{v}^n+\mathbf{V})\Big](\mathbf{q}-\mathbf{v}^n)\,dx\;=\;\int\limits_{\Omega}\Big[(\mathbf{v}+\mathbf{V})\nabla(\mathbf{v}+\mathbf{V})\Big](\mathbf{q}-\mathbf{v})\,dx.$$

Using this all in (29), we get (11).

We need to show that (11) is satisfied not only for all $\mathbf{q}\in\mathcal{M}$ but for all $\mathbf{q}\in\mathcal{K}$ now. In order to do that, it is sufficient to show that $\mathcal{M}$ is dense in $\mathcal{K}$ in the norm of V. Let $\epsilon>0$ and $\mathbf{q}\in\mathcal{K}$ be such that

$$\int\limits_{\Gamma_2}\Big[\mathrm{dist}(\mathbf{V}(x)+\mathbf{q}(x),K_\alpha(x))\Big]^a dS_x\leq c_0-\epsilon$$

at first. There exists $\xi>0$ so that if $\mathbf{q}_1\in\mathcal{V}$, $\|\mathbf{q}_1-\mathbf{q}\|_V\leq\xi$ then $\mathbf{q}_1\in\mathcal{K}$. Let $\mathbf{q}^m$ be the orthogonal (in V) projection of $\mathbf{q}$ onto the subspace of V which is generated by the functions $\mathrm{e}^1,\ldots,\ \mathrm{e}^m$. Then $\|\mathbf{q}^m-\mathbf{q}\|_V\to0$ if $m\to+\infty$. This means that $\mathbf{q}^m\in\mathcal{K}$ for m large enough. Thus, $\mathbf{q}^m\in\mathcal{M}$ for m large enough and hence $\mathbf{q}$ can be approximated with an arbitrary accuracy (in the norm of V) by a function from $\mathcal{M}$. We can get the same result for all $\mathbf{q}\in\mathcal{K}$ if we let $\epsilon\to0+$.

We have proved the following theorem:

Theorem 3. *There exists a solution of Problem 1.*

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Received for publication October 17, 2001