A MIXED PROBLEM FOR THE WAVE EQUATION IN COORDINATE DOMAINS. I. MIXED PROBLEM FOR THE WAVE EQUATION IN A QUADRANT*

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A formal solution from the class $W_2^1(R_+^2)$ to the mixed problem for the classic wave equation in the coordinate corner $R_+^3 = \{(t,x,y)|t > 0, x > 0, y > 0\}$ with boundary conditions of an oblique derivative type is found in the first part of this article. The solution is found for all values of parameters of the boundary conditions such that the Lopatinsky condition is fulfilled on every boundary. The energy inequality (an a priori estimation in $W_2^1(R_+^2)$) without loss of generality is proved provided that the right hand sides are finite and the uniform Lopatinsky condition is fulfilled.

The second part of this article will be published in the next issue of journal.

In recent years mixed problems for hyperbolic equations and systems in domains with non-smooth boundaries attract attention of mathematicians since mathematical modelling of many actual processes generates such problems (examples can be found, say, in [31]).

We note that to the present time problems for hyperbolic systems with boundary conditions which do not contain t-derivatives of solution are well investigated ([9, 19, 21]). Such problems admit application of results of the well-developed theory of elliptic equations for domains with singularities on boundaries [22], whereas the appearance of a t-derivative in a boundary operator generates additional and essential difficulties.

In the authors' opinion, in this situation it makes sense to investigate the simplest (in the sense of formulation) mixed problem for the wave equation in the quadrant $R_+^2 = \{(x,y)|x,y > 0\}$, boundary conditions are of an oblique derivative type, and to select its singularities. In the first chapter of the work we give some results on investigation of this problem.

Using ideas of Miyatake ([28]), M. Taniguchi ([36]) has proved the well-posedness of the above-mentioned problem for the wave equation in a particular case: $a = \beta$, $b = \alpha$ (see §1); and H. Reisman [32] has proved its well-posedness in W_2^1 in the case when instead of the oblique derivative the Dirichlet condition is given on one of the boundaries. The main element in Reisman's investigation is application of properties of an operator similar to the Kreiss's "symmetrizer" [23].

G. Eskin in [12] has obtained a solution of the problem in dual variables of the Fourier-Laplace transform for the case of the higher orders boundary operators. However, the obtained in [12] a priori estimation of solution has an essential disadvantage: it is the loss of smoothness.

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As a result of long-standing researches, authors formed the opinion that the complete investigation of the problem under consideration becomes possible if ideas of both theory of elliptic equations (the behavior of solution in a neighborhood of the corner point) and theory of hyperbolic equations (the time influence) are applied. So, it is reasonable, on the one hand, to use advantages of the natural for hyperbolic problems method of energy integrals ("symmetrization" [15]) and, on the other hand, to find the image of the solution trace on a bound which satisfies certain boundary problem for analytic functions. A. P. Calderon [7] seems to be the first who began to use regularly properties of a boundary operator (Calderon's operator).

In § 2, 3 of the first chapter with the help of potentials there is obtained a solution to the above mentioned problem provided that the right hand sides are finite and the uniform Lopatinsky condition is fulfilled. In passing the authors have managed to answer a question which occupied them for a long time: "Why do we have the loss of smoothness as coefficients come to the domain where just Lopatinsky conditions are fulfilled?" It occurs that in this case we have to add an analog of a side wave into the formula of solution, this causes the deterioration of its differential properties.

Finally, § 4 of the first chapter is dedicated to the deducing of an a priori estimate of solution in $W_2^1(R_+^2)$ without loss of smoothness provided that the uniform Lopatinsky condition is fulfilled and the right hand sides satisfy some additional requirements.

Now we begin to detail the announced results of the present paper.

1. Formulation of the problem. The main results

We consider the following test formulation: in the domain $R_+^3 = \{(t, x, y) | t > 0, x > 0, y > 0\}$ we seek the solution to the wave equation

$$u_{tt} - u_{xx} - u_{yy} = f(t, x, y), \quad (t, x, y) \in R_+^3, \tag{1.1}$$

which satisfies the boundary conditions:

$$u_t - au_x - bu_y = 0,$$

$$x = 0, \quad (t, y) \in R_+^2 = \{(t, y) | t > 0, y > 0\}; \tag{1.2}$$

$$u_t - \alpha u_y - \beta u_x = 0,$$

$$y = 0, (t, x) \in R_+^2 \tag{1.3}$$

and the initial data:

$$u|_{t=0} = \phi(x, y), u_t|_{t=0} = \psi(x, y),$$

$$(x, y) \in R_+^2, \tag{1.4}$$

where a, b, α, β are real numbers.

It is known [15, 28, 33] that the uniform Lopatinsky condition is fulfilled for problem (1.1)–(1.4) if:

$$1)\ a > 0, \quad |b| < 1; \tag{1.5}$$

$$2)\ \alpha > 0, \quad |\beta| < 1. \tag{1.6}$$

The following assertions are valid.

Theorem 1. *Let the coefficients of problem (1.1)–(1.4) satisfy inequalities (1.5), (1.6), the right hand sides, finite with respect to x, y, satisfy, if necessary, conditions (2.26), (2.29), (2.31). Then we have the integral representation of the solution*

$$u(t,x,y) = \frac{\theta(t-\sqrt{x^2+y^2})}{2\pi\sqrt{t^2-x^2-y^2}} \underset{t,x,y}{*} f(t,x,y) + \frac{\partial}{\partial t}\left[\frac{\theta(t-\sqrt{x^2+y^2})}{2\pi\sqrt{t^2-x^2-y^2}} \underset{x,y}{*} \varphi(x,y)\right] +$$

$$+\frac{\theta(t-\sqrt{x^2+y^2})}{2\pi\sqrt{t^2-x^2-y^2}} \underset{x,y}{*} \psi(x,y) - \left(\frac{\partial}{\partial x} + \frac{1}{a}\frac{\partial}{\partial t} - \frac{b}{a}\frac{\partial}{\partial y}\right)\left[\frac{\theta(t-\sqrt{x^2+y^2})}{2\pi\sqrt{t^2-x^2-y^2}} \underset{t,y}{*} v(t,y)\right] -$$

$$-\left(\frac{\partial}{\partial y} + \frac{1}{\alpha}\frac{\partial}{\partial t} - \frac{\beta}{\alpha}\frac{\partial}{\partial x}\right)\left[\frac{\theta(t-\sqrt{x^2+y^2})}{2\pi\sqrt{t^2-x^2-y^2}} \underset{t,x}{*} z(t,x)\right],$$

where $\theta(z)$ is the Heaviside function; $v(t,y)$ is determined, for example, as follows:

$$v(t,y) = \int\limits_0^{+\infty}\int\limits_0^{+\infty}\left[\frac{\theta(t-\sqrt{z_1^2+(z_2-y)^2})}{\sqrt{t^2-z_1^2-(z_2-y)^2}}K(t,y,z_1,z_2) + \right.$$

$$\left. + \frac{\theta(t-\sqrt{z_1^2+(z_2+y)^2})}{\sqrt{t^2-z_1^2-(z_2+y)^2}}M(t,y,z_1,z_2)\right] \underset{t}{*} f(t,z_1,z_2)\,dz_1\,dz_2 +$$

$$+\frac{\partial}{\partial t}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\left[\frac{\theta(t-\sqrt{z_1^2+(z_2-y)^2})}{\sqrt{t^2-z_1^2-(z_2-y)^2}}K(t,y,z_1,z_2) + \right.$$

$$\left. + \frac{\theta(t-\sqrt{z_1^2+(z_2+y)^2})}{\sqrt{t^2-z_1^2-(z_2+y)^2}}M(t,y,z_1,z_2)\right] \varphi(z_1,z_2)\,dz_1\,dz_2 +$$

$$+\int\limits_0^{+\infty}\int\limits_0^{+\infty}\left[\frac{\theta(t-\sqrt{z_1^2+(z_2-y)^2})}{\sqrt{t^2-z_1^2-(z_2-y)^2}}K(t,y,z_1,z_2) + \right.$$

$$\left. + \frac{\theta(t-\sqrt{z_1^2+(z_2+y)^2})}{\sqrt{t^2-z_1^2-(z_2+y)^2}}M(t,y,z_1,z_2)\right] \psi(z_1,z_2)\,dz_1\,dz_2, \tag{1.7}$$

$$K(t,y,z_1,z_2) = \frac{1}{\pi}\frac{z_1 t(tz_1 + \frac{1}{a}(z_1^2+(z_2-y)^2) + \frac{b}{a}t(y-z_2)) +}{(tz_1 + \frac{1}{a}(z_1^2+(z_2-y)^2) + \frac{b}{a}t(y-z_2))^2 +}$$

$$\frac{+(y-z_2)(y-z_2-\frac{b}{a}z_1)(t^2-z_1^2-(z_2-y)^2)}{+(y-z_2-\frac{b}{a}z_1)^2(t^2-z_1^2-(z_2-y)^2)},$$

$$M(t,y,z_1,z_2) = \frac{1}{\pi}\mathrm{Re}\left\{\frac{[-\frac{\beta}{\alpha}tz_1 + \frac{1}{\alpha}(z_1^2+(z_2+y)^2) - t(y+z_2) +}{[-\frac{\beta}{\alpha}tz_1 + \frac{1}{\alpha}(z_1^2+(z_2+y)^2) + t(y+z_2) +}\right.$$

$$\frac{+i(-\frac{\beta}{\alpha}(y+z_2) + z_1)\sqrt{t^2-z_1^2-(z_2+y)^2}]}{+i(-\frac{\beta}{\alpha}(y+z_2) - z_1)\sqrt{t^2-z_1^2-(z_2+y)^2}]} \times$$

$$\times \frac{[-i(y+z_2)+\frac{z_1 t}{\sqrt{t^2-z_1^2-(z_2+y)^2}}]}{[tz_1+\frac{1}{a}(z_1^2+(z_2+y)^2)+\frac{b}{a}t(y+z_2)+i(y+z_2-\frac{b}{a}z_1)\sqrt{t^2-z_1^2-(z_2+y)^2}]}\Bigg\};$$

formulae for $z(t,x)$ look analogously with the corresponding replacement of a by α, b by β, y by x, the symbol $*$ stands for the convolution with respect to corresponding variables which are specified below.

We introduce the following notations:

$$\|f(t,x,y)\|^2_{L_2(R^2_+)} = \iint\limits_{R^2_+} f^2(t,x,y)\,dx\,dy,$$

$$\|u(t,x,y)\|^2_{W^1_2(R^2_+)} = \iint\limits_{R^2_+} (u^2(t,x,y)+u^2_x(t,x,y)+u^2_y(t,x,y)+u^2_t(t,x,y))\,dx\,dy.$$

Theorem 2. *Provided that the conditions of Theorem 1 are fulfilled, the solution to problem (1.1)–(1.4) satisfies the a priori estimation*

$$\|u(t,x,y)\|_{W^1_2(R^2_+)} \le C_1(t)\|u(0)\|_{W^1_2(R^2_+)} + C_2(t)\max_{0\le\tau\le t}\|f(\tau)\|_{L_2(R^2_+)},$$

where $C_i(t)>0$, $i=1,2$.

Remark 1.1. In this paper we give a method which allows to obtain an a priori estimate without loss of smoothness for the case when the boundary operators in (1.2), (1.3) are homogeneous and have the higher order. It suffices to reduce this problem to a mixed problem for the vector wave equation ([17]).

2. Construction of the formal solution to problem (1.1)–(1.4) and its uniqueness

We take into consideration very important in elliptic theory agreement on the coefficients a and α of boundary conditions (1.2), (1.3) [14]: $a\neq 0$, $\alpha\neq 0$. We also assume that the initial data of the problem satisfy the requirements: $f(t,x,y)\in C^\infty([0,\infty);C^\infty_0(R^2_+))$, $\varphi(x,y),\psi(x,y)\in C^\infty_0(R^2_+)$.

We denote

$$u(t,0,y)=v(t,y),\quad u(t,x,0)=z(t,x),\quad u(t,0,0)=H(t)$$

and continue all the considered functions by zero outside of the domains of their definition. Then, applying the Fourier–Laplace transform to general equation (1.1) and integrating by parts, we obtain the relation:

$$\hat u(s,\xi,\eta)(\xi^2+\eta^2+s^2)=\hat f(s,\xi,\eta)+s\hat\varphi(\xi,\eta)+\hat\psi(\xi,\eta)+$$

$$+(\overline{b}+\overline{\beta})\hat H(s)+\hat v(s,\eta)(i\xi-\overline{a}s+\overline{b}i\eta)+\hat z(s,\xi)(i\eta-\overline{\alpha}s+\overline{\beta}i\xi), \tag{2.1}$$

where

$$\overline{a}=\frac{1}{a},\quad \overline{b}=-\frac{b}{a},\quad \overline{\alpha}=\frac{1}{\alpha},\quad \overline{\beta}=-\frac{\beta}{\alpha}. \tag{2.2}$$

In view of finiteness with respect to x and y of the functions $f(t,x,y)$, $\varphi(x,y,)$, $\psi(x,y)$, the accepted restriction on the growth with respect to t of the functions f, H, and u, and the boundedness of the velocity of perturbations propagation which enter equation (2.1), the functions $\hat{u}(s,\xi,\eta)$, $\hat{f}(s,\xi,\eta)$; $\hat{\varphi}(\xi,\eta)$, $\hat{\psi}(\xi,\eta)$, $\hat{v}(s,\eta)$, $\hat{z}(s,\xi)$, $\hat{H}(s)$ are analytic, correspondingly, in the domains $\operatorname{Re} s > \sigma$, $\operatorname{Im}\xi > 0$, $\operatorname{Im}\eta > 0$; $\operatorname{Im}\xi > 0$, $\operatorname{Im}\eta > 0$; $\operatorname{Re} s > \sigma$, $\operatorname{Im}\eta > 0$; $\operatorname{Re} s > \sigma$, $\operatorname{Im}\xi > 0$; $\operatorname{Re} s > \sigma$.

Let $x < 0$. We apply the inverse Fourier transform with respect to x, Jordan lemma [26] and the Cauchy theorem to equation (2.1). As a result we have

$$e^{\sqrt{\eta^2+s^2}x}\hat{v}(s,\eta)(-\sqrt{\eta^2+s^2}-\overline{a}s+\overline{b}i\eta)+(i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2})\times$$
$$\times\int_0^\infty \exp\{-|x-p|\sqrt{\eta^2+s^2}\}\,\hat{z}(s,p)\,dp+$$
$$+\int_0^\infty \exp\{-|x-p|\sqrt{\eta^2+s^2}\}[\hat{f}(s,p,\eta)+s\hat{\varphi}(s,p)+\hat{\psi}(s,p)+(\overline{b}+\overline{\beta})\hat{H}(s)]\,dp=0, \tag{2.3}$$

the branch $\sqrt{z}$ is chosen such that $\operatorname{Re}\sqrt{z} > 0$.

Passing to the limit $x \to -0$ in formula (2.3), we will have the relation which shows the connection between boundary values $v(t,y)$ and $z(t,x)$:

$$\hat{v}(s,\eta)(-\sqrt{\eta^2+s^2}-\overline{a}s+\overline{b}i\eta)+(i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2})\int_0^\infty \exp\{-p\sqrt{\eta^2+s^2}\}\,\hat{z}(s,p)\,dp+$$
$$+\int_0^\infty \exp\{-p\sqrt{\eta^2+s^2}\}[\hat{f}(s,p,\eta)+s\hat{\varphi}(s,p)+\hat{\psi}(s,p)+(\overline{b}+\overline{\beta})\hat{H}(s)]\,dp=0. \tag{2.4}$$

Then, by the analytic continuation principle [26, 34], the integral

$$\int_0^\infty \exp\{-p\sqrt{\eta^2+s^2}\}\,\hat{z}(s,p)\,dp,$$

provided that

$$i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2}\neq 0, \quad \Im\eta=0,$$

is an analytic function in the upper half-plane $\operatorname{Im}\eta > 0$ iff

$$\frac{\hat{v}(s,\eta)(\sqrt{\eta^2+s^2}+\overline{a}s-\overline{b}i\eta)-\hat{F}(s,i\sqrt{\eta^2+s^2},\eta)}{i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2}}=$$
$$=\frac{\hat{v}(s,-\eta)(\sqrt{\eta^2+s^2}+\overline{a}s+\overline{b}i\eta)-\hat{F}(s,i\sqrt{\eta^2+s^2},-\eta)}{-i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2}}, \quad \eta\in R. \tag{2.5}$$

Here

$$\widehat{F}(k,m,n)=\hat{f}(k,m,n)+k\hat{\varphi}(m,n)+\hat{\psi}(m,n)+(\overline{b}+\overline{\beta})\widehat{H}(k).$$

We reduce equation (2.5) to the Riemann boundary problem. We put

$$v_0(s,\eta^2)=\hat{v}(s,\eta).$$

Consequently, the function $v_0(s,\lambda)$ is defined on the whole complex plane with a cut along the positive real axis. We choose the branch $\sqrt{\lambda}$ such that $\sqrt{\lambda}>0$ at $\lambda>0$ and introduce the notations:

$$\begin{aligned} v_0^+(s,\lambda) &= \lim_{\varepsilon\to+0} v_0(s,\lambda+i\varepsilon)=\hat{v}(s,\sqrt{\lambda}),\\ v_0^-(s,\lambda) &= \lim_{\varepsilon\to-0} v_0(s,\lambda-i\varepsilon)=\hat{v}(s,-\sqrt{\lambda}). \end{aligned}$$

Then it follows from (2.5) that $v_0(s,\lambda)$ is the solution to boundary problem (K): on the complex plane cut along the positive real axis we seek the piece-wise smooth analytic function $v_0(s,\lambda)$ satisfying the conjunction condition at $\lambda>0$:

$$\text{(K)}\qquad v_0^+(s,\lambda)\frac{\sqrt{\lambda+s^2}+\overline{a}s-\overline{b}i\sqrt{\lambda}}{i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2}}-v_0^-(s,\lambda)\frac{\sqrt{\lambda+s^2}+\overline{a}s+\overline{b}i\sqrt{\lambda}}{-i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2}}=$$

$$=\frac{\hat{F}(s,i\sqrt{\lambda+s^2},\sqrt{\lambda})}{i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2}}-\frac{\hat{F}(s,i\sqrt{\lambda+s^2},-\sqrt{\lambda})}{-i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2}}. \tag{2.6}$$

Remark 2.1. By analogy, we formulate a boundary problem to find $z_0(s,\lambda)$. The conjunction condition, for example, looks like:

$$z_0^+(s,\lambda)\frac{\sqrt{\lambda+s^2}+\overline{\alpha}s-\overline{\beta}i\sqrt{\lambda}}{i\sqrt{\lambda}-\overline{a}s-\overline{b}\sqrt{\lambda+s^2}}-z_0^-(s,\lambda)\frac{\sqrt{\lambda+s^2}+\overline{\alpha}s+\overline{\beta}i\sqrt{\lambda}}{-i\sqrt{\lambda}-\overline{a}s-\overline{b}\sqrt{\lambda+s^2}}=$$

$$=\frac{\hat{F}(s,\sqrt{\lambda},i\sqrt{\lambda+s^2})}{i\sqrt{\lambda}-\overline{a}s-\overline{b}\sqrt{\lambda+s^2}}-\frac{\hat{F}(s,-\sqrt{\lambda},i\sqrt{\lambda+s^2})}{-i\sqrt{\lambda}-\overline{a}s-\overline{b}\sqrt{\lambda+s^2}}. \tag{2.6$'$}$$

Preliminary to solving problem (2.6) we have to check if the coefficients

$$\frac{\sqrt{\lambda+s^2}+\overline{a}s-\overline{b}i\sqrt{\lambda}}{i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2}}\quad\text{and}\quad\frac{\sqrt{\lambda+s^2}+\overline{a}s+\overline{b}i\sqrt{\lambda}}{-i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2}}$$

turn into zero on the contour $\lambda\geq 0$. It obviously suffices to find real zeros of the functions $\sqrt{\eta^2+s^2}+\overline{a}s-\overline{b}i\eta$ and $i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2}$. It is easy to derive that complex roots $\eta=\eta_1+\eta_2$ of the equation

$$\sqrt{\eta^2+s^2}+\overline{a}s-\overline{b}i\eta=0 \tag{2.7}$$

are as follows

$$\eta^{1,2}=\frac{-\overline{a}\overline{b}i\pm\sqrt{\overline{a}^2-\overline{b}^2-1}}{\overline{b}^2+1}s,\quad\text{and}\quad \overline{a}s_1+\overline{b}\eta_2<0, \tag{2.8}$$

where $s=s_1+is_2$.

We select two cases

$$1)\quad \overline{a}^2-\overline{b}^2-1>0, \tag{2.9}$$

$$2)\quad \overline{a}^2-\overline{b}^2-1\leq 0. \tag{2.10}$$

If inequality (2.9) is valid, then expression (2.8) for the solutions of (2.7) can be rewritten in the form:

$$\eta_1^{1,2} = \frac{\overline{a}\overline{b}s_2 \pm s_1\sqrt{\overline{a}^2 - \overline{b}^2 - 1}}{\overline{b}^2 + 1},$$
$$\eta_2^{1,2} = \frac{-\overline{a}\overline{b}s_1 \pm s_2\sqrt{\overline{a}^2 - \overline{b}^2 - 1}}{\overline{b}^2 + 1}. \tag{2.11}$$

Of natural interest for us is a variant when the roots lie in the upper half-plane, i.e., the relation is fulfilled

$$-\overline{a}\overline{b}s_1 \pm s_2\sqrt{\overline{a}^2 - \overline{b}^2 - 1} \geq 0. \tag{2.12}$$

Analysing conditions (2.9), (2.11), (2.12), we conclude that for

$$\overline{a}^2 - \overline{b}^2 - 1 > 0, \quad \overline{a} < 0 \tag{2.13}$$

roots of equation (2.7) are on the real axis.

In the case (2.10), the real and complex components of the roots to (2.7) are derived from the formulae:

$$\eta_1^{1,2} = \frac{\overline{a}\overline{b} \pm \sqrt{1 - \overline{a}^2 + \overline{b}}}{\overline{b}^2 + 1}s_2,$$
$$\eta_2^{1,2} = \frac{-\overline{a}\overline{b} \pm \sqrt{1 - \overline{a}^2 + \overline{b}^2}}{\overline{b}^2 + 1}s_1 \geq 0. \tag{2.14}$$

Consequently, again at $\overline{a} = -1$ at least one of the roots is real.

Remark 2.2. It is known [28, 33] that for the problem with one boundary ($x = 0$) $\sqrt{\eta^2 + s^2} + \overline{a}s - \overline{b}i\eta$ is the Lopatinsky determinant and its turning into zero at $\eta \in R$, $s_1 = \text{Re}s > \sigma$ is equivalent to the assertion that the corresponding mixed problem with conditions (1.1), (1.2), (1.4) is ill-posed in the sense of L_2.

Remark 2.3. Thus, if

$$\begin{aligned} &1)\quad \overline{a}^2 - \overline{b}^2 - 1 > 0, \quad \overline{a} < 0; \\ &2)\quad \overline{a} = -1 \end{aligned} \tag{2.15}$$

then for the problem with one boundary the Lopatinsky condition is not fulfilled (see also [15]).

The equation

$$i\eta - \overline{\alpha}s - \overline{\beta}\sqrt{s^2 + \eta^2} = 0,$$

is solved analogously, and as it follows from (2.6) it is sufficient to suppose that $\overline{\beta} \neq 0$ and to consider the equality

$$\sqrt{\eta^2 + s^2} + \frac{\overline{\alpha}}{\overline{\beta}}s - \frac{1}{\overline{\beta}}i\eta = 0. \tag{2.16}$$

So, the solution of equation (2.16) is real if one of the equalities is true:

$$\begin{aligned} &1)\quad \overline{\alpha}^2 - \overline{\beta}^2 - 1 > 0, \quad \overline{\alpha}\overline{\beta} < 0; \\ &2)\quad \overline{a} = -\overline{\beta}. \end{aligned} \tag{2.17}$$

In what follows we will suppose that the coefficients of the boundary conditions $\overline{a}$, $\overline{b}$, $\overline{\beta}$, $\overline{\alpha}$ are such that the Lopatinsky condition is valid and inequalities (2.17) are broken. Such assumption allows to reduce problem (K) to the canonical Riemann problem [13, 30], after representation of condition (2.6) in the classical form:

$$\text{(R)} \qquad v_0^+(s,\lambda) = \frac{(\sqrt{\lambda+s^2}+\overline{a}s+\overline{b}i\sqrt{\lambda})(i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2})}{(\sqrt{\lambda+s^2}+\overline{a}s-\overline{b}i\sqrt{\lambda})(-i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2})} v_0^-(s,\lambda)+$$

$$+\frac{\hat{F}(s,i\sqrt{\lambda+s^2},\sqrt{\lambda})}{i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2}} - \frac{\hat{F}(s,i\sqrt{\lambda+s^2},-\sqrt{\lambda})(i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2})}{(\sqrt{\lambda+s^2}+\overline{a}s-\overline{b}i\sqrt{\lambda})(-i\sqrt{\lambda}-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda+s^2})}.$$

Let us derive the index of problem (2.18) [13]. The formula [5] is true

$$\sqrt{\eta^2+s^2} = r + is_1s_2/r, \tag{2.19}$$

where $r = ((((s_1^2-s_2^2+\eta^2)+4s_1^2s_2^2)^{1/2}+s_1^2-s_2^2+\eta^2)/2)^{1/2}$. Consequently, as η increases, $\mathrm{Re}\big(\sqrt{\eta^2+s^2}\big)+\overline{a}s_1$ increases too. Besides,

$$\sqrt{\eta^2+s^2}+\overline{a}s+\overline{b}i\eta-(\eta+\overline{a}s+bi\eta)\to 0 \quad \text{as} \quad \eta\to+\infty. \tag{2.20}$$

From relations (2.19), (2.20) we obtain, for example, the increment of the argument as η varies from 0 to $+\infty$ at

$$\overline{a} > -1, \quad \overline{b}\geq 0 \tag{2.21}$$

$$\left[\Delta\arg\frac{\sqrt{\eta^2+s^2}+\overline{a}s+\overline{b}i\eta}{\sqrt{\eta^2+s^2}+\overline{a}s-\overline{b}i\eta}\right]_{\eta=0}^{\eta=+\infty} = 2\ \mathrm{arctg}\, b. \tag{2.22}$$

Below we derive the increment of argument of coefficient for the problem (R) and the index for case (2.21)

$$1)\quad \underline{\overline{\beta}\geq 0, \quad \overline{\alpha} > -\overline{\beta}},$$

$$\frac{1}{2\pi}\left[\Delta\arg\frac{(\sqrt{\eta^2+s^2}+\overline{a}s+\overline{b}i\eta)(i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2})}{(\sqrt{\eta^2+s^2}+\overline{a}s-\overline{b}i\eta)(-i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2}}\right]_{\eta=0}^{\eta=+\infty} = \frac{\mathrm{arctg}\overline{b}-\mathrm{arctg}\frac{1}{\overline{\beta}}}{\pi},$$

$$a)\quad \overline{b} < \frac{1}{\overline{\beta}}, \kappa = [\frac{\Delta}{2\pi}] = -1;$$

$$b)\quad \overline{b} \geq \frac{1}{\overline{\beta}}, \quad \kappa = 0;$$

$$2)\quad \underline{\overline{\beta}<0, \quad \overline{\alpha} < -\overline{\beta}}, \quad \frac{\Delta}{2\pi} = \frac{\mathrm{arctg}\overline{b}-\mathrm{arctg}\frac{1}{\overline{\beta}}}{2\pi}, \quad \kappa = 0;$$

$$3)\quad \underline{\overline{\beta}\geq 0, \overline{\alpha} < \overline{-\beta}}, \quad \frac{\Delta}{2\pi} = 1 + \frac{\mathrm{arctg}\overline{b}-\mathrm{arctg}\frac{1}{\overline{\beta}}}{\pi},$$

$$a)\quad \overline{b} < \frac{1}{\overline{\beta}}, \quad \kappa = 0;$$

$$b)\quad \overline{b} \geq \frac{1}{\overline{\beta}}, \quad \kappa = 1;$$

$$4)\quad \underline{\overline{\beta}<0,\ \overline{\alpha}>-\overline{\beta}},\qquad \frac{\Delta}{2\pi}=\frac{\operatorname{arctg}\overline{b}-\operatorname{arctg}\frac{1}{\overline{\beta}}}{2\pi}-1, \kappa=-1.$$

For another values of the parameters $\overline{a}$, $\overline{b}$, $\overline{\alpha}$, $\overline{\beta}$ calculations are carried out in a similar way. We note that κ can take the values -2, -1, 0, 1, 2.

Using the factorization method [13] and the canonical function $X(s,\eta)$

$$\ln X(s,\eta)=\frac{\eta^2+s^2}{2\pi i}\int\limits_0^{+\infty}\frac{\ln G(s,\lambda)2\lambda}{(\lambda^2+s^2)(\lambda^2-\eta^2)}d\lambda,$$

where

$$G(s,\eta)=\frac{(\sqrt{\eta^2+s^2}+\overline{a}s+\overline{b}i\eta)(i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2})}{(\sqrt{\eta^2+s^2}+\overline{a}s-\overline{b}i\eta)(-i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2})},$$

theorems on asymptotic representation of Cauchy integrals in a neighborhood of the end of the integration interval [13, 20, 30], we write the solution to problem (R) in the form:

1) $\dfrac{1}{2}\le\dfrac{\Delta}{2\pi}-\kappa<1,$

a) $\kappa\ge-1,$

$$\hat{v}(s,\eta)=\frac{X(s,\eta)(\eta^2+s^2)^{\kappa+1}}{2\pi i}\int\limits_0^{+\infty}\frac{h(s,\lambda)2\lambda\,d\lambda}{(\lambda^2+s^2)^{\kappa+1}X^+(s,\lambda)(\lambda^2-\eta^2)}+\hat{K}(s,\eta), \tag{2.24}$$

where

$$h(s,\lambda)=\frac{\hat{F}(s,i\sqrt{\lambda^2+s^2},\lambda)}{\sqrt{\lambda^2+s^2}+\overline{a}s-\overline{b}i\lambda}-\frac{\hat{F}(s,i\sqrt{\lambda^2+s^2},-\lambda)(i\lambda-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda^2+s^2})}{(-i\lambda-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda^2+s^2})(\sqrt{\lambda^2+s^2}+\overline{a}s-\overline{b}i\lambda)},$$

$\hat{K}(s,\eta)$ is the solution to the corresponding homogeneous problem (at $\kappa=-1$, $\hat{K}(s,\eta)=0$);

b) $\kappa=-2$

$$\hat{v}(s,\eta)=\frac{X(s,\eta)(\eta^2+s^2)^{-1}}{2\pi i}\int\limits_0^{+\infty}\frac{h(s,\lambda)2\lambda\,d\lambda}{(\lambda^2+s^2)^{-1}X^+(s,\lambda)(\lambda^2-\eta^2)}, \tag{2.25}$$

besides

$$\int\limits_0^{\infty}\frac{h(s,\lambda)\lambda\,d\lambda}{X^+(s,\lambda)}=0; \tag{2.26}$$

2) $0\le\dfrac{\Delta}{2\pi}-\kappa<\dfrac{1}{2},$

a) $\kappa\ge0,$

$$\hat{v}(s,\eta)=\frac{X(s,\eta)(\eta^2+s^2)^{\kappa}}{2\pi i}\int\limits_0^{+\infty}\frac{h(s,\lambda)2\lambda\,d\lambda}{(\lambda^2+s^2)^{\kappa}X^+(s,\lambda)(\lambda^2-\eta^2)}+\check{K}(\eta,s); \tag{2.27}$$

b) $\kappa = -1$,

$$\hat{v}(s,\eta) = \frac{X(s,\eta)(\eta^2+s^2)^{-1}}{2\pi i} \int\limits_0^{+\infty} \frac{h(s,\lambda)2\lambda\,d\lambda}{(h^2+s^2)^{-1}X^+(s,\lambda)(\lambda^2-\eta^2)}, \tag{2.28}$$

provided that

$$\int\limits_0^{\infty} \frac{h(s,\lambda)\lambda\,d\lambda}{X^+(s,\lambda)} = 0; \tag{2.29}$$

c) $\kappa = -2$,

$$\hat{v}(s,\eta) = \frac{X(s,\eta)(\eta^2+s^2)^{-2}}{2\pi i} \int\limits_0^{+\infty} \frac{h(s,\lambda)2\lambda\,d\lambda}{(\lambda^2+s^2)^{-1}X^+(s,\lambda)(\lambda^2-\eta^2)} \tag{2.30}$$

and

$$\int\limits_0^{\infty} \frac{h(s,\lambda)\lambda\,d\lambda}{(\lambda^2+s^2)^{-2}X^+(s,\lambda)(\lambda^2+s^2)^k} = 0, \quad k = 1,2. \tag{2.31}$$

Remark 2.4. It is obvious that if the index κ is negative, then we can eliminate a singularity of the analytic function $\hat{v}(s,\eta)$ at the point $\eta^2 = -s^2$, subtracting corresponding members of the Laurent series, however, it will lower its smoothness. G. Eskin also pointed at this fact, connecting the index of solution smoothness with the index of the boundary Riemann problem with a shift considered in [12].

Direct calculations show that if the uniform Lopatinsky condition (1.5), (1.6) is fulfilled, then κ is negative excluding the case $a > 0$, $-1 < b \le 0$; $\alpha > 0$, $-1 < \beta \le 0$, $\dfrac{a}{b} \ge \dfrac{\beta}{\alpha}$ with $\kappa = 0$. However, $v(t,y) \in W^{1/2}(R^2_+)$ implies the existence of only trivial solution to the homogeneous boundary problem, formulae (2.24)–(2.32) present a trace of $u(t,x,y)$ on the edge $x = 0$ in dual coordinates s and η.

In a similar way, we derive the function $\hat{z}(s,\xi)$ and, in view of (2.1), $\hat{u}(s,\xi,\eta)$, the solution to investigated problem (1.1)–(1.4).

Remark 2.5. The suggested method after insignificant changes can be used in order to find exact solutions to another problems, for example, when a, b, α, β are complex or the Dirichlet conditions are taken instead of (1.2), (1.3).

3. Integral representation of functions $v(t,y)$ and $z(t,x)$

It suffices to consider only one function (say, $v(t,y)$). By relations (2.24) (2.27) and Sokhotsky-Plemelj formulae, we have

$$\hat{v}(s,y) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} e^{iy\eta}\hat{v}(s,\eta)d\eta,$$

where

$$v(s,\eta)=\begin{cases}\dfrac{1}{2}h(s,\eta)+\dfrac{X^{+}(s,\eta)(\eta^2+s^2)^k}{2\pi i}\displaystyle\int\limits_0^{+\infty}\dfrac{h(s,\lambda)2\lambda d\lambda}{(\lambda^2+s^2)^kX^{+}(s,\lambda)(\lambda^2-\eta^2)}, & \eta>0,\\[2ex] -\dfrac{1}{2}h(s,-\eta)\dfrac{(\sqrt{\eta^2+s^2}+\overline{a}s+\overline{b}i\eta)(i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2})}{(\sqrt{\eta^2+s^2}+\overline{a}s-\overline{b}i\eta)(-i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2})}+ & \\[2ex] +\dfrac{X^{+}(s,\eta)(\eta^2+s^2)^k}{2\pi i}\displaystyle\int\limits_0^{+\infty}\dfrac{h(s,\lambda)2\lambda d\lambda}{(\lambda^2+s^2)^kX^{+}(s,\lambda)(\lambda^2-\eta^2)}, & \eta<0,\end{cases}$$

k equals either $\kappa+1$ or κ. Consequently,

$$\hat{v}(s,y)=\frac{1}{2\pi}\int\limits_{-\infty}^{0}e^{-iy\eta}d\eta\Big[-\frac{1}{2}h(s,-\eta)\frac{(\sqrt{\eta^2+s^2}+\overline{a}s+\overline{b}i\eta)(i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2})}{(\sqrt{\eta^2+s^2}+\overline{a}s-\overline{b}i\eta)(-i\eta-\overline{\alpha}s-\overline{\beta}\sqrt{\eta^2+s^2})}+$$

$$+\frac{X^{+}(s,\eta)(\eta^2+s^2)^k}{2\pi i}\int\limits_0^{+\infty}\frac{h(s,\lambda)2\lambda d\lambda}{(\lambda^2+s^2)^kX^{+}(s,\lambda)(\lambda^2-\eta^2)}\Big]+$$

$$+\frac{1}{2\pi}\int\limits_0^{\infty}e^{-iy\eta}d\eta\Big[\frac{1}{2}h(s,\eta)+\frac{X^{+}(s,\eta)(\eta^2+s^2)^k}{2\pi i}\int\limits_0^{+\infty}\frac{h(s,\lambda)2\lambda d\lambda}{(\lambda^2+s^2)^kX^{+}(s,\lambda)(\lambda^2-\eta^2)}\Big].$$

Let first $y<0$. Changing the order of integration and using an idea from the proof of Jordan lemma [34], we obtain:

$$\hat{v}(s,y)=\frac{1}{2\pi}\int\limits_0^{\infty}e^{-iy\lambda}\Big[\frac{1}{2}h(s,\lambda)-\frac{1}{2}h(s,\lambda)\Big]d\lambda+$$

$$+\frac{1}{2\pi}\int\limits_0^{\infty}e^{iy\lambda}\Bigg[-\frac{1}{2}h(s,\lambda)\frac{(\sqrt{\lambda^2+s^2}+\overline{a}s-\overline{b}i\lambda)(-i\lambda-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda^2+s^2})}{(\sqrt{\lambda^2+s^2}+\overline{a}s+\overline{b}i\lambda)(i\lambda-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda^2+s^2})}+$$

$$+\frac{1}{2}h(s,\lambda)\frac{X^{+}(s,-\lambda)}{X^{+}(s,\lambda)}\Bigg]d\lambda.$$

From definition of the canonical function $X(s,\eta)$ (2.23) it follows that

$$\hat{v}(s,y)=0,\quad \text{if}\quad y<0. \tag{3.1}$$

Let us consider the case $y>0$. As in previous situation, we come to the relation

$$\hat{v}(s,y)=\frac{1}{2\pi}\int\limits_0^{\infty}e^{-iy\lambda}h(s,\lambda)d\lambda+\frac{1}{2\pi}\int\limits_0^{\infty}e^{iy\lambda}\Bigg(-h(s,\lambda)\times$$

$$\times \frac{(\sqrt{\lambda^2+s^2}+\overline{a}s-\overline{b}i\lambda)(-i\lambda-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda^2+s^2})}{(\sqrt{\lambda^2+s^2}+\overline{a}s+\overline{b}i\lambda)(i\lambda-\overline{\alpha}s-\overline{\beta}\sqrt{\lambda^2+s^2})}\Bigg) d\lambda.$$

Making an obvious change of variable in the last integral, we have

$$\hat{v}(s,y) = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{-iy\lambda} h(s,\lambda)d\lambda \quad \text{for} \quad y>0. \tag{3.2}$$

Equalities (3.1) and (3.2) give the representation which defines the function $\hat{v}(s,y)$ at every point, exception for $y=0$:

$$\hat{v}(s,y) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int\limits_{-\infty}^{+\infty} e^{-iy\lambda} h(s,\lambda)d\lambda, & y>0, \\ 0, & y<0. \end{cases} \tag{3.3}$$

Remark 3.1. If $\kappa+1<0$ (or $\kappa<0$), then analysis of formulae (2.25) (or (2.28), (2.30)) leads again to representation (3.3) for the function $\hat{v}(s,y)$, although in this case we have to take into account the existence of a pole of the function $\hat{v}(s,y)$ at the point $\eta = is$.

Finally, taking into account an asymptotic representation of the Cauchy integral [30], we can find the function $\hat{v}(s,y)$ at the point $y=0$ [2]:

$$\hat{H}(s) = \lim_{y\to+0}\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{-y\lambda}\hat{v}(s,\lambda)d\lambda. \tag{3.4}$$

It occurs that $\hat{H}(s)\equiv 0$ (see also [31]).

By (3.3) and (3.4), the function is completely defined by the formulae

$$\hat{v}(s,y) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int\limits_{-\infty}^{+\infty} e^{-iy\lambda} h(s,\lambda)d\lambda, & y>0, \\ 0, & y\le 0, \end{cases} \qquad \hat{H}(s)\equiv 0. \tag{3.5}$$

Now we directly proceed to solving the main problem of this section, i.e, seeking of the function $v(t,y)$. In relation (3.5) we pass from the dual coordinates s,λ to the initial Cartesian t and y. For this purpose we use Cargniard-de Hoop method (see [11]).

Because of analogy, it suffices to consider only the first addendum from (3.5) (see (2.24)). We have:

$$I = \int\limits_{-\infty}^{+\infty} e^{-iy\lambda}\, d\lambda \left[\int\limits_0^{+\infty}\int\limits_0^{+\infty} \frac{e^{-\sqrt{\lambda^2+s^2}z_1+i\lambda z_2}\hat{F}(s,z_1,z_2)}{\sqrt{\lambda^2+s^2}+\overline{a}s-\overline{b}i\lambda}\, dz_1dz_2\right]. \tag{3.6}$$

Changing the variable $\lambda = sk$ and the order of integration, we come to

$$I = \int\limits_0^{+\infty}\int\limits_0^{+\infty} \hat{F}(s,z_1,z_2)\, dz_1dz_2 \int\limits_0^{\frac{1}{s}\infty} \frac{e^{-s\sqrt{k^2+1}z_1+skiz_2-skiy}}{\sqrt{k^2+1}+\overline{a}-\overline{b}ik}\, dk +$$

$$+\int\limits_0^{+\infty}\int\limits_0^{+\infty}\hat F(s,z_1,z_2)\,dz_1dz_2\int\limits_{-\frac{1}{s}\infty}^{0}\frac{e^{-s\sqrt{k^2+1}z_1+skiz_2-skiy}}{\sqrt{k^2+1}+\overline{a}-\overline{b}ik}\,dk=I_1+I_2.$$

Let us investigate the first integral, namely, the inner part of this integral

$$J_1=\int\limits_0^{\frac{1}{s}\infty}\frac{e^{-s(\sqrt{k^2+1}z_1-kiz_2+kiy)}}{\sqrt{k^2+1}+\overline{a}-\overline{b}ik}\,dk.$$

We introduce a new integration variable $t>0$ by the formula

$$\sqrt{k^2+1}z_1-kiz_2+kiy=t. \tag{3.9}$$

Equation (3.9) has two roots

$$k_{1,2}=\frac{-it(y-z_2)\pm z_1\sqrt{t^2-z_1^2-(z_2-y)^2}}{z_1^2+(z_2-y)^2}.$$

We choose the plus before the radical and thus obtain the explicit relation between t and k:

$$k=\frac{-it(y-z_2)+z_1\sqrt{t^2-z_1^2-(z_2-y)^2}}{z_1^2+(z_2-y)^2}. \tag{3.10}$$

The contour of integration (see Fig. 1) consists of:

1) a half-line which connects the points 0 and $\frac{1}{s}\infty$, its equation is

$$k=t\frac{1}{s},\quad t\ge 0;$$

2) a curve parametrically given by (3.10);

3) a curve satisfying

$$\mathrm{Re}(-s\sqrt{k^2+1}z_1+skiz_2-skiy)<0$$

at its points.

At $y>z_2$ the integration contour looks like.

We require the fulfilment of the uniform Lopatinsky condition, i.e., $\overline{a}>|\overline{b}|$. It occurs that in this case the integrand in J_1 does not have singularities on the interval $(-i,i)$. The Cauchy theorem gives

$$J_1=\left[\int\limits_{z_1}^{\sqrt{z_1^2+(z_2-y)^2}}+\int\limits_{\sqrt{z_1^2+(z_2-y)^2}}^{\infty}\right]e^{-st}\frac{\frac{z_1t}{\sqrt{t^2-z_1^2-(z_2-y)^2}}-i(y-z_2)}{tz_1+\overline{a}(z_1^2+(z_2-y)^2)-\overline{b}t(y-z_2)-}$$

$$\frac{dt}{-i(y-z_2+\overline{b}z_1)\sqrt{t^2-z_1^2-(z_2-y)^2}}. \tag{3.11}$$

By analogy, for $y>z_2$ on the contour.

$$J_2=\int\limits_{-\frac{1}{s}\infty}^{0}\frac{e^{-s\sqrt{k^2+1}z_1+skiz_2-skiy}}{\sqrt{k^2+1}+\overline{a}-\overline{b}ik}\,dk=\int\limits_0^{\frac{1}{s}\infty}\frac{e^{-s(\sqrt{k^2+1}z_1+kiz_2-kiy)}}{\sqrt{k^2+1}+\overline{a}+\overline{b}ik}\,dk$$

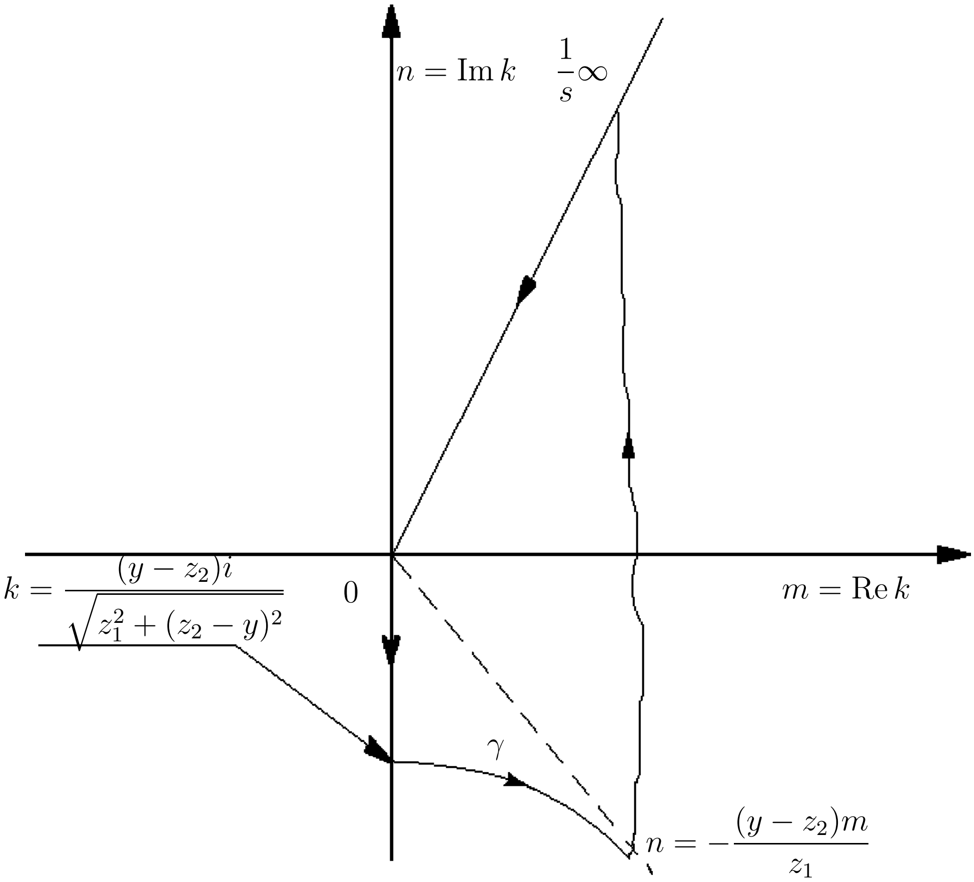

Figure 1. The integration contour for J_1 at $y > z_2$.

$$= \int\limits_{z_1}^{\sqrt{z_1^2+(z_2-y)^2}} + \int\limits_{\sqrt{z_1^2+(z_2-y)^2}}^{\infty} e^{-st} \frac{\frac{z_1 t}{\sqrt{t^2-z_1^2-(z_2-y)^2}} + i(y-z_2)}{tz_1 + \overline{a}(z_1^2+(z_2-y)^2) - \overline{b}t(y-z_2) +}$$

$$\frac{dt}{+\, i(y-z_2+\overline{b}z_1)\sqrt{t^2-z_1^2-(z_2-y)^2}}. \tag{3.12}$$

From (3.11) and (3.12) it follows that

$$I = \frac{1}{\pi}\int\limits_0^{+\infty}\int\limits_0^{+\infty} \hat{F}(s,z_1,z_2)dz_1dz_2 \int\limits_{\sqrt{z_1^2+(z_2-y)^2}}^{\infty} e^{-st}\mathrm{Re}\frac{i(y-z_2)+}{tz_1+\overline{a}(z_1^2+(z_2-y)^2)-\overline{b}t(y-z_2)+}$$

$$\frac{+\frac{z_1 t}{\sqrt{t^2-z_1^2-(z_2-y)2}}}{+\, i(y-z_2+\overline{b}z_1)\sqrt{t^2-z_1^2-(z_2-y)^2}}dt.$$

Consequently, a perturbation on $y = 0$ which we will call a "direct wave" (in the variables s and η it corresponds to the first addendum in (3.5)) is derived as follows:

$$v_1(t,y) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} \left[\frac{\theta(t-\sqrt{z_1^2+(z_2-y)^2})}{\sqrt{t^2-z_1^2-(z_2-y)^2}}K(t,y,z_1,z_2)\right] \underset{t}{*} f(t,z_1,z_2)\,dz_1\,dz_2 +$$

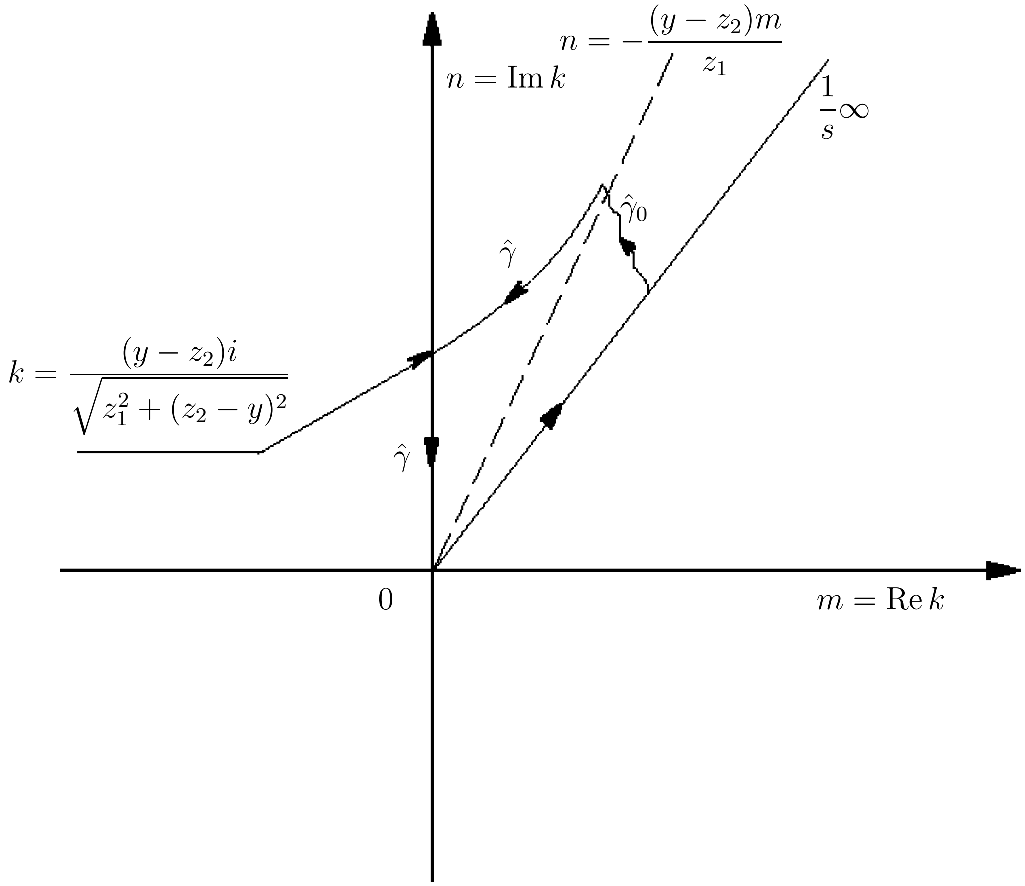

Figure 2. The integration contour for J_2 at $y > z_2$.

$$+\int\limits_0^{+\infty}\int\limits_0^{+\infty} \frac{\theta(t-\sqrt{z_1^2+(z_2-y)^2})}{\sqrt{t^2-z_1^2-(z_2-y)^2}} K(t,y,z_1,z_2)\psi(z_1,z_2)\,dz_1\,dz_2+$$

$$+\frac{\partial}{\partial t}\int\limits_0^{+\infty}\int\limits_0^{+\infty} \frac{\theta(t-\sqrt{z_1^2+(z_2-y)^2})}{\sqrt{t^2-z_1^2-(z_2-y)^2}} K(t,y,z_1,z_2)\varphi(z_1,z_2)\,dz_1\,dz_2. \tag{3.13}$$

Here $\theta(z)$ is the Heaviside function, $\frac{\partial}{\partial t}$ is the operator of generalized differentiation with respect to t, and the kernel $K(t,y,z_1,z_2)$ is of the form

$$K(t,y,z_1,z_2) = \frac{1}{\pi}\frac{z_1 t(tz_1+\overline{a}(z_1^2+(z_2-y)^2)) - \overline{b}t(y-z_2))+}{(tz_1+\overline{a}(z_1^2+(z_2-y)^2)-\overline{b}t(y-z_2))^2+}$$

$$\frac{+(y-z_2)(y-z_2+\overline{b}z_1)(t^2-z_1^2-(z_2-y)^2)}{+(y-z_2+\overline{b}z_1)(t^2-z_1^2-(z_2-y)^2)}.$$

The same method is used to find a "reflected wave" (the second addendum in (3.5); $\alpha/\beta > 1/|\beta|$) which takes into account also the influence of the solution behavior on the edge $x = 0$. Briefly this perturbation is characterized in such a way

$$v_2(t,y) = \int\limits_0^{+\infty}\int\limits_0^{+\infty} \left[\frac{\theta(t-\sqrt{z_1^2+(z_2-y)^2})}{\sqrt{t^2-z_1^2-(z_2+y)^2}} M(t,y,z_1,z_2)\right] \underset{t}{*} f(t,z_1,z_2)\,dz_1\,dz_2+$$

$$+\frac{\partial}{\partial t}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\frac{\theta(t-\sqrt{z_1^2+(z_2+y)^2})}{\sqrt{t^2-z_1^2-(z_2+y)^2}}M(t,y,z_1,z_2)\varphi(z_1,z_2)\,dz_1\,dz_2+$$

$$+\int\limits_0^{+\infty}\int\limits_0^{+\infty}\frac{\theta(t-\sqrt{z_1^2+(z_2+y)^2})}{\sqrt{t^2-z_1^2-(z_2+y)^2}}M(t,y,z_1,z_2)\psi(z_1,z_2)\,dz_1\,dz_2. \tag{3.14}$$

Here

$$M(t,y,z_1,z_2)=\frac{1}{\pi}\mathrm{Re}\left\{\frac{[\overline{\beta}tz_1+\overline{\alpha}(z_1^2+(z_2+y)^2)-t(y+z_2)+}{[\overline{\beta}tz_1+\overline{\alpha}(z_1^2+(z_2+y)^2)+t(y+z_2)+}\right.$$

$$\frac{+i(\overline{\beta}(y+z_2)+z_1)\sqrt{t^2-z_1^2-(z_2+y)^2}]\times}{+i(\overline{\beta}(y+z_2)-z_1)\sqrt{t^2-z_1^2-(z_2+y)^2}]\times}$$

$$\left.\frac{\times[-i(y+z_2)+\frac{z_1t}{\sqrt{t^2-z_1^2-(z_2+y)^2}}]}{\times[tz_1+\overline{a}(z_1^2+(z_2+y)^2)-\overline{b}t(y+z_2)+i(y+z_2+\overline{b}z_1)\sqrt{t^2-z_1^2-(z_2+y)^2}]}\right\}.$$

Uniting (3.13) and (3.14), we come to the important conclusion which gives a qualitative characteristic of the phenomenon: the function $v(t,y)$ which describes the behavior of solution on $x=0$ is a superposition of two waves: direct and reflected from the edge $\Gamma=\{(t,0,0)|t\geq 0\}$.

Remark 3.2. Let the condition $\overline{a}>|\overline{b}|$ be broken. Then the integrand in J_1 (3.8) can have poles of the first order at $(-i,i)$.

Let consider, for example, the case $\overline{a}>0$, $\overline{b}>0$ and $\overline{a}<\overline{b}$. Then the singularity is placed on the interval $(0,-i)$

$$k=\frac{i(-\overline{b}\overline{a}-\sqrt{1-\overline{a}^2+\overline{b}^2})}{\overline{b}^2+1}=k_0.$$

and we have to add

$$-\pi i\operatorname*{Res}_{k=k_0}\left[\frac{e^{-s(\sqrt{k^2+1}z_1-kiz_2+kiy)}}{\sqrt{k^2+1}+\overline{a}-\overline{b}ik}\right]=-\pi i\frac{e^{-s((\overline{b}ik_0-\overline{a})z_1+k_0i(y-z_2))}}{\frac{k_0}{\overline{b}ik_0-\overline{a}}-\overline{b}i}=$$

$$=e^{-s\left(\frac{-\overline{a}+\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}z_1+\frac{\overline{b}\overline{a}+\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}(y-z_2)\right)}\frac{\pi(\overline{a}-\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2})}{(\overline{b}^2+1)\sqrt{1-\overline{a}^2+\overline{b}^2}}$$

into (3.11), $y-z_2-\dfrac{b\overline{a}+\sqrt{1-\overline{a}+\overline{b}^2}}{\overline{b}^2+1}\sqrt{z_1^2+(z_2-y)^2}\geq 0$. In common J_1 and J_2 give a supplement into (3.8):

$$e^{-s(\frac{-\overline{a}+\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}z_1+\frac{\overline{b}\overline{a}+\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}(y-z_2))}\frac{2\pi(\overline{a}-\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2})}{(\overline{b}^2+1)\sqrt{1-\overline{a}^2+\overline{b}^2}}.$$

As a result, formula (3.13) has to be complemented by the addend (from the f-th side):

$$\frac{\overline{a}-\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{(\overline{b}^2+1)\sqrt{1-\overline{a}^2+\overline{b}^2}}\int\limits_0^{\infty}\int\limits_0^{\infty}f\left(t-\frac{-\overline{a}+\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}z_1-\right.$$

$$-\frac{\overline{b}\overline{a}+\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}(y-z_2)z_1, z_2\Bigg)\theta\Bigg(y-z_2-\frac{b\overline{a}+\sqrt{1-\overline{a}+\overline{b}^2}}{\overline{b}^2+1}\sqrt{z_1^2+(z_2-y)^2}\Bigg)dz_1dz_2. \tag{3.16}$$

Initial deviation and velocity also bring their contributions:

$$\frac{\overline{a}-\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{(\overline{b}^2+1)\sqrt{1-\overline{a}^2+\overline{b}^2}}\int_0^\infty\int_0^\infty \delta(t-\frac{-\overline{a}+\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}z_1-$$

$$-\frac{\overline{b}\overline{a}+\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}(y-z_2))\psi(z_1,z_2)\theta\Bigg(y-z_2-\frac{b\overline{a}+\sqrt{1-\overline{a}+\overline{b}^2}}{\overline{b}^2+1}\sqrt{z_1^2+(z_2-y)^2}\Bigg)dz_1dz_2+ \tag{3.17}$$

$$+\frac{\overline{a}-\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{(\overline{b}^2+1)\sqrt{1-\overline{a}^2+\overline{b}^2}}\frac{\partial}{\partial t}\int_0^\infty\int_0^\infty \delta(t-\frac{-\overline{a}+\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}z_1-$$

$$-\frac{\overline{b}\overline{a}+\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}(y-z_2))\varphi(z_1,z_2)\theta\Bigg(y-z_2-\frac{b\overline{a}+\sqrt{1-\overline{a}+\overline{b}^2}}{\overline{b}^2+1}\sqrt{z_1^2+(z_2-y)^2}\Bigg)z_1dz_2=$$

$$=\frac{\overline{a}-\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{(\overline{b}^2+1)\sqrt{1-\overline{a}^2+\overline{b}^2}}\Bigg\{\int_0^{t\gamma}dz_1\psi(z_1,y-t\frac{b^2+1}{\overline{a}\overline{b}+\sqrt{1-\overline{a}^2+\overline{b}^2}}+$$

$$+\frac{-\overline{a}+\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{a}\overline{b}+\sqrt{1-\overline{a}^2+\overline{b}^2}}z_1)+\frac{\partial}{\partial t}\Bigg(\int_0^{t\gamma}dz_1\varphi(z_1,y-t\frac{b^2+1}{\overline{a}\overline{b}+\sqrt{1-\overline{a}^2+\overline{b}^2}}+$$

$$+\frac{-\overline{a}+\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{a}\overline{b}+\sqrt{1-\overline{a}^2+\overline{b}^2}}z_1)\Bigg)\Bigg\},$$

where $\gamma=\left(\frac{-\overline{a}+\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}+\frac{(\overline{b}\overline{a}+\sqrt{1-\overline{a}^2+\overline{b}^2})^2/(\overline{b}^2+1)^2}{(1-(\overline{b}\overline{a}+\sqrt{1-\overline{a}^2+\overline{b}^2})^2/(\overline{b}^2+1)^2)^{1/2}}\right)^{-1}$.

It is interesting to note that the planes

$$t-\frac{-\overline{a}+\overline{b}\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}x-\frac{\overline{b}\overline{a}+\sqrt{1-\overline{a}^2+\overline{b}^2}}{\overline{b}^2+1}y=\text{const}$$

are the characteristics of the wave equation (1.1).

In the aggregate, formulae (3.16), (3.17) are an analog of a "side wave" ([24]; A. Yu. Chinilov in [10] has analyzed its properties for the half-plane case (see also [15])).

Remark 3.3. If $\overline{\alpha}>|\overline{\beta}|$, then in (3.14), the representation of the reflected wave, addends which correspond to the side wave are absent.

That completes the proof of Theorem I.

4. A priori estimates of solutions

Proceeding to obtaining of estimates, we note the important fact: under conditions on the right hand sides from §2 the generalized solution (see Theorem 1) becomes the classical solution of the problem since after usual changes of variables one can carry out differentiation in the integrals as many times as one wants.

To prove Theorem 2 we reduce problem (1.1)–(1.4) to a mixed problem for the symmetric system [12]:

$$\left\{A_0\frac{\partial}{\partial t}-B_0\frac{\partial}{\partial x}-C_0\frac{\partial}{\partial y}\right\}U=0,\quad t>0,\ (x,y)\in R^2_+, \tag{4.1}$$

$$u_1-au_2-bu_3=0,\quad x=0,\ (t,y)\in R^2_+, \tag{4.2}$$

$$u_1-\beta u_2-\alpha u_3=0,\quad y=0,\ (t,x)\in R^2_+, \tag{4.3}$$

$$U=\begin{pmatrix}\psi(x,y)\\ \dfrac{\partial}{\partial x}\varphi(x,y)\\ \dfrac{\partial}{\partial y}\varphi(x,y)\end{pmatrix},\quad t=0,\ (x,y)\in R^2_+. \tag{4.4}$$

Here the matrices A_0, B_0, C_0 are as follows

$$A_0=\begin{pmatrix}k & l & m\\ l & k & 0\\ m & 0 & k\end{pmatrix},\quad B_0=\begin{pmatrix}l & k & 0\\ k & l & m\\ 0 & m & -l\end{pmatrix},\quad C_0=\begin{pmatrix}m & 0 & k\\ 0 & -m & l\\ k & l & m\end{pmatrix},$$

$$U=\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}=\begin{pmatrix}u_t\\ u_x\\ u_y\end{pmatrix},$$

and k,l,m are some real numbers satisfying two inequalities

$$\begin{aligned}&1)\quad k>0,\\ &2)\quad k^2-m^2-l^2>0.\end{aligned} \tag{4.5}$$

Note that conditions (4.5) provide the matrix A_0 with the positive definiteness.

From system (4.1) we easily derive the principal identity:

$$\frac{d}{dt}\left\{\iint\limits_{R^2_+}(A_0U,U)\,dxdy\right\}+\int\limits_{R^+_1}(B_0U,U)\Big|_{x=0}dy+$$

$$+\int\limits_{R^+_1}(C_0U,U)\Big|_{y=0}dx=0, \tag{4.6}$$

Let us turn again to formulae (3.13), (3.14) and transform the expression for the direct wave (accounted is yet the perturbation caused by the right hand side $f(t,x,y)$ of the equation):

$$v_1(t,y)=\frac{1}{\pi}\int\limits_0^{+\infty}\int\limits_0^{+\infty}dz_1dz_2\frac{z_1t(z_1t+\overline{a}(z_1^2+(z_2-y)^2)-\overline{b}t(y-z_2))+}{(z_1t+\overline{a}(z_1^2+(z_2-y)^2)-\overline{b}t(y-z_2))^2+}$$

$$\frac{+(y-z_2)(y-z_2+\overline{b}z_1)(t^2-z_1^2-(z_2-y)^2)}{+(y-z_2+\overline{b}z_1)^2(t^2-z_1^2-(z_2-y)^2)}\frac{\theta(t-\sqrt{z_1^2+(z_2-y)^2})}{\sqrt{t^2-z_1^2-(z_2-y)^2}} \underset{t}{*} f(t,z_1,z_2)=$$

$$=\frac{1}{\pi}\int\limits_0^t d\tau \iint\limits_{\substack{z_1^2+(z_2-y)^2\leq(t-\tau)^2,\\ z_1>0}} dz_1dz_2 \frac{z_1(t-\tau)(z_1(t-\tau)+\overline{a}(z_1^2+(z_2-y)^2)-\overline{b}(t-\tau)(y-z_2))+}{(z_1(t-\tau)+\overline{a}(z_1^2+(z_2-y)^2)-\overline{b}(t-\tau)(y-z_2))^2+}$$

$$\frac{+(y-z_2)(y-z_2+\overline{b}z_1)((t-\tau)^2-z_1^2-(z_2-y)^2)}{+(y-z_2-\overline{b}z_1)^2((t-\tau)^2-z_1^2-(z_2-y)^2)}\frac{f(\tau,z_1,z_2)}{\sqrt{(t-\tau)^2-z_1^2-(z_2-y)^2}}.$$

Making the change of variables

$$\begin{cases} z_1 &= \xi_1(t-\tau), \\ z_2-y &= \xi_2(t-\tau), \end{cases}$$

we come to

$$v_1(t,y)=\frac{1}{\pi}\int\limits_0^t (t-\tau)\,d\tau \iint\limits_{\substack{\xi_1^2+\xi_2^2\leq 1\\ \xi_1\geq 0}} \frac{\xi_1(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)+\xi_2(\xi_2-\overline{b}\xi_1)(1-\xi_1^2-\xi_2^2)}{(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)^2+(\xi_2-\overline{b}\xi_1)^2(1-\xi_1^2-\xi_2^2)}\times$$

$$\times\frac{f(\tau,(t-\tau)\xi_1,y+(t-\tau)\xi_2)}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_1 d\xi_2. \tag{4.7}$$

We seek for zeros of the determinant of the kernel from presentation (4.7), i.e. of the function $\frac{\xi_1(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)+\xi_2(\xi_2-\overline{b}\xi_1)(1-\xi_1^2-\xi_2^2)}{(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)^2+(\xi_2-\overline{b}\xi_1)^2(1-\xi_1^2-\xi_2^2)}$.

Let first $\xi_1^2+\xi_2^2\neq 1$. Then the denominator in (4.7) comes into zero if simultaneously

$$\begin{cases} \xi_2-b\xi_1=0, \\ \xi_1-\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2=0, \end{cases} \tag{4.8}$$

what is possible if

$$\begin{cases} \xi_1 &= 0, \\ \xi_2 &= 0, \end{cases} \quad \text{or} \quad \overline{a}\neq 0, \begin{cases} \xi_1 &= -\dfrac{1}{\overline{a}}, \\ \xi_2 &= -\dfrac{b}{\overline{a}}. \end{cases} \tag{4.9}$$

It is apparent that of interest is only the second case. It appears that under $\overline{a}^2>\overline{b}^2+1$, $\overline{a}<0$ (see(2.13)) the kernel of the presentation has a singularity of the first order inside the domain of integration.

If $\xi_1^2+\xi_2^2=1$, then the system

$$\begin{cases} \xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2=0, \\ \xi_1^2+\xi_2^2=1, \end{cases} \tag{4.10}$$

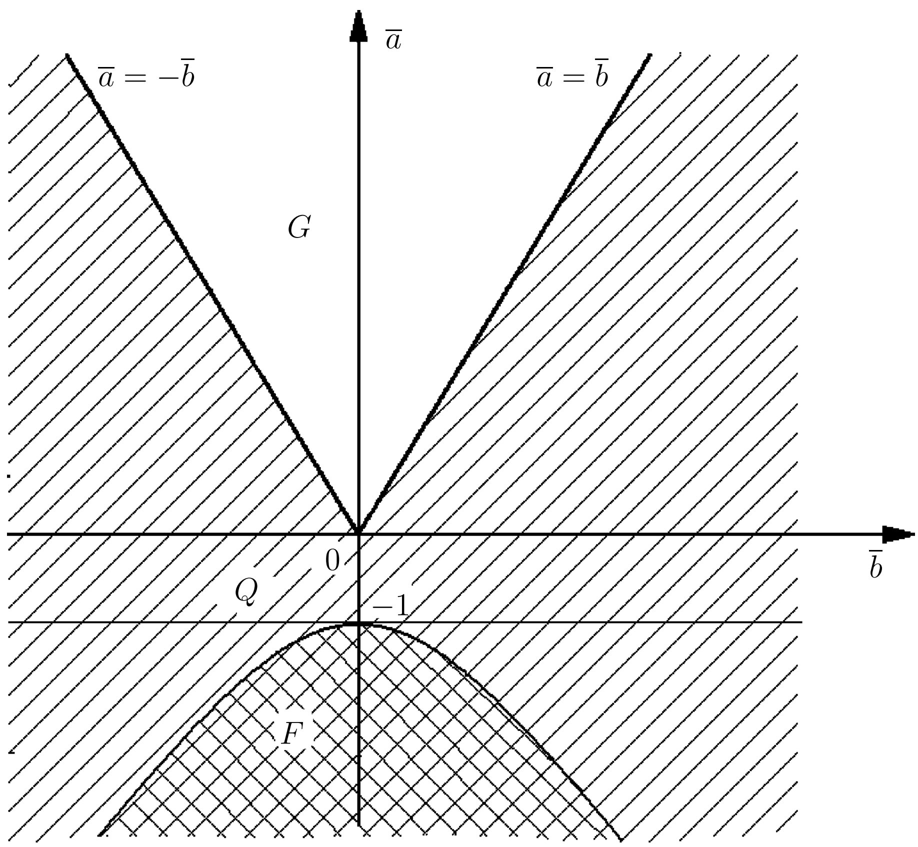

Figure 3. The representation of the domains G, Q, F.

provided that $\overline{a}^2 \le 1 + \overline{b}^2$, has two solutions:

$$\begin{cases} \xi_1 = \dfrac{-\overline{a} \pm \overline{b}\sqrt{1 - \overline{a}^2 + \overline{b}^2}}{1 + \overline{b}^2}, \\ \xi_2 = \dfrac{-\overline{a}\overline{b} \pm \sqrt{1 - \overline{a}^2 + \overline{b}^2}}{1 + \overline{b}^2}. \end{cases} \tag{4.11}$$

It is easy to verify that in the domain $G = \{(\overline{a}, \overline{b}) \big| \overline{a} > |\overline{b}|\}$ conditions $\xi_2^2 + \xi_1^2 = 1$, $\xi_1 \ge 0$ are broken.

By results of analysis of formulae (4.8)–(4.11) we divide R^2, $(\overline{a}, \overline{b})$–coordinates plane into three parts (see the Fig. 3), they are

1) the above described domain G, its inherent property is that the closed semicircle $\xi_1^2 + \xi_2^2 \le 1$, $\xi_1 \ge 0$ does not contain singularities of the integrand function;

2) a closed domain Q bounded by parts of the lines $\overline{a} = -\overline{b}$, $\overline{a} = \overline{b}$ and the hyperbolic curve $\overline{a}^2 = \overline{b}^2 + 1$, $\overline{a} < 0$. Here singularities of the integrand function from (4.6) are situated strictly on the semicircle $\xi_1^2 + \xi_2^2 = 1$, $\xi_1 \ge 0$; the line $\overline{a} = -1$ belongs to Q;

3) a domain F such that if $(\overline{a}, \overline{b}) \in F$, then a singularity of the integrand function is an interior point of the unit semicircle.

We will concentrate our attention on the case $(\overline{a},\overline{b})\in G$. From (4.7) for the wave $v_1(t,y)$ we obtain:

$$v_1(t,y)=\frac{1}{\pi}\int\limits_0^t(t-\tau)\,d\tau\int\limits_0^1 d\xi_1\int\limits_{-\sqrt{1-\xi_1^2}}^{\sqrt{1-\xi_1^2}}\frac{\xi_1(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)+}{(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)^2+}$$

$$\frac{+\xi_2(\xi_2-\overline{b}\xi_1)(1-\xi_1^2-\xi_2^2)}{+(\xi_2-\overline{b}\xi_1)^2(1-\xi_1^2-\xi_2^2)}\frac{f(\tau,(t-\tau)\xi_1,y+(t-\tau)\xi_2)}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_2=$$

$$=\frac{1}{\pi}\int\limits_0^t(t-\tau)\,d\tau\int\limits_0^1 d\xi_1\int\limits_0^{\sqrt{1-\xi_1^2}}\left[\frac{\xi_1(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)+\xi_2(\xi_2-\overline{b}\xi_1)(1-\xi_1^2-\xi_2^2)}{(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)^2+(\xi_2-\overline{b}\xi_1)^2(1-\xi_1^2-\xi_2^2)}-\right.$$

$$\left.-\frac{\xi_1}{\xi_1+\overline{a}+\overline{b}\sqrt{1-\xi_1^2}}\right]\frac{f(\tau,(t-\tau)\xi_1,y+(t-\tau)\xi_2)}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_2+$$

$$+\frac{1}{\pi}\int\limits_0^t(t-\tau)\,d\tau\int\limits_0^1 d\xi_1\int\limits_0^{\sqrt{1-\xi_1^2}}\frac{\xi_1}{(\xi_1+\overline{a}+\overline{b}\sqrt{1-\xi_1^2})\sqrt{(1-\xi_1^2-\xi_2^2)}}\times$$

$$\times f(\tau,(t-\tau)\xi_1,y+(t-\tau)\xi_2)\,d\xi_2+$$

$$+\frac{1}{\pi}\int\limits_0^t(t-\tau)\,d\tau\int\limits_0^1 d\xi_1\int\limits_{-\sqrt{1-\xi_1^2}}^{0}\left[\frac{\xi_1(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)+\xi_2(\xi_2-\overline{b}\xi_1)(1-\xi_1^2-\xi_2^2)}{(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)^2++(\xi_2-\overline{b}\xi_1)^2(1-\xi_1^2-\xi_2^2)}-\right.$$

$$\left.-\frac{\xi_1}{\xi_1+\overline{a}-\overline{b}\sqrt{1-\xi_1^2}}\right]\frac{f(\tau,(t-\tau)\xi_1,y+(t-\tau)\xi_2)}{\sqrt{1-\xi_1^2-\xi_2^2}}\,d\xi_2+$$

$$+\frac{1}{\pi}\int\limits_0^t(t-\tau)\,d\tau\int\limits_0^1 d\xi_1\int\limits_{-\sqrt{1-\xi_1^2}}^{0}\frac{\xi_1}{(\xi_1-\overline{a}+\overline{b}\sqrt{1-\xi_1^2})\sqrt{(1-\xi_1^2-\xi_2^2)}}\times$$

$$\times f(\tau,(t-\tau)\xi_1,y+(t-\tau)\xi_2)\,d\xi_2=I_1+I_2+I_3+I_4. \tag{4.12}$$

Functions in I_1 and I_3 do not possess singularities, so estimates for them can be found in traditional manner with the use of the generalized Minkowski and Hölder equalities:

$$\|I_i(t)\|_{L_2;y}\le\tilde{C}_i\int\limits_0^t(t-\tau)^{1/2}\|f(\tau,x,y)\|_{L_2;(x,y)}d\tau,\quad i=1,3, \tag{4.13}$$

where

$$\tilde{C}_1^2=\frac{1}{\pi^2}\int\limits_0^1 d\xi_1\left\{\int\limits_0^{\sqrt{1-\xi_1^2}}d\xi_2\left[\frac{\xi_1(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)+\xi_2(\xi_2-\overline{b}\xi_1)(1-\xi_1^2-\xi_2^2)}{(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)^2+(\xi_2-\overline{b}\xi_1)^2(1-\xi_1^2-\xi_2^2)}-\right.\right.$$

$$
\left.\left.-\frac{\xi_1}{\xi_1+\overline{a}+\overline{b}\sqrt{1-\xi_1^2}}\right]\frac{1}{\sqrt{1-\xi_1^2-\xi_2^2}}\right\}^2,
$$

$$
\tilde{C}_3^2=\frac{1}{\pi^2}\int\limits_0^1 d\xi_1\left\{\int\limits_{-\sqrt{1-\xi_1^2}}^0 d\xi_2\left[\frac{\xi_1(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)+\xi_2(\xi_2-\overline{b}\xi_1)(1-\xi_1^2-\xi_2^2)}{(\xi_1+\overline{a}(\xi_1^2+\xi_2^2)+\overline{b}\xi_2)^2+(\xi_2-\overline{b}\xi_1)^2(1-\xi_1^2-\xi_2^2)}-\right.\right.
$$

$$
\left.\left.-\frac{\xi_1}{\xi_1+\overline{a}-\overline{b}\sqrt{1-\xi_1^2}}\right]\frac{1}{\sqrt{1-\xi_1^2-\xi_2^2}}\right\}^2.
$$

The situation with the integrals I_2 and I_4 is somewhat more difficult. Obviously it suffices to consider one of them, let it be I_2. Using the known theorem on convolution operator norm in L_2 [25, 35] and the generalized Minkowski inequality, we obtain the estimate:

$$
||I_2(t)||_{L_2;y}\leq\frac{1}{\pi}\int\limits_0^t(t-\tau)d\tau\int_0^1 d\xi_1\frac{\xi_1}{\xi_1+\overline{a}+\overline{b}\sqrt{1-\xi_1^2}}\Big|\sup_{\eta\in R}\hat{M}(\eta)\Big|\,\|f(\tau,(t-\tau)\xi_1,y)\|_{L_2;y}. \quad (4.14)
$$

Here the function $M(\xi_2)$ is determined by the formula:

$$
M(\xi_2)=\begin{cases}\dfrac{1}{\sqrt{1-\xi_1^2-\xi_2^2}}, & \text{if}\quad 0<\xi_2<\sqrt{1-\xi_1^2},\\ 0, & \text{if}\quad \xi_2<0.\end{cases}
$$

or, in other way,

$$
M(\xi_2)=\frac{1}{2}[K_1(\xi_2)+K_2(\xi_2)],
$$

where

$$
K_1(\xi_2)=\begin{cases}(1-\xi_1^2-\xi_2^2)^{-1/2},\ |\xi_2|<\sqrt{1-\xi_1^2},\\ 0,\ |\xi_2|>\sqrt{1-\xi_1^2},\end{cases}
$$

and

$$
K_2(\xi_2)=\begin{cases}\operatorname{sgn}\xi_2(1-\xi_1^2-\xi_2^2)^{-1/2},\ |\xi_2|<\sqrt{1-\xi_1^2},\\ 0,\ |\xi_2|>\sqrt{1-\xi_1^2}.\end{cases}
$$

In [6] we found formulae, the result of application of the Fourier transform to the functions $K_1(\xi_2)$ and $K_2(\xi_2)$:

$$
\hat{K}_1(\eta)=\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)J_0(\sqrt{1-\xi_1^2}|\eta|), \quad (4.15)
$$

$$
\hat{K}_2(\eta)=i\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)\operatorname{sgn}\eta H_0(\sqrt{1-\xi_1^2}|\eta|). \quad (4.16)
$$

Here $J_0(x)=\sum\limits_{k=0}^{\infty}\dfrac{(-1)^k(\frac{x}{2})^{2k}}{k!(k+1)!}$ is a zero order Bessel function, $H_0(x)=\sum\limits_{k=0}^{\infty}\dfrac{(-1)^k(\frac{x}{2})^{2k+1}}{[\Gamma(k+\frac{3}{2})]^2}$ is a zero order Struve function, and $\Gamma(x)$ is a gamma-function [3]. In [1, 3] asymptotic expansions of these functions at infinity are given.

$$
J_0(x)=\sqrt{\frac{2}{\pi x}}\{\cos(x-\frac{\pi}{4}+O(x^{-3/2})\}\quad\text{as}\quad x\to+\infty, \quad (4.17)
$$

$$Y_0(x) = \sqrt{\frac{2}{\pi x}}\{\sin(x - \frac{\pi}{4}) + O(x^{-3/2})\} \quad \text{as} \quad x \to +\infty,$$

$Y_0(x)$ is the Neumann function;

$$H_0(x) = Y_0(x) + \frac{1}{\pi}\sum_{k=0}^{m-1} \frac{\Gamma(k+1/2)}{\Gamma(\frac{1}{2}-k)(\frac{x}{2})^{2k+1}} + R_m(x),$$

where $R_m(x) = O(x^{-2m-1})$. From (4.14)–(4.17) the desired estimate follows:

$$\|I_2(t)\|_{L_2;y} \leq \check{C} \int_0^t (t-\tau)^{1/2}\|f(\tau,x,y)\|_{L_2;(x,y)}d\tau. \tag{4.18}$$

Summing up equalities (4.13), (4.18), we have

$$\|v_1(t,y)\|_{L_2;y} \leq \hat{C} \int_0^t (t-\tau)^{1/2}\|f(\tau,x,y)\|_{L_2;(x,y)}d\tau. \tag{4.19}$$

With account of perturbations caused by the initial deviation and velocity, a more refined version for formula (4.19) is:

$$\|v_1(t,y)\|_{L_2;y} \leq A \int_0^t (t-\tau)^{1/2}\|f(\tau,x,y)\|_{L_2;(x,y)}d\tau + \tag{4.20}$$

$$+At^{1/2}\|\psi\|_{L_2;(x,y)} + Bt^{-1/2}\|\varphi\|_{L_2;(x,y)} +$$

$$+Ct^{1/2}\|\varphi_x\|_{L_2;(x,y)} + Dt^{1/2}\|\varphi_y\|_{L_2;(x,y)}.$$

In a similar way we investigate the reflected wave. Thus, we come to the aim of our consideration, the following relation:

$$\|v(t,y)\|_{L_2;y} \leq \|v_1(t,y)\|_{L_2;y} + \|v_2(t,y)\|_{L_2;y} \leq$$

$$\leq \tilde{A}\int_0^t (t-\tau)^{1/2}\|f(\tau,x,y)\|_{L_2;y}d\tau + \tilde{A}t^{1/2}\|\psi\|_{L_2;(x,y)} + \tilde{B}t^{-1/2}||\varphi||_{L_2;(x,y)} +$$

$$+\tilde{C}t^{1/2}\|\varphi_x\|_{L_2;(x,y)} + \tilde{D}t^{1/2}\|\varphi_y\|_{L_2;(x,y)}. \tag{4.21}$$

Inequality (4.21) and identity (4.6) yield an estimate with loss of smoothness.

Remark 4.1. (regarding to inequality (4.21)). For consideration of the perturbation generated by the initial deviation $\varphi(x,y)$ estimation (4.20) becomes inconvenient since it contains $Bt^{-1/2}\|\varphi\|_{L_2;(x,y)}$. But in the final part while obtaning the estimation one can reason in other way using the following relation:

$$\int_0^1 \hat{P}(\eta,\xi_1)d\xi_1 \int_0^{t\xi_1} \hat{\varphi}_z(z,\eta)dz \leq Ct^{1/2}\|\hat{\varphi}_z(z,\eta)\|_{L_2;z}.$$

Consequently, the following estimation holds for small t:

$$\|v(t,y)\|_{L_2;y} \leq \tilde{A}\int_0^t (t-\tau)\|f(\tau,x,y)\|_{L_2;(x,y}d\tau + \tilde{A}t\|\psi\|_{L_2;(x,y)}+$$

$$+\tilde{C}t^{1/2}\|\varphi_x\|_{L_2;(x,y)} + \tilde{D}t^{1/2}\|\varphi_y\|_{L_2;(x,y)}. \tag{4.22}$$

Inequalities (4.21), (4.22) and identity (4.6) allow to obtain a priori estimation with the loss of smoothness.

It turns out, however, that if $\bar{a} > |\bar{b}|$ and $\bar{\alpha} > |\bar{\beta}|$, then the additional requirement on smoothness of right parts of the problem are not required since a priori estimation without loss of smoothness are valid.

Actually it suffices to apply the properties of the Fourier-Laplace transform of the function $\dfrac{\theta(t)\,\theta(t-\sqrt{z_1^2+z_2^2})}{\sqrt{t^2-z_1^2-z_2^2}}$, i.e., $L_{t\to s}\, F_{z_2\to\eta}\, \dfrac{\theta(t)\,\theta(t-\sqrt{z_1^2+z_2^2})}{\sqrt{t^2-z_1^2-z_2^2}}$. This leads, for example, to the following inequality:

$$\sup_{\substack{\eta\in R\\ s_2=\mathrm{Im}\in R}} \left(\int_0^\infty dz_1|\eta|^2|\hat{H}(s,z_1,\eta,)|^2\right)^{1/2} < \infty, \tag{4.23}$$

where

$$H(t,z_1,z_2) = \frac{z_1t(z_1t+\overline{a}(z_1^2+z_2^2)-\overline{b}tz_2)+z_2(z_2+\overline{b}z_1)(t^2-z_1^2-z_2^2)}{(z_1t+\overline{a}(z_1^2+z_2^2)-\overline{b}tz_2)^2+(z_2+\overline{b}z_1)^2(t^2-z_1^2-z_2^2)}\times$$

$$\times\frac{\theta(t)\,\theta(t-\sqrt{z_1^2+z_2^2})}{\sqrt{t^2-z_1^2-z_2^2}}.$$

Thus Theorem 2 is completely proved.

Remark 4.2. If at least for one boundary just Lopatinsky condition is fulfilled, then in the solution representation there appears a side wave (see formulae (3.16), (3.17)) what implies the loss of smoothness in a priori estimate. However, the same situation takes place for the case with one boundary ([15, 28, 33]).

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