ON THE WEAK SOLUTIONS OF THE EQUATION RELATED TO THE DIAMOND OPERATOR*

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Рассматривается функция Грина оператора $\oplus^k$, определенного следующим образом:

$$\oplus^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^4 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^4\right]^k,$$

где $p+q=n$ — размерность пространства C^n векторов $x=(x_1,x_2,\ldots,x_n)$ с n комплексными компонентами x_j, k — целое неотрицательное число. Выполнено исследование функции Грина, которая затем применяется для построения слабого решения уравнения $K(x)$, такого что

$$\oplus^k K(x) = f(x),$$

где f — обобщенная функция.

1. Introduction

The operator $\oplus^k$ can be factorized in the following form

$$\oplus^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \left[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k \left[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k \tag{1.1}$$

where $i=\sqrt{-1}$ and $p+q=n$. The operator $\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2$ has first been introduced by A. Kananthai [4] and is named the Diamond operator which is denoted by

$$\diamondsuit^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k. \tag{1.2}$$

Let us denote the operators

$$L_1^k = \left[\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right]^k \tag{1.3}$$

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and

$$L_2^k = \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right]^k. \tag{1.4}$$

Thus the operator $\oplus^k$, iterated k-times defined by (1.1) can be written in the form

$$\oplus^k = \Diamond^k L_1^k L_2^k. \tag{1.5}$$

In this work, we obtain the Green function of the operator $\oplus^k$, i. e. $\oplus^k G(x) = \delta$ where δ is the Dirac-delta distribution and $G(x)$ is the Green function and $x \in R^n$.

Moreover, we find the weak solution of the equation

$$\oplus^k K(x) = f(x) \tag{1.6}$$

where f is a given generalized function and $K(x)$ is an unknown and $x \in R^n$.

2. Preliminary

Definition 2.1. *Let $x = (x_1, x_2, ..., x_n) \in R^n$*

Let us denote by

$$u = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \tag{2.1}$$

the nondegenerated quadratic form, whereas $p + q = n$ is the dimension of R^n.

Let $\Gamma_+ = \{x \in R^n : x_1 > 0$ and $u > 0\}$ and $\overline{\Gamma_+}$ denotes its closure.

For any complex number α, we define the function

$$R_\alpha^H(u) = \begin{cases} \dfrac{u^{\frac{(\alpha-n)}{2}}}{K_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 \quad \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{(n-1)}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)}.$$

The function R_α^H is called The Ultra-Hyperbolic Kernel of Marcel Riesz and was introduced by Y. Nozaki (see [3], p. 72).

It is well known that R_α^H is an ordinary function if $\operatorname{Re}(\alpha) \geq n$ and is a distribution of α if $\operatorname{Re}(\alpha) < n$. Let us supp $R_\alpha^H(u)$ denote the support of $R_\alpha^H(u)$. Assume $R_\alpha^H(u) \subset \overline{\Gamma_+}$.

Definition 2.2. *Let $x = (x_1, x_2, ..., x_n)$ be a point of the Euclidean space R^n and*

$$v = \sum_{i=1}^{n} x_i^2. \tag{2.3}$$

Define the function

$$R^e_\alpha(v) = \frac{v^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, \tag{2.4}$$

where α is any complex number and the constant $H_n(\alpha)$ is given by the formula

$$H_n(\alpha) = \frac{\pi^{\frac{1}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \tag{2.5}$$

Now the function $R^e_\alpha(v)$ is called the Elliptic Kernel of Marcel Riesz.

Definition 2.3. *Let $x = (x_1, x_2, ..., x_n)$ be a point of the C^n and let*

$$w = x_1^2 + x_2^2 + ... + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2), \tag{2.6}$$

where $i = \sqrt{-1}$ and $p + q = n$ is the dimension of R^n.

Define the function

$$S_\alpha(w) = \frac{w^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, \tag{2.7}$$

where α is any complex number and $H_n(\alpha)$ is defined as the formula (2.5).

Definition 2.4. *Define the function*

$$T_\alpha(z) = \frac{z^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, \tag{2.8}$$

where

$$z = x_1^2 + x_2^2 + ... + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + ... + x_{p+q}^2) \tag{2.9}$$

and $i = \sqrt{-1}$, $p + q = n$ and $H_n(\alpha)$ is defined as (2.5).

Lemma 2.1. *The convolution $R^H_{2k}(u) * (-1)^k R^e_{2k}(v)$ is an elementary solution of the operator remove off $\diamondsuit^k$ where $\diamondsuit^k$ is defined by (1.2) and $R^H_{2k}(u)$ and $R^e_{2k}(v)$ are defined by (2.2) and (2.4) respectively with $\alpha = 2k$.*

Proof. The elementary solution of $\diamondsuit^k$ is the solution of the equation $\diamondsuit^k K(x) = \delta$ where δ is the Dirac-delta distribution, $K(x)$ is an unknown and $x \in R^n$. Now we need to prove that

$$K(x) = R^H_{2k}(u) * (-1)^k R^e_{2k}(v).$$

To prove this , see ([4], p. 33).

Lemma 2.2. (i) *The function $K(x) = S_2(w)$ is the solution of the equation $L_1 K(x) = 0$ where L_1 is defined by (1.3) and $S_2(w)$ is defined by (2.7) with $\alpha = 2$.*

(ii) *The function $K(x) = (-1)^k(-i)^{\frac{q}{2}} S_{2k}(w)$ is an elementary solution of the operator L_1^k, where L_1^k is the operator iterated k times defined by (1.3) and $S_{2k}(w)$ is defined by (2.7) with $\alpha = 2k$.*

Proof. (i) Now $L_1 = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$.

We need to show that $L_1 S_2(w) = 0$. Now if α is real, we have for $1 \le r \le p$

$$\frac{\partial}{\partial x_r} S_\alpha(w) = \frac{\partial}{\partial x_r}\left(\frac{w^{\frac{\alpha-n}{2}}}{H_n(\alpha)}\right) = \frac{(\alpha-n)}{2}\frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)} 2x_r = (\alpha-n)x_r \frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)},$$

$$\frac{\partial^2}{\partial x_r^2}S_\alpha(w) = (\alpha-n)\frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)} + \frac{\alpha-n}{H_n(\alpha)}(\alpha-n-2)w^{\frac{\alpha-n-4}{2}}x_r^2.$$

Thus

$$\sum_{r=1}^{p}\frac{\partial^2}{\partial x_r^2}S_\alpha(w) = p\frac{\alpha-n}{H_n(\alpha)}w^{\frac{\alpha-n-2}{2}} + \frac{\alpha-n}{H_n(\alpha)}(\alpha-n-2)w^{\frac{\alpha-n-4}{2}}\sum_{r=1}^{p}x_r^2.$$

Similarly

$$i\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}S_\alpha(w) = \frac{q(\alpha-n)}{H_n(\alpha)}w^{\frac{\alpha-n-2}{2}} - i\frac{\alpha-n}{H_n(\alpha)}(\alpha-n-2)w^{\frac{\alpha-n-4}{2}}\sum_{j=p+1}^{p+q}x_j^2.$$

Thus

$$L_1S_\alpha(w) = \frac{(p+q)}{H_n(\alpha)}(\alpha-n)w^{\frac{\alpha-n-2}{2}} + \frac{(\alpha-n)(\alpha-n-2)}{H_n(\alpha)}w^{\frac{\alpha-n-4}{2}}\left(\sum_{i=1}^{p}x_i^2 - i\sum_{j=p+1}^{q+p}x_j^2\right) =$$

$$= \frac{n(\alpha-n)}{H_n(\alpha)}w^{\frac{\alpha-n-2}{2}} + \frac{(\alpha-n)(\alpha-n-2)}{H_n(\alpha)}w^{\frac{\alpha-n-2}{2}} = (\alpha-2)(\alpha-n)\frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)}. \tag{2.10}$$

For $\alpha = 2$, we have $L_1S_2 = 0$. That is $K(x) = S_2(w)$ is a solution of the homogeneous equation $L_1K(x) = 0$.

(ii) To show that $K(x) = (-1)^k(-i)^{\frac{q}{2}}S_{2k}(w)$ is an elelmentary solution of L_1^k, that is $L_1^k(-1)^k(-i)^{\frac{q}{2}}S_{2k}(w) = \delta$. At first we need to show that $L_1^k(-1)^kS_\alpha(w) = S_{\alpha-2k}(w)$ and $S_{-2k}(w) = (-1)^k(i)^{\frac{q}{2}}L_1^k\delta$.

Now, from (2.10) and (2.5)

$$L_1S_\alpha(w) = (\alpha-2)(\alpha-n)\frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha)} = \frac{(\alpha-2)(\alpha-n)w^{\frac{\alpha-n-2}{2}}}{\pi^{\frac{n}{2}}\frac{2^\alpha\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}}.$$

By direct calculation with the property of Gamma function we obtain

$$L_1S_\alpha(w) = -\frac{w^{\frac{\alpha-n-2}{2}}}{\pi^{\frac{n}{2}}\cdot 2^{\alpha-2}\frac{\Gamma(\frac{\alpha-2}{2})}{\Gamma(\frac{n-(\alpha-2)}{2})}} = -\frac{w^{\frac{\alpha-n-2}{2}}}{H_n(\alpha-2)} = -S_{\alpha-2}(w).$$

By keeping on operating the operator L_1 k-times to the function $S_\alpha(w)$, we obtain

$$L_1^kS_\alpha(w) = (-1)^kS_{\alpha-2k}(w)$$

or

$$L_1^k(-1)^kS_\alpha(w) = S_{\alpha-2k}(w). \tag{2.11}$$

Then we show that $S_{-2k} = (-1)^k(i)^{\frac{q}{2}}\delta$.

Now

$$S_{-2k}(w) = \lim_{\alpha\to-2k}S_\alpha(w) = \lim_{\alpha\to-2k}\left[\frac{w^{\frac{\alpha-n}{2}}}{H_n(\alpha)}\right] = \frac{\lim_{\alpha\to-2k}\left[w^{\frac{\alpha-n}{2}}\right]}{\lim_{\alpha\to-2k}\left[\Gamma(\frac{\alpha}{2})\right]}\cdot\pi^{\frac{-n}{2}}\cdot\lim_{\alpha\to-2k}2^{-\alpha}\Gamma\left(\frac{n-\alpha}{2}\right). \tag{2.12}$$

Now consider $\lim\limits_{\alpha\to-2k}[w^{\frac{\alpha-n}{2}}]$. We have $w = x_1^2+x_2^2+...+x_p^2-i(x_{p+1}^2+x_{p+2}^2+...+x_{p+q}^2)$. By changing the variables, let $x_1 = y_1$, $x_2 = y_2$, ..., $x_p = y_p$ and $x_{p+1} = \frac{y_{p+1}}{\sqrt{-i}}, x_{p+2} = \frac{y_{p+2}}{\sqrt{-i}}, ..., x_{p+q} = \frac{y_{p+q}}{\sqrt{-i}}$.

Thus $w = y_1^2 + y_2^2 + ... + y_p^2 + y_{p+1}^2 + ... + y_{p+q}^2$, where $y_i (i = 1, 2, ..., n)$ is real and $p+q = n$.

Let $r^2 = w = y_1^2 + y_2^2 + ... + y_n^2$ and consider the distribution w^λ, where λ is a complex parameter. Since $< w^\lambda, Q >= \int\limits_Q w^\lambda Q(x)dx$, where $Q(x)$ is the element of the space D of the infinitely differentiable functions with compact supports and $x \in R^n$, $dx = dx_1dx_2...dx_n$. Thus

$$< w^\lambda, Q >= \int\limits_{R^n} r^{2\lambda}\frac{\partial(x_1, x_2, ..., x_n)}{\partial(y_1, y_2, ..., y_n)} \cdot Qdy_1dy_2...dy_n =$$

$$= \frac{1}{(-i)^{\frac{q}{2}}}\int\limits_{R^n} r^{2\lambda}Qdy_1dy_2...dy_n = \frac{1}{(-i)^{\frac{q}{2}}} < r^{2\lambda}, Q > .$$

Now, by Gelfand and Shilov (see [1], p. 271), $< w^\lambda, Q >$ hase simple poles at $\lambda = \frac{-n}{2} - k$ and for $k = 0$ the residue of $r^{2\lambda}$ at $\lambda = \frac{-n}{2}$ is given by $\operatorname*{res}\limits_{\lambda=\frac{-n}{2}} r^{2\lambda} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\delta(x)$.

Thus

$$\operatorname*{res}_{\lambda=\frac{-n}{2}} < w^\lambda, Q >= \frac{1}{(-i)^{\frac{q}{2}}}\operatorname*{res}_{\lambda=\frac{-n}{2}} < r^{2\lambda}, Q >= \frac{1}{(-i)^{\frac{q}{2}}}\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} < \delta(x), Q >$$

or

$$\operatorname*{res}_{\lambda=\frac{-n}{2}} w^\lambda = \frac{1}{(-i)^{\frac{q}{2}}}\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\delta(x). \tag{2.13}$$

Now we find $\operatorname*{res}\limits_{\lambda=\frac{-n}{2}-k} w^\lambda$ for k is nonnegative integer by, Gelfand and Shilov (see [1], p. 272) we have

$$w^\lambda = \frac{1}{4^k(\lambda+1)(\lambda+2)...(\lambda+k)(\lambda+\frac{n}{2})(\lambda+\frac{n}{2}+1)...(\lambda+\frac{n}{2}+k-1)}L_1^k w^{\lambda+k}.$$

Thus

$$\operatorname*{res}_{\lambda=\frac{-n}{2}-k} w^\lambda = \operatorname*{res}_{\lambda=\frac{-n}{2}} L_1 w^\lambda \cdot \frac{1}{4^k(\lambda+1)...(\lambda+k)(\lambda+\frac{n}{2})...(\lambda+\frac{n}{2}+k-1)}\Bigg|_{\lambda=\frac{-n}{2}-k}$$

by (2.12) we have

$$\operatorname*{res}_{\lambda=\frac{-n}{2}-k} w^\lambda = \frac{1}{(-i)^{\frac{q}{2}}}\frac{2\pi^{\frac{n}{2}}}{4^k k!\Gamma(\frac{n}{2}+k)}L_1^k\delta(x). \tag{2.14}$$

Thus

$$\lim_{\alpha\to-2k}[w^{\frac{\alpha-n}{2}}] = \lim_{\lambda\to\frac{-n}{2}-k} w^\lambda.$$

Now from (2.12), we have

$$S_{-2k}(w) = \frac{\lim\limits_{\alpha\to-2k}(\alpha+2k)w^{\frac{\alpha-n}{2}}}{\lim\limits_{\alpha\to-2k}(\alpha+2k)\Gamma\left(\frac{\alpha}{2}\right)}\pi^{\frac{-n}{2}}2^{2k}\Gamma\left(\frac{n}{2}+k\right) = \frac{\operatorname*{res}\limits_{\alpha=-2k} w^{\frac{\alpha-n}{2}}}{\operatorname*{res}\limits_{\alpha=-2k}\Gamma\left(\frac{\alpha}{2}\right)}\pi^{\frac{-n}{2}}4^k\Gamma\left(\frac{n}{2}+k\right).$$

Now

$$\operatorname*{res}_{\alpha=-2k}\Gamma\left(\frac{\alpha}{2}\right)=\frac{2(-1)^k}{k!}.$$

Thus by (2.14), we obtain

$$S_{-2k}(w)=\frac{(-1)^k}{(-i)^{\frac{q}{2}}}\frac{2\pi^{\frac{n}{2}}\pi^{\frac{-n}{2}}k!4^k\Gamma\left(\frac{n}{2}+k\right)}{2\cdot 4^k k!\Gamma\left(\frac{n}{2}+k\right)}L_1^k\delta(x)=$$

$$=\frac{(-1)^k}{(-i)^{\frac{q}{2}}}L_1^k\delta(x)=(-1)^k(i)^{\frac{q}{2}}L_1^k\delta(x).$$

Thus

$$S_0(w)=(i)^{\frac{q}{2}}\delta(x). \tag{2.15}$$

From (2.11) and (2.15), we obtain

$$L_1^k(-1)^kS_{2k}(w)=S_{2k-2k}(w)=S_0(w)=(i)^{\frac{q}{2}}\delta(x)$$

or

$$L_1^k(-1)^k(-i)^{\frac{q}{2}}S_{2k}(w)=\delta.$$

It follows that $K(x)=(-1)^k(-i)^{\frac{q}{2}}S_{2k}(w)$ is an elementary solution of the operator L_1^k. Similary $K(x)=(-1)^k(i)^{\frac{q}{2}}T_{2k}(z)$ is an elementary solution of the operator L_2^k where z is defined by (2.9) and T_{2k} is defined by (2.8) with $\alpha=2k$.

3. Main results

Theorem 3.1. *Given the equation*

$$\oplus^k K(x)=\delta \tag{3.1}$$

where $\oplus^k$ is the operator iterated k-times defined by (1.1), δ is the Dirac-delta distribution, $x=(x_1,x_2,...,x_n)\in R^n$ and k is a nonnegative integer. Then the convolution

$$K(x)=R_{2k}^H(u)*(-1)^kR_{2k}^e(v)*(-1)^k(-i)^{\frac{q}{2}}S_{2k}(w)*(-1)^k(i)^{\frac{q}{2}}T_{2k}(z) \tag{3.2}$$

is an elementary solution or the Green function of the equation (3.1) where $R_{2k}^H(u), R_{2k}^e(v), S_{2k}(w)$ and $T_{2k}(z)$ are defined by (2.2), (2.4), (2.7) and (2.8) respectively with $\alpha=2k$.

Proof. By (1.5) the equation (3.1) can be written as

$$\oplus^k K(x)=\diamondsuit^k L_1^k L_2^k K(x)=\delta. \tag{3.3}$$

Since the function $R_{2k}^H(u), R_{2k}^e(v), S_{2k}(w)$ and $T_{2k}(z)$ are tempered distributions (see [5], p. 34, Lemma 2.1) and the convolution of functions in (3.2) exists and is a tempered distribution (see [5], p. 35, Lemma 2.2 and [2], pp. 156–159). Now convolving both sides of (3.3) by $R_{2k}^H(u)*(-1)^kR_{2k}^e(v)*(-1)^k(-i)^{\frac{q}{2}}S_{2k}(w)*(-1)^k(i)^{\frac{q}{2}}T_{2k}(z)$ we obtain

$$\diamondsuit^k[R_{2k}^H(u)*(-1)^kR_{2k}^e(v)]*L_1^k[(-1)^k(-i)^{\frac{q}{2}}S_{2k}(w)]*L_2^k[(-1)^k(i)^{\frac{q}{2}}T_{2k}(z)]*K(x)=$$

$$=[R_{2k}^H(u)*(-1)^kR_{2k}^e(v)*(-1)^k(-i)^{\frac{q}{2}}S_{2k}(w)*(-1)^k(i)^{\frac{q}{2}}T_{2k}(z)]*\delta.$$

By Lemma 2.1 and Lemma 2.2 (*ii*), we obtain (3.2) as required, we call the solution $K(x)$ in (3.2) the Green function of the operator $\oplus^k$ we denote the Green function

$$G(x) = R_{2k}^H(u) * (-1)^k R_{2k}^e(v) * (-1)^k (-i)^{\frac{q}{2}} S_{2k}(w) * (-1)^k (i)^{\frac{q}{2}} T_{2k}(z). \tag{3.4}$$

Theorem 3.2. *Given the equation*

$$\oplus^k K(x) = f(x) \tag{3.5}$$

where $\oplus^k$ *is defined by (1.1) and* $f(x)$ *is a generalized function, then* $K(x) = G(x) * f(x)$ *is a weak solution for (3.5) where* $G(x)$ *is a Green function of* $\oplus^k$ *defined by (3.4).*

Proof. Convolving both sides of (3.5) by $G(x)$ defined by (3.4) we obtain

$$G(x) * \oplus^k K(x) = G(x) * f(x)$$

or

$$\oplus^k G(x) * K(x) = G(x) * f(x).$$

By Theorem 3.1, we have

$$\delta * K(x) = G(x) * f(x)$$

or

$$K(x) = G(x) * f(x)$$

as required.

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