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STRUCTURAL STABILITY IN GENERALIZED SEMI-INFINITE OPTIMIZATION*

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Обсуждаются свойства многообразий и вопросы непрерывости допустимых ограничений $M[h, g, u, v]$, а также соответствующее поведение (f, h, g, u, v) при слабых возмущениях. Формулируются теоремы о многообразиях, непрерывности, универсальности, устойчивости и структурной устойчивости. Кратко описываются возможные расширения на случаи неограниченности и недифференцируемости, указываются такие структурные границы, при которых полученные результаты могут трактоваться в терминах задач оптимального управления для обыкновенных дифференциальных уравнений.

1. Introduction

Under suitable assumptions, the following fields of problems from science, engineering and control lead to generalized semi-infinite ($\mathcal{GSI}$) optimization: ∘ optimizing the layout of a special assembly line, ∘ maneuverability of a robot, ∘ time minimal heating or cooling of a ball of some homogeneous material, ∘ approximation of a thermo-couple characteristic in chemical engineering, ∘ structure and stability in optimal control of ordinary differential equations. For motivations and references see, e.g., [59, 60]. In future, $\mathcal{GSI}$ applications may also be expected in optimal experimental design ([9]). The $\mathcal{GSI}$ problems under consideration have the form

$$\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v) \qquad \begin{cases} \text{Minimize} \quad f(x) \ \text{on} \ M_{\mathcal{GSI}}[h,g], \quad \text{where} \\[2ex] M_{\mathcal{GSI}}[h,g] \ := \ \{\, x \in I\!R^n \mid \ h_i(x) = 0 \ \ (i \in I), \\ \qquad\qquad\qquad\qquad g(x,y) \geq 0 \ \ (y \in Y(x))\,\}. \end{cases}$$

The semi-*infinite* character comes from the perhaps infinite number of elements of $Y(= Y(x))$ [10, 45], while the *generalized* character comes from the x-dependence of $Y(\cdot)$. We suppose these index sets to be finitely constrained ($\mathcal{F}$):

$$Y(x) \ = \ M_{\mathcal{F}}[u(x,\cdot), v(x,\cdot)] := \{\, y \in I\!R^q \mid u_k(x,y) = 0 \ (k \in K), \ v_\ell(x,y) \geq 0 \ (\ell \in L)\}(x \in I\!R^n).$$

Let $h = (h_i)_{i \in I}$, $u = (u_k)_{k \in K}$, $v = (v_\ell)_{\ell \in L}$, where $h_i : I\!R^n \to I\!R$, $i \in I := \{1, \ldots, m\}$, $u_k : I\!R^n \times I\!R^q \to I\!R$, $k \in K := \{1, \ldots, r\}$, $v_\ell : I\!R^n \times I\!R^q \to I\!R$, $\ell \in L := \{1, \ldots, s\}$ $(m < n;\ r < q)$.

We assume that $f : I\!R^n \to I\!R$, $g : I\!R^n \times I\!R^q \to I\!R$, $h_i\,(i \in I)$, $u_k\,(k \in K)$, $v_\ell\,(\ell \in L)$ are once continuously differentiable (C^1). By $Df(x)$, $D^T f(x)$ we denote the row- (column) vector of the first-order partial derivatives $\frac{\partial}{\partial x_\kappa} f(x)$, and $D_x g(x,y)$, $D_y g(x,y)$ consist of $\frac{\partial}{\partial x_\kappa} g(x,y)$ and $\frac{\partial}{\partial y_\sigma} g(x,y)$. Let a given set $\mathcal{U}^0 \subset I\!R^n$, $M_{\mathcal{GSI}}[h,g] \cap \mathcal{U}^0 \neq \emptyset$, be *bounded* and *open.*

Assumption A$_{\mathcal{U}^0}$: $\cup_{x \in \overline{\mathcal{U}^0}} Y(x)$ is *bounded (hence, by continuity, compact).*

In *generalized* semi-infinite optimization, the feasible set $M_{\mathcal{GSI}}[h,g]$ need not be closed [24]. The following assumption, however, ensures closedness:

Assumption B$_{\mathcal{U}^0}$: *For all $x \in \overline{\mathcal{U}^0}$, the linear independence constraint qualification (LICQ) is fulfilled for $M_{\mathcal{F}}[u(x,\cdot), v(x,\cdot)]$, i.e., linear independence of*

$$D_y u_k(\overline{x},\overline{y}),\ k \in K, \quad D_y v_\ell(\overline{x},\overline{y}),\ \ell \in L_0(\overline{x},\overline{y})$$

(considered as a family), where $L_0(\overline{x},\overline{y}) := \{\, \ell \in L \mid v_\ell(\overline{x},\overline{y}) = 0 \,\}$ consists of active indices.

Using differential topology [17, 20], these assumptions admit local linearization of $Y(x)$ ($x \in \overline{\mathcal{U}^0}$) by finitely many C^1-diffeomorphisms $\phi_x^j : \mathcal{V}^j \to S^j$ ($j \in J$) in such a way that the image sets Z^j of indices are *x-independent* squares (in a linear subspace). Herewith, $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ becomes locally (in $\overline{\mathcal{U}^0}$) equivalently expressed as an **ordinary** *semi-infinite* optimization problem $\mathcal{P}_{\mathcal{OSI}}(f,h,g^0,u^0,v^0)$, where $M_{\mathcal{OSI}}[h,g^0] \cap \overline{\mathcal{U}^0} = M_{\mathcal{GSI}}[h,g] \cap \overline{\mathcal{U}^0}$, f being unaffected [57, 59].

On the upper stage of variable x, we shall use a constraint qualification, too. This *cq* geometrically means the existence of an (at $M[h] = h^{-1}(\{0\})$) tangential, "inwardly" pointing direction at x:

Definition. We say that the **extended Mangasarian-Fromovitz constraint qualification (EMFCQ)** is fulfilled at a given $\overline{x} \in M_{\mathcal{GSI}}[h,g]$, if the conditions $EMF_{1,2}$ are satisfied:

EMF$_1$. $Dh_i(\overline{x})$, $i \in I$, are linearly independent.

EMF$_2$. There exists an "*EMF-vector*" $\zeta \in I\!R^n$ such that

$$Dh_i(\overline{x})\,\zeta \,=\, 0 \quad \text{for all } i \in I,$$

$$D_x g_j^0(\overline{x},z)\,\zeta \,>\, 0 \quad \text{for all } z \in I\!R^q,\ j \in J,\ \text{with}\ \ (\phi_{\overline{x}}^j)^{-1}(z) \in Y_0(\overline{x}),$$

where $Y_0(\overline{x}) := \{\, y \in Y(\overline{x}) \mid g(\overline{x},y) = 0 \,\}$ consists of *active* indices. **EMFCQ** is said to be fulfilled for $M_{\mathcal{GSI}}[h,g]$ on $\overline{\mathcal{U}^0}$, if EMFCQ is fulfilled for all $x \in M_{\mathcal{GSI}}[h,g] \cap \overline{\mathcal{U}^0}$.

For further information and versions of EMFCQ see [15, 20, 24, 26, 40], but also [7, 18].

Let a local minimum $\hat{x}$ of $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ be given and EMFCQ be fulfilled there. Then, we can state the existence of *Lagrange multipliers* λ_i, μ_κ such that the conditions

$$Df(\hat{x}) \,=\, \sum_{i \in I} \lambda_i Dh_i(\hat{x}) \,+\, \sum_{\kappa \in \{1,\ldots,\hat{\kappa}\}} \mu_\kappa D_x g^0_{j^\kappa}(\hat{x}, z^\kappa),$$

$$\mu_\kappa \,\geq\, 0 \ \ (\kappa \in \{1,\ldots,\hat{\kappa}\})$$

are satisfied, referring to *ordinary* semi-infinite ($\mathcal{OSI}$) data [15, 57, 59]. Now, we call $\hat{x}$ a $\mathcal{G}$-$\mathcal{O}$ *Kuhn-Tucker point.* Here, the points $z^\kappa \in Z^{j^\kappa}$ are suitable active indices. Referring to *all* the given $\mathcal{GSI}$ data now, a further evaluation yields the following **Kuhn-Tucker conditions** with corresponding *Lagrange multipliers* λ_i, μ_κ, $\alpha_{\kappa,k}$, $\beta_{\kappa,\ell}$ [57, 59]:

KT$_1$. $Df(\hat{x}) = \sum_{i\in I} \lambda_i Dh_i(\hat{x}) + \sum_{\kappa\in\{1,\ldots,\hat{\kappa}\}} \mu_\kappa D_x g(\hat{x}, y^\kappa) -$

$$- \sum_{k\in K} \alpha_{\kappa,k} Du_k(\hat{x}, y^\kappa) - \sum_{\substack{\ell\in L_0(\hat{x},y^\kappa) \\ \kappa\in\{1,\ldots,\hat{\kappa}\}}} \beta_{\kappa,\ell} D_x v_\ell(\hat{x}, y^\kappa),$$

KT$_2$. $\mu_\kappa,\ \beta_{\kappa,\ell} \geq 0 \ \ (\ell \in L_0(\hat{x}, y^\kappa),\ \kappa \in \{1,\ldots,\hat{\kappa}\})$.

Again, the $y^\kappa \in Y_0(\hat{x})$ are active. Now, we call $\hat{x}$ a $\mathcal{G}$ *Kuhn-Tucker point.* Under general assumptions, the **necessary optimality condition** $\mathrm{KT}_{1,2}$ was for the first time proved by Jongen, Rückmann and Stein [24]. Note, that the linear combination KT_1 contains the derivatives of *all* the defining functions. The foregoing conditions can also be stated as growth (angular) conditions over *tangent cones* [32, 57, 59]. These growth conditions estimate scalar products against 0; they give rise to deduce first-order **sufficient** *optimality conditions.* In fact, let LICQ be satisfied at a given point $\hat{x}$ as an element of $M[h]$, and $M[h]\cap\overline{\mathcal{U}^0}$ be *star-shaped* with *star point* $\hat{x}$. Moreover, let the functions $g_j^0(\cdot, z)$ $(z \in Z^j,\ j \in J)$ be *quasi-concave* and f be *pseudo-convex* on $M[h]\cap\overline{\mathcal{U}^0}$. This means the following implications for all $x \in M[h]\cap\overline{\mathcal{U}^0}$ [16, 32]:

$$g_j^0(x,z) \geq g_j^0(\hat{x},z) \quad\Longrightarrow\quad D_x g_j^0(\hat{x},z)\,(x-\hat{x}) \geq 0,$$
$$Df(\hat{x})\,(x-\hat{x}) \geq 0 \quad\Longrightarrow\quad f(x) \geq f(\hat{x})\,.$$

Then, $\hat{x}$ turns out to be a local minimizer of $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ [57, 59]; cf. [29]. Concerning structural frontiers in ($\mathcal{F}$) *nonconvex* optimization see [28]. Before we introduce the second-order condition of *strong stability* we state under our basic Assumptions $\mathrm{A}_{\mathcal{U}^0}$, $\mathrm{B}_{\mathcal{U}^0}$:

Lemma [59]. Let $\hat{x} \in M_{\mathcal{GSI}}[h,g]\cap\overline{\mathcal{U}^0}$ be given, and EMFCQ be fulfilled at $\hat{x}$. Then, $\hat{x}$ is a $\mathcal{G}$-$\mathcal{O}$ *Kuhn-Tucker point* for $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$, if and only if the extended Mangasarian-Fromovitz constraint qualification on $M_{\mathcal{GSI}}[h,(g,-f+f(\hat{x}))]$, called $\widehat{EMFCQ}$, is violated at $\hat{x}$.

Proof: This result results from Farkas' Lemma for infinite systems [15, 53, 59]. ∎

We prepare our introduction of strong stability of a stationary point by assuming that f,h,g,u,v are C^2 and putting for any bounded open neighbourhood $\mathcal{V} \subseteq I\!R^q$ of $\bigcup_{x\in\overline{\mathcal{U}^0}} Y(x)$ and any subset $\mathcal{M} \subseteq I\!R^n$:

$$\mathrm{norm}_{\mathcal{GSI}}[(f,h,g,u,v),\mathcal{M}] \;:=$$
$$\sup\left\{ \sup_{x\in\mathcal{M}} \max_{\substack{\gamma\in\{f\}\cup \\ \{h_\nu|\nu\in I\}}} \left\{ |\gamma(x)| + \sum_{i=1}^n |\tfrac{\partial\gamma}{\partial x_i}(x)| + \sum_{\substack{i=1\\j=1}}^n |\tfrac{\partial^2\gamma}{\partial x_i\partial x_j}(x)| \right\},\right.$$
$$\sup_{\substack{x\in\mathcal{M}\\ y\in\overline{\mathcal{V}}}} \max_{\substack{\eta\in\{g\}\cup\\ \{u_\nu|\nu\in K\}\cup\\ \{v_\nu|\nu\in L\}}} \left\{ |\eta(x)| + \sum_{i=1}^n |\tfrac{\partial\eta}{\partial x_i}(x,y)| + \sum_{j=1}^q |\tfrac{\partial\eta}{\partial y_j}(x,y)| + \sum_{\substack{i=1\\j=1}}^n |\tfrac{\partial^2\eta}{\partial x_i\partial x_j}(x)| + \right.$$
$$\left.\left. + \sum_{i=1}^n \sum_{j=1}^q |\tfrac{\partial^2\eta}{\partial x_i\partial y_j}(x)| + \sum_{\substack{i=1\\j=1}}^q |\tfrac{\partial^2\eta}{\partial y_i\partial y_j}(x)| \right\}\right\}.$$

In cases of $\mathcal{F}$ or $\mathcal{OSI}$ optimization, replacing $\overline{\mathcal{V}}$ by J, Y or disregarding u,v, we denote by $\mathrm{norm}_{\mathcal{F}}[\cdot,\cdot]$, $\mathrm{norm}_{\mathcal{OSI}}[\cdot,\cdot]$. Because of continuity properties stated in Section 2, the next condition is well-defined [59].

Definition. Suppose a feasible point $\hat{x}^u \in M_{\mathcal{GSI}}[h,g] \cap \mathcal{U}^0$ for $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ (of class C^2). Now, $\mathcal{P}_{\mathcal{OSI}}(f,h,g^0,u^0,v^0)$ be locally (in $\overline{\mathcal{U}^0}$) representing $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$, and $\hat{x}^u$ be a $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point of $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$. Then, we say that $\hat{x}^u$ is ($\mathcal{G}$-$\mathcal{O}$) **strongly stable**, if for some $\overline{\epsilon} > 0$ with $B(\hat{x}^u, \overline{\epsilon}) \subseteq \mathcal{U}^0$ and for each $\epsilon \in (0, \overline{\epsilon}]$ there is some $\delta > 0$ such that for each C^2-function $(\tilde{f}, \tilde{h}, \widetilde{g^0})$ with $\mathrm{norm}_{\mathcal{OSI}}[(f-\tilde{f}, h-\tilde{h}, g^0-\widetilde{g^0}), B(\hat{x}^u,\epsilon)] \leq \delta$ the open ball $B(\hat{x}^u,\epsilon)$ contains an *ordinary Kuhn-Tucker point* $\hat{x}^d$ of $\mathcal{P}^*_{\mathcal{OSI}}(\tilde{f},\tilde{h},\widetilde{g^0}) := \mathcal{P}_{\mathcal{OSI}}(\tilde{f},\tilde{h},\widetilde{g^0},u^0,v^0)$, which is unique in $B(\hat{x}^u,\overline{\epsilon})$. Referring to a $\mathcal{G}$ Kuhn-Tucker point $\hat{x}^u$ and to $\mathrm{norm}_{\mathcal{GSI}}[(f - \tilde{f}, h-\tilde{h}, g-\tilde{g}, u-\tilde{u}, v-\tilde{v}), B(\hat{x}^u,\epsilon)]$, we get the condition of ($\mathcal{G}$) **strong stability** of $\hat{x}^u$.

Here, "*u*" (and "*d*") indicates *(un)disturbed*. For our preferred ($\mathcal{G}$-$\mathcal{O}$) strong stability expressed by original $\mathcal{GSI}$ data, see [59]. In Section 3, we utilize an algebraical characterization of strong stability in the tradition of Kojima [30] and Rückmann [48].

2. Stability of the Feasible Set.

The following theorems underline the importance of *EMFCQ* for concluding that $M_{\mathcal{GSI}}[h,g,u,v] := M_{\mathcal{GSI}}[h,g]$ is a topological manifold with boundary, it behaves continuous and stable under perturbations of our defining C^1-functions. With these perturbations we remain inside of suitable open neighbourhoods of (h,g,u,v) in the sense of the **strong** or **Whitney topology** C^1_S that takes into account asymptotic effects (for topologies C^k_S, $k \in \mathbb{N} \cup \{\infty\}$, cf. [17, 20]). We call a given subset $M \subseteq \mathbb{R}^n$ a **Lipschitzian manifold** (with boundary) of dimension κ, if for each $\overline{x} \in M$ there are open neighbourhoods $\mathcal{W}^1 \subseteq \mathbb{R}^n$ of $\overline{x}$, $\mathcal{W}^2 \subseteq \mathbb{R}^n$ of 0_n, and a bijective "chart" $\varphi : \mathcal{W}^1 \to \mathcal{W}^2$, $\varphi(\overline{x}) = 0_n$, with Lipschitzian continuity of φ, φ^{-1} such that φ carries $M \cap \mathcal{W}^1$ to the relatively open set $(\{0_{n-\kappa}\} \times \mathbb{R}^\kappa) \cap \mathcal{W}^2$ or to the set $(\{0_{n-\kappa}\} \times \{w \in \mathbb{R} \,|\, w \geq 0\} \times \mathbb{R}^{\kappa-1}) \cap \mathcal{W}^2$ with (relative) boundary. So, Lipschitzian manifolds can locally be linearized, however, without preserving "angulars" in the boundary. In $\mathcal{F}$ optimization, that preservation is guaranteed by the stronger condition LICQ, using C^1-smooth linearizing charts. In this sense, we find qualified versions of the following topological results for $Y(x)$, [19, 59].

Manifold Theorem [59]. *Let EMFCQ be fulfilled in* $\overline{\mathcal{U}^0}$ *for* $M_{\mathcal{GSI}}[h,g]$. *Then, there is an open neighbourhood* $\mathcal{W} \subseteq \mathbb{R}^n$ *of* $\overline{\mathcal{U}^0}$ *such that* $M_{\mathcal{GSI}}[h,g] \cap \mathcal{W}$ *is a Lipschitzian manifold (with boundary) of the dimension* $n-m$. *Moreover, then we can represent the (relative) boundary:*

$$(\partial M_{\mathcal{GSI}}[h,g]) \cap \mathcal{W} \;=\; \{\, x \in \mathcal{W} \,|\, h_i(x) = 0 \ (i \in I),\ \min_{y \in Y(x)} g(x,y) = 0 \,\}.$$

Proof: Assumption $\mathrm{B}_{\mathcal{U}^0}$, delivers diffeomorphisms ϕ^j_x for all x of some open neighbourhood $\mathcal{W}$ of $\overline{\mathcal{U}^0}$. These transformations guarantee that the insight from [26] on $\mathcal{OSI}$ optimization can be carried over for our $\mathcal{GSI}$ problem.

For the properties of **upper** and **lower semi-continuity**, **continuity** (in Hausdorff-metric), **genericity** (implying density) and **transversality** (absense of tangentiality), considered for functions or sets next, we refer to [3, 17, 20, 26, 59].

Continuity Theorem [58, 59]. *Let EMFCQ be fulfilled in* $\overline{\mathcal{U}^0}$ *for* $M_{\mathcal{GSI}}[h,\, g]$. *Moreover, let the closure* $\overline{\mathcal{W}} \subseteq \mathbb{R}^n$ *of some open set* $\mathcal{W} \subseteq \mathcal{U}^0$ *be representable as a feasible set from* $\mathcal{F}$ *optimization which fulfills LICQ, and let the intersection of its boundary* $\partial\mathcal{W}$ *with* $M_{\mathcal{GSI}}[h,g]$ *be transversal. Then, there is an open* C^1_S*-neighbourhood* $\mathcal{O} \subseteq (C^1(\mathbb{R}^n,\mathbb{R}))^m \times C^1(\mathbb{R}^{n+q},\mathbb{R}) \times (C^1(\mathbb{R}^{n+q},\mathbb{R}))^r \times (C^1(\mathbb{R}^{n+q},\mathbb{R}))^s$ *of* (h,g,u,v) *such that* $\mathsf{M}^{\mathcal{W}} : (\tilde{h},\tilde{g},\tilde{u},\tilde{v}) \mapsto M_{\mathcal{GSI}}[\tilde{h},\tilde{g},\tilde{u},\tilde{v}] \cap \overline{\mathcal{W}}$, *is upper and lower semi-continuous at all* $(\tilde{h},\tilde{g},\tilde{u},\tilde{v}) \in \mathcal{O}$. *If, moreover,* $\mathcal{W}$ *is bounded, then* $\mathcal{O}$ *can be chosen so that* $\mathcal{O}$ *is mapped to* $\mathcal{P}_c(\mathbb{R}^n)$ *by* $\mathsf{M}^{\mathcal{W}}$, *and* $\mathsf{M}^{\mathcal{W}}$ *is continuous.*

Proof: These assertions are consequences of the *continuous* dependence of the $\mathcal{OSI}$ functional data g^0, u^0, v^0 on the $\mathcal{GSI}$ data g, u, v and, then, of [26], Theorem 2.2. We apply this theorem on $M_{\mathcal{OSI}}[h, g^0, u^0, v^0] := M_{\mathcal{OSI}}[h, g^0]$. In the proof of *Genericity Theorem* below, we investigate the continuous dependence $\Psi_R : (\tilde{h}, \tilde{g}, \tilde{u}, \tilde{v}) \mapsto (\tilde{h}, \tilde{g}^0, \tilde{u}^0, \tilde{v}^0)$.

In [59], also a global version and a version on $(\tilde{x}, \tilde{u}, \tilde{v}) \mapsto Y^{\tilde{u},\tilde{v}}(\tilde{x})$ are presented for the previous theorem. The following result refers to the straightforward generalization **ELICQ** of LICQ that is a stronger condition than EMFCQ [26, 53, 59]. (The double usage of $\mathcal{F}$ should not lead to any confusion. For a global result see [59].)

Genericity Theorem [59].

(a) *Let* $\mathsf{C}^\infty := (C^\infty(\mathbb{R}^n, \mathbb{R}))^m \times C^\infty(\mathbb{R}^n \times \mathbb{R}^q, \mathbb{R}) \times (C^\infty(\mathbb{R}^n \times \mathbb{R}^q, \mathbb{R}))^r \times (C^\infty(\mathbb{R}^n \times \mathbb{R}^q, \mathbb{R}))^s$ *be endowed with the* C^∞_S*-topology. Furthermore, let its subspace* $\mathsf{C}^\infty_{\mathsf{loc}}$ *of all* $(h, g, u, v) \in \mathsf{C}^\infty$ *with validity of Assumptions* $A_{\mathcal{U}^0}, B_{\mathcal{U}^0}$ *be endowed with the* C^∞_S*-relative topology. Then, there exists a generic subset* $\mathcal{E} \subseteq \mathsf{C}^\infty_{\mathsf{loc}}$ *such that ELICQ is satisfied for each* $(h, g, u, v) \in \mathcal{E}$.

(b) *Let* $\mathsf{C}^1 := (C^1(\mathbb{R}^n, \mathbb{R}))^m \times C^1(\mathbb{R}^n \times \mathbb{R}^q, \mathbb{R}) \times (C^1(\mathbb{R}^n \times \mathbb{R}^q, \mathbb{R}))^r \times (C^1(\mathbb{R}^n \times \mathbb{R}^q, \mathbb{R}))^s$ *be endowed with the* C^1_S*-topology. Furthermore, let its subspace* $\mathsf{C}^1_{\mathsf{loc}}$ *of all* $(h, g, u, v) \in \mathsf{C}^1$ *with validity of* $A_{\mathcal{U}^0}, B_{\mathcal{U}^0}$ *be endowed with the* C^1_S*-relative topology. Then, there exists an open and dense subset* $\mathcal{F} \subseteq \mathsf{C}^1_{\mathsf{loc}}$ *such that EMFCQ is satisfied for each* $(h, g, u, v) \in \mathcal{F}$. *The set* $\mathcal{F}$ *can just be defined by the fulfillment of EMFCQ.*

Outline of Proof: The first insight on the desired *subset* $\mathcal{E}$ of C^∞-functions follows from the $\mathcal{OSI}$ result [26], Theorem 2.4, that applies Multi-Jet Transversality Theorem [17, 20] and additional reflections. For that theorem our u^0, v^0 are kept *fixed*, focussing topological interest on (h, g^0) $(h = h^0)$; here, the part of some constant set $\mathcal{Y}$ is taken by the union of the sets Z^j $(j \in J)$. Without loss of generality, J consists of one single element. Now, we can state that there is a generic set $\mathcal{E}^{\mathcal{O}}$ of $\mathcal{OSI}$ data functions (h, g^0), which (by definition of genericity) is the intersection of countably many open and dense subsets $\mathcal{E}^{\mathcal{O},\nu}$ $(\nu \in \mathbb{N})$.

However, for the tracing back of the $\mathcal{OSI}$ genericity (or, below, openess and density) to $\mathcal{GSI}$ optimization, we utilize that the problem representation is *continuous*. In fact, by *Implicit Function Theorem in Banach Spaces* [20, 37], the inserted local coordinate transformations continuously depend on $(\tilde{h}, \tilde{g}, \tilde{u}, \tilde{v})$. Let us regard this continuous dependence (representation) as a function Ψ_R locally mapping $(\tilde{h}, \tilde{g}, \tilde{u}, \tilde{v}) \in \mathsf{C}^\infty$ into the space of all C^∞-functions $(\tilde{h}, \tilde{g}^0, \tilde{u}^0, \tilde{v}^0)$. With respect to $\tilde{h}$, the mapping Ψ_R is constant. Using Ψ_R we find $\mathcal{E}$ as the intersection of the countably many *open* sets $\mathcal{E}^\nu := \Psi_R^{-1}(\mathcal{E}^{\mathcal{O},\nu})$ $(\nu \in \mathbb{N})$.

Now, let us consider an element $(h, g, u, v) \in \mathsf{C}^\infty_{\mathsf{loc}}$. After sufficiently small perturbations it still remains in $\mathsf{C}^\infty_{\mathsf{loc}}$. Let also some $\nu \in \mathbb{N}$ be given. In the $\mathcal{OSI}$ problem, however, we consider *separate (de-coupled)* perturbations $g^0_j \to \widetilde{g^0_j}$ $(j \in J)$ (before we really turn to one single inequality j). Therefore, the "problem representation" Ψ_R is *not surjective*. Actually, as for some $x \in \overline{\mathcal{U}^0}$ and two (or more) different $j^1, j^2 \in J$ the sets $(\phi^{j^1}_x)^{-1}(Z^{j^2}_0(x))$, $(\phi^{j^2}_x)^{-1}(Z^{j^2}_0(x))$ might have a nonempty intersection, these perturbations cannot always be traced back to a perturbation $g \to \tilde{g}$ of the given $\mathcal{GSI}$ problem. The following perturbation technique, however, will be helpful to get rid with such a difficulty, and it will finally guide us to the asserted density.

By definition of ϕ^j_x $(j \in J)$ (linearization) the implicitly disturbed sets $\widetilde{Z^j}$ can be chosen as Z^j. Moreover, because of the locally finite covering structure underlying Ψ_R, no difficulty arises. In view of that locally "fix" u^0, v^0 and of the constant property of Ψ_R with respect to $\tilde{h}$, we delete $u^0, v^0, \tilde{h}$ in the definition of Ψ_R. So, we get a mapping called Ψ^*_R. First of all, we add to g one j-independent, arbitrarily C^∞_S-small positive function g in an arbitrarily small neighbourhood of the compact set $\bigcup_{x \in M_{\mathcal{GSI}}[h,g] \cap \overline{\mathcal{U}^0}} (\phi_x^{j^1})^{-1}(Z_0^{j^1}(x)) \cap (\phi_x^{j^2})^{-1}(Z_0^{j^2}(x))$, making active indices y inactive there. Then, $g^* := g + \mathsf{g}$ is a globally defined C^∞-function. Now, for each $\nu \in I\!N$ we find a (componentwise) arbitrarily C^∞_S-close approximation $(\widetilde{h^\nu}, \widetilde{g^{\nu 0}}, \widetilde{u^{\nu 0}}, \widetilde{v^{\nu 0}}) \in \mathcal{E}^{\mathcal{O},\nu}$ of (h^0, g^0, u^0, v^0), where the approximation $\widetilde{g^{\nu 0}}$ coincides with $g^{*0} := \Psi_R^*(g^*, u, v)$ in $\cup_{j \in J} Z^j$. Here, we may choose the C^1-function $(\widetilde{u^{\nu 0}}, \widetilde{v^{\nu 0}}) := (u^0, v^0)$. Hence, that perturbed function $\widetilde{g^{\nu 0}}$ can continuously be traced back under ${\Psi_R^*}^{-1}$ to one C^∞-function $\tilde{g}^\nu$, i.e., $\{(\tilde{g}^\nu, u, v)\} = {\Psi_R^*}^{-1}(\{\widetilde{g^{\nu 0}}\})$. So we are in a position to state, that (h, g, u, v) can arbitrarily well be C^∞_S-approximated by $(\tilde{h}, \tilde{g}, \tilde{u}, \tilde{v}) := (\widetilde{h^\nu}, \tilde{g}^\nu, u, v) \in \mathcal{E}^\nu$. This means that $\mathcal{E}^\nu$ is *dense*, too. Altogether, we have shown that $\mathcal{E}$ is **generic**.

Preparation: This (relative) genericity implies (relative) density [20], because of the "C^∞_S-*openess*" of both LICQ and (y-) boundedness. Now, we use the fact that EMFCQ follows from ELICQ, and the C^1_S-density of $C^\infty(I\!R^k, I\!R)$ in $C^1(I\!R^k, I\!R)$ $(k \in I\!N)$. Moreover, we take account of our preparation and of the perturbational "C^1_S-openess" of EMFCQ.

We underline "$\mathcal{F}$" or "$\mathcal{GSI}$ *open" properties*: LICQ and EMFCQ remain preserved under sufficiently slight data perturbation.

Next, we refer to the *same* underlying dimensions n, q in x- or y-space, and numbers r, s of functions u_k, v_ℓ. Two feasible sets $M_{\mathcal{GSI}}[h^1, g^1, u^1, v^1]$, $M_{\mathcal{GSI}}[h^2, g^2, u^2, v^2]$ are called **(topologically) equivalent**, notation: $M_{\mathcal{GSI}}[h^1, g^1, u^1, v^1] \sim_M M_{\mathcal{GSI}}[h^2, g^2, u^2, v^2]$, if there is a homeomorphism $\varphi_M \,(= \varphi_{M_{\mathcal{GSI}}}) : I\!R^n \to I\!R^n$ such that

$$\varphi_M(M_{\mathcal{GSI}}[h^1, g^1, u^1, v^1]) = M_{\mathcal{GSI}}[h^2, g^2, u^2, v^2].$$

The given feasible set $M_{\mathcal{GSI}}[h, g] \,(= M_{\mathcal{GSI}}[h, g, u, v])$ is called **(topologically) stable**, if there is an open C^1_S-neighbourhood $\mathcal{O}$ of (h, g, u, v) such that for each $(\tilde{h}, \tilde{g}, \tilde{u}, \tilde{v}) \in \mathcal{O}$ we have $M_{\mathcal{GSI}}[h, g, u, v] \sim_M M_{\mathcal{GSI}}[\tilde{h}, \tilde{g}, \tilde{u}, \tilde{v}]$ (see [12, 26, 53, 59]. Let us make the *boundedness* (hence, compactness) assumption that $M_{\mathcal{GSI}}[h, g]$ lies in $\mathcal{U}^0$.

Stability Theorem [58, 59]. *The feasible set $M_{\mathcal{GSI}}[h, g] \subset \mathcal{U}^0$ is topologically stable, if and only if EMFCQ is fulfilled for $M_{\mathcal{GSI}}[h, g]$.*

Proof: We trace back to the $\mathcal{OSI}$ situation again, given by [26], Theorem 2.3, now. As being the case in the proof of *Genericity Theorem*, technicalities arise. Moreover, in [26] the equality constraint functions h are assumed to be C^2. All these difficulties can be governed: In Section 3 we prove *Characterization Theorem* on the lower level sets of the whole $\mathcal{GSI}$ optimization problem; that theorem implies our *Stability Theorem*. We note that under our overall boundedness assumptions, $M_{\mathcal{GSI}}[h, g]$ is a lower level set of $\mathcal{P}_{\mathcal{GSI}}(f, h, g, u, v)$ for a sufficiently high f-level. Already to point out the essential ideas for the **sufficiency part**, "$\Longleftarrow$", proved in a *constructive* way, and for the **necessity part**, "$\Longrightarrow$", proved in an *indirect* way, we look at Figures 1, 2, respectively. For both parts differential topology and Morse theory are helpful. While for the *necessity part* some algebraic topology [19, 51] is essential to evaluate unstable situations, for the *sufficiency part* flows [1] are important. To construct a homeomorphism φ_M, we first of all C^1-transform (in $\mathcal{U}$) the sets $M[h], M[\tilde{h}]$ to some manifold $M[\hat{h}]$. Here $\hat{h}$ is of class C^2 (or C^∞) [47, 53]. Now, we may suppose $I = \emptyset$. Finally, we homeomorphically map the feasible set $M_{\mathcal{GSI}}[g]$ onto the feasible set $M_{\mathcal{GSI}}[\tilde{g}]$ by steering the boundary $\partial M_{\mathcal{GSI}}[g]$ onto $\partial M_{\mathcal{GSI}}[\tilde{g}]$ along an EMF-vector field.

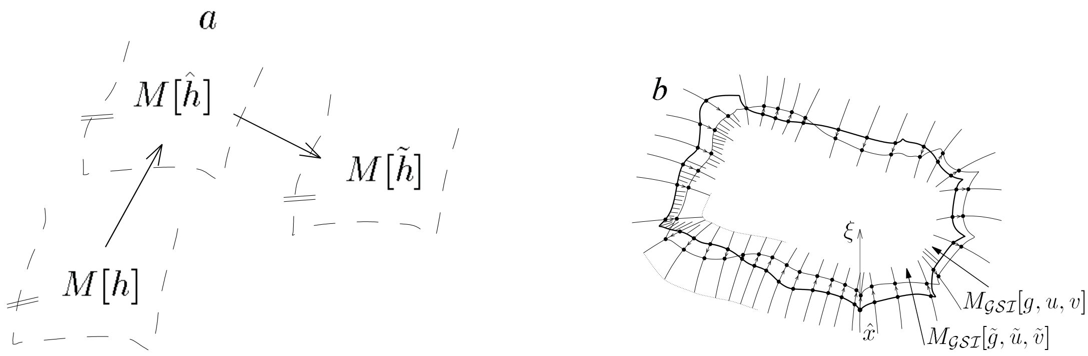

Fig. 1. Proof of sufficiency part, Stability Theorem

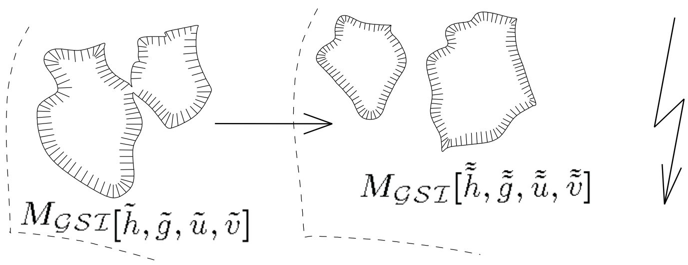

Fig. 2. Proof of necessity part, Stability Theorem

3. Structural Stability and its Characterization.

3.1. Structural Stability of the Problem.

Under Assumptions $\mathrm{A}_{\mathcal{U}^0}$, $\mathrm{B}_{\mathcal{U}^0}$, we still refer to the bounded set $M_{\mathcal{GSI}}[h,g]$, but additionally take f into consideration. We establish the structure of the entire problem $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ by all its lower level sets

$$L^{\tau}_{\mathcal{GSI}}(f,h,g,u,v) \;:=\; \{\, x\in \mathbb{R}^n \mid x\in M_{\mathcal{GSI}}[h,g,u,v],\;\; f(x)\le\tau\,\} \quad (\tau\in\mathbb{R}).$$

In the tradition of Guddat, Jongen, Rückmann and Weber, we observe this structure under data perturbation and define *structural stability.* Here, *descent* has to be preserved, if the level varies. Let us still assume that the defining functions are C^2. Then, this global stability can essentially be *characterized* by EMFCQ of $M_{\mathcal{GSI}}[h,g]$ and by strong stability of all considered stationary points.

Two problems $\mathcal{P}_{\mathcal{GSI}}(f^1,h^1,\ g^1,u^1,v^1)$, $\mathcal{P}_{\mathcal{GSI}}(f^2,h^2,g^2,u^2,v^2)$ (with defining C^2-functions) are called **structurally equivalent**:

$$\mathcal{P}_{\mathcal{GSI}}(f^1,h^1,g^1,u^1,v^1)\;\sim_{\mathcal{P}}\;\mathcal{P}_{\mathcal{GSI}}(f^2,h^2,g^2,u^2,v^2)$$

if there are continuous functions $\varphi_{\mathcal{P}}\,(=\varphi_{\mathcal{P}_{\mathcal{GSI}}}) : \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ and $\psi\,(=\psi_{\mathcal{GSI}}) : \mathbb{R}\to\mathbb{R}$ with the three properties $\mathcal{E}_{\mathcal{GSI}\ 1,2,3}$ (Fig. 3):

$\mathbf{E}_{\mathcal{GSI}\ 1}$. $\varphi_{\mathcal{P},\tau} : \mathbb{R}^n\to\mathbb{R}^n$ is a homeomorphism, where $\varphi_{\mathcal{P},\tau}(x):=\varphi_{\mathcal{P}}(\tau,x)$, for every $\tau\in\mathbb{R}$.

$\mathbf{E}_{\mathcal{GSI}\ 2}$. $\psi : \mathbb{R}\to\mathbb{R}$ is a monotonically increasing homeomorphism.

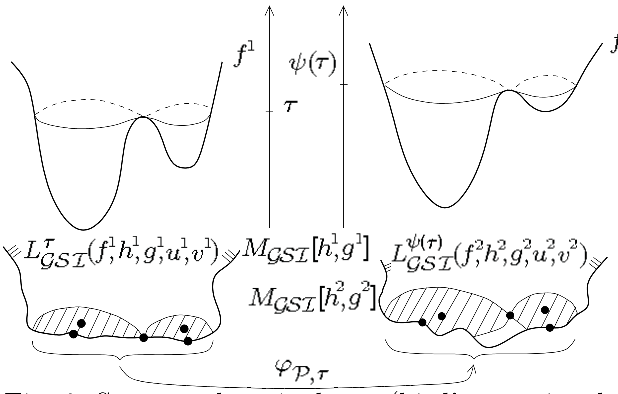

Fig. 3. Structural equivalence (bird's-eye view below)

$\mathbf{E}_{\mathcal{GSI}\ 3}$. $\varphi_{\mathcal{P},\tau}(L^{\tau}_{\mathcal{GSI}}(f^1,h^1,g^1,u^1,v^1)) = L^{\psi(\tau)}_{\mathcal{GSI}}(f^2,h^2,g^2,u^2,v^2)$ for all $\tau \in I\!R$.

Considering the first problem as *undisturbed* and the second one as slightly *disturbed*, we arrive at *structural stability* [11, 23, 27, 53, 59]; cf. also [1, 4, 20, 50]: $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ (with defining C^2-functions) is called **structurally stable**, if there exists a C^2_S-neighbourhood $\mathcal{O}$ of (f,h,g,u,v) such that for each $(\tilde{f},\tilde{h},\tilde{g},\tilde{u},\tilde{v}) \in \mathcal{O}$

$$\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v) \sim_{\mathcal{P}} \mathcal{P}_{\mathcal{GSI}}(\tilde{f},\tilde{h},\tilde{g},\tilde{u},\tilde{v})$$

.

3.2. Characterization Theorem.

Under Assumptions $\mathrm{A}_{\mathcal{U}^0}$ and $\mathrm{B}_{\mathcal{U}^0}$ we state:

Characterization Theorem (or Structural Stability Theorem; [59]).

Let $M_{\mathcal{GSI}}[h,g] \subset \mathcal{U}^0$ *hold for problem* $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ *(with defining* C^2*-functions).*

Then, $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ *is structurally stable, if and only if the three conditions* $\mathcal{C}_{\mathcal{GSI}\ 1,2,3}$ *are fulfilled:*

$\mathbf{C}_{\mathcal{GSI}\,1}$. *EMFCQ holds for* $M_{\mathcal{GSI}}[h,g]$.

$\mathbf{C}_{\mathcal{GSI}\,2}$. *All the* $\mathcal{G}$-$\mathcal{O}$ *Kuhn-Tucker points* $\overline{x}$ *of* $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ *are* ($\mathcal{G}$-$\mathcal{O}$) *strongly stable.*

$\mathbf{C}_{\mathcal{GSI}\,3}$. *For each two different* $\mathcal{G}$-$\mathcal{O}$ *Kuhn-Tucker points* $\overline{x}^1 \neq \overline{x}^2$ *of* $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ *the corresponding critical values are different (separate), too:* $f(\overline{x}^1) \neq f(\overline{x}^2)$.

In this main result, we could also make a further assumption, excluding certain inequality constraints z from the relative boundary ∂Z^j $(j \in J)$. Then we could identify the $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points by some $\mathcal{G}$ Kuhn-Tucker points. However, for validity of Characterization Theorem, such an assumption is *not* necessary [59].

3.3. Proof of Characterization Theorem.

Preparations. For preparation, let us recall the proof of *Genericity Theorem*, taking into account the parametrical dependences on the defining data $(\tilde{g},\tilde{u},\tilde{v})$ (by construction, $\tilde{h}$ may be disregarded). Now, we make again applications of *Implicit Function Theorem in Banach spaces*, such that, in particular, we state a *continuous dependence* of $(\tilde{g}^0,\tilde{u}^0,\tilde{v}^0)$ on $(\tilde{g},\tilde{u},\tilde{v})$. Consequently, small perturbations on the data of $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ cause slight perturbations on the data of $\mathcal{P}_{\mathcal{OSI}}(f,h,g^0,u^0,v^0)$. The *reverse question* arises: *Can small perturbations of the* $\mathcal{OSI}$ *data be reconstructed under the problem representation from slight perturbations of* *the given $\mathcal{GSI}$ problem?* We give a conditionally positive answer. However, this answer will be fitting for the *perturbational* argumentations on Characterization Theorem:

Item 1. For representing $\mathcal{OSI}$ problem(s), $\tilde{u}^0, \tilde{v}^0$ are of special linearly affine form and, under sufficiently small perturbations of the $\mathcal{GSI}$ problem, we may treat them as *fixed*. Hence, besides the perturbations $(f,h) \to (\tilde{f},\tilde{h})$, for $\mathcal{P}_{\mathcal{OSI}}(f,h,g^0,u^0,v^0)$ we are concerned with $g^0 \to \widetilde{g^0}$ only. We therefore introduce the simplifying notation $\mathcal{P}^*_{\mathcal{OSI}}(f,h,g^0) := \mathcal{P}_{\mathcal{OSI}}(f,h,g^0,u^0,v^0)$.

Item 2. Subsequently, we mainly perform *local perturbations* for $\mathcal{P}^*_{\mathcal{OSI}}(f,h,g^0)$. Hereby, we treat the finitely many functions g^0_j $(j \in J)$ separately in small *disjoint* open sets $\mathcal{V}^*_j$ $(j \in J)$ such that their perturbations $g^0_j \to \widetilde{g^0_j}$ can be reconstructed by one single C^2-function $\tilde{g}$ (given below). Therefore, we would need the perturbationally stable

Assumption F*: *For all $j^1, j^2 \in J$, $j^1 \neq j^2$, we have*

$$\bigcup_{x \in M_{\mathcal{GSI}}[h,g] \cap \overline{\mathcal{U}^0}} \left((\phi^{j^1}_x)^{-1}(Z^{j^1}_0(x)) \cap (\phi^{j^2}_x)^{-1}(Z^{j^2}_0(x)) \right) = \emptyset.$$

We are going to exploit the condition from Assumption F* *after* perturbations. However, if we may *suitably* choose our perturbed functions $\widetilde{g^0}$, then Assumption F* is naturally fulfilled (after perturbation), and we need not make it in the unperturbed situation. Now, under problem representation and joined by u, v, this function $\tilde{g}$ generates $\widetilde{g^0_j}$ locally in $\mathcal{V}^*_j$ $(j \in J)$. Then, for each $j \in J$, small perturbational (global) effects outside of $\mathcal{V}^*_j$ $(j \in J)$ have no influence. They can be ignored. The announced function is

$$\tilde{g}(x,y) := \begin{cases} \widetilde{g^0_j}(x, \phi^j_x(y)), & \text{if } y \in (\phi^j_x)^{-1}(Z^j) \text{ and } (x, \phi^j_x(y)) \in \mathcal{V}^*_j, \ j \in J \\ \\ g(x,y), & \text{else.} \end{cases}$$

Item 3. Below we must consider a certain *global perturbation* of $\mathcal{P}^*_{\mathcal{OSI}}(f,h,g^0)$ to receive C^∞-data or, finally, some (global) "open and dense" property. Therefore, we apply on the one hand the perturbation technique from the proof of Genericity Theorem. On the other hand, whenever it is possible to turn from the $\mathcal{GSI}$ problem to an $\mathcal{OSI}$ (or $\mathcal{F}$) one, then we are back in the situation of Item 2 in order to perform local perturbations.

For our proof of Characterization Theorem, the *algebraical characterization of* ($\mathcal{G}$-$\mathcal{O}$) *strong stability* of a $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point $\overline{x}$ is important. It was given by Rückmann [48] for $\mathcal{OSI}$ optimization and extended in [59] for our $\mathcal{GSI}$ one. Here, we assume EMFCQ at $\overline{x}$. That sophisticated characterization refers to (restricted) Hessians of Lagrange functions, and it bases on a case study referring to the *reduction ansatz*. This **RA** demands strong stability in the sense of $\mathcal{F}$ optimization [30] for the local minimizers of the problem from the lower (y-) stage. Herewith, RA admits local representation of $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ around $\hat{x}$ by Implicit Function Theorem [48, 59]; see [14, 61]. These cases are:

I ELICQ and RA are fulfilled at $\hat{x}$.

II EMFCQ – but *not* ELICQ – and *GRA* are fulfilled at $\hat{x}$.

III EMFCQ – but *not* GRA – is fulfilled at $\hat{x}$.

In any case, we can also classify the *type* of the strongly stable stationary point $\overline{x}$: While in case I a saddle point, a local minimizer or local maximizer is detected by the "stationary index" of $\hat{x}$ (a topological invariant), in cases II, III we have a strict local minimizer throughout [59]; cf. [31, 48, 53].

Sufficiency Part. Let $\mathcal{C}_{\mathcal{GSI}1,2,3}$ be satisfied. We equivalently represent $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ by $\mathcal{P}_{\mathcal{GSI}}(f,h,$
$g^0,u^0,v^0)$, and straightforwardly interpret $\mathrm{C}_{\mathcal{GSI}\,1,2,3}$ as $\mathcal{OSI}$ conditions $\mathrm{C}_{\mathcal{OSI}\,1,2,3}$. These conditions are the ($\mathcal{OSI}$) constraint qualification EMFCQ, strong stability of all Kuhn-Tucker points in the sense of $\mathcal{OSI}$ optimization, and separateness of the values of these $\mathcal{OSI}$ stationary points. Under slight perturbations of the $\mathcal{GSI}$ data, u^0,v^0 do not (and need not) vary. Now, we are prepared for $\mathcal{OSI}$ explanations and, finally, $\mathcal{F}$ constructions from [23, 27, 53] in our $\mathcal{GSI}$ context. We briefly repeat main ideas of construction. In [27, 53], detailed information on the techniques can be found together with illustrations.

An easy counterexample shows that the separateness condition $\mathrm{C}_{\mathcal{GSI}\,3}$ is not generally avoidable for establishing structural stability (see [20, 53]). Here, two connected sets, say: (arcwise) components, would have to be mapped onto one connected component, contradicting homeomorphy. A similar reasoning made for another counterexample shows that, in general, the τ- (level-) dependence of the intended homeomorphisms also cannot be avoided. Moreover, each $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point $\hat{x}^u$ has to be mapped to the corresponding stationary point $\hat{x}^d$ of the slightly perturbed problem $\mathcal{P}_{\mathcal{GSI}}(\tilde{f},\tilde{h},\tilde{g},\tilde{u},\tilde{v})$. Finally, we conclude from the overall boundedness assumption, from EMFCQ and strong stability, that the number of $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points is *finite*: $\hat{x}^u_\sigma$ $(\sigma\in\{1,\ldots,\sigma^0\})$ [27, 53, 59].

We start the construction by transforming the C^2-manifold $M[h]$ to the C^2-manifold $M[\tilde{h}]$ in a suitable bounded, open neighbourhood of $M_{\mathcal{GSI}}[h,g]$. Therefore, first we make a **local construction** by a graph (or implicit function) argumentation. Locally around the stationary points, the transformation is C^2. Then, we complete the whole transformation by means of a **global construction**. Here, we use the *Morse theoretical technique* of walking along trajectories of a vector field in $I\!R^{n+1}$. Outside of (local) neighbourhoods of the stationary points, the transformation is C^1. There, this means a (by 1) diminished order of differentiability, which does not cause any difficulty. From now on, we may assume that there are no inequalities, i.e., $I=\emptyset$. Next, we dynamically construct the *level shift* ψ. In fact, we integrate a C^∞-vector field such that each critical value $f(\hat{x}^u_\sigma)$ gets shifted in $I\!R$ to the corresponding critical value $\tilde{f}(\hat{x}^d_\sigma)$ $(\sigma\in\{1,\ldots,\sigma^0\})$. Now, we may think $\psi=Id_{I\!R}$, referring to $f\circ\psi$ otherwise. There are disjoint open neighbourhoods $B(\hat{x}^u_\sigma,\epsilon)$ (balls) around $\hat{x}^u_\sigma$ such that the smaller neighbourhoods $B(\hat{x}^u_\sigma,\frac{\epsilon}{2})$ contain $\hat{x}^d_\sigma$ $(\sigma\in\{1,\ldots,\sigma^0\})$. We *assume* that the unperturbed and the perturbed lower level sets *coincide* in *all* the sets $B(\hat{x}^u_\sigma,\epsilon)\setminus\overline{B(\hat{x}^u_\sigma,\frac{\epsilon}{2})}$ $(\sigma\in\{1,\ldots,\sigma^0\})$. This assumption will not restrict generality.

Based on the foregoing reductions of I,ψ and the previous assumption, we go on constructing $\varphi_{\mathcal{P},\tau}$ $(\tau\in I\!R^n)$ in a local-global way. Firstly, we realize which undisturbed sets have to be homeomorphically mapped onto which corresponding sets from the disturbed situation (*mapping task*). We distinguish three situations given by levels $\tau<\overline{\tau}, \tau=\overline{\tau}$, or $\tau>\overline{\tau}$. Herewith, we learn that some area from outside of the feasible set possibly has to be "carried in". Moreover, outside of the stationary points, the intersections of the level sets with the boundaries are *transversal*. Our further construction will be raised on these intersections (*fundamental domains*).

Outside of $B(\hat{x}^u_\sigma,\epsilon)$ $(\sigma\in\{1,\ldots,\sigma^0\})$, we use *EMF-technique* indicated in the sufficiency part on *Stability Theorem*. Here, we use our Lemma from Section 1, and apply this dynamical $\widehat{EMF}$*-technique* on $L^\tau_{\mathcal{OSI}}(f,g^0)$ $(=L^\tau_{\mathcal{GSI}}(f,g,u,v))$ and on $L^\tau_{\mathcal{OSI}}(\tilde{f},\tilde{g}^0)$ (see Figure 4(II)). By differential geometry, this **global construction** is glued together in $\cup_{\sigma=1}^{\sigma^0}(B(\hat{x}^u_\sigma,\epsilon)\setminus\overline{B(\hat{x}^u_\sigma,\frac{\epsilon}{2})})$

with the **local construction** sketched next. We may refer to one unperturbed stationary point $\hat{x}^u (= \hat{x}^u_\sigma) \in \{\hat{x}^u_1, \ldots, \hat{x}^u_{\sigma^0}\}$ and corresponding perturbed point $\hat{x}^d$. Now, we are **inside** of $B(\hat{x}^u, \epsilon)$. We restrict to $n \in \{2, 3\}$, because higher dimensions can be **reduced** to those small dimensions by successive hyperplane intersection.

Case 1. $\hat{x}^u$ *is lying in the interior* $M_{\mathcal{OSI}}[g^0] (= M_{\mathcal{GSI}}[g, u, v])$ *:*
Then, $\hat{x}^d$, being sufficiently slightly perturbed, lies in the interior of $M_{\mathcal{OSI}}[\tilde{g}^0]$. Both stationary points are *nondegenerate* [19], and for each τ we transform the τ-levels around $\hat{x}^u$ onto the local τ-levels at $\hat{x}^d$. In fact, this local construction can be made by a C^1-diffeomorphism using Morse theory [27, 53].

Case 2. $\hat{x}^u$ *is placed on the boundary of* $M_{\mathcal{OSI}}[g^0]$ *:*
Then, $\hat{x}^d$ may lie on the boundary or in the interior of $M_{\mathcal{OSI}}[\tilde{g}^0]$. Without loss of generality we assume the second (boundary) case. Actually, using an **implantation** of a suitable level structure we turn from stationary points at the boundary to *fictive* stationary points in the interior. This level structure is locally given by *fictive* objective functions $\hat{f}^u$ and $\hat{f}^d$. (In case 1, those fictive points naturally exist.) For performing this implantation of $\hat{f}^u$, $\hat{f}^d$ we need precise knowledge of the configurations around the boundary points $\hat{x}^u, \hat{x}^d$. These configurations are characterizable by the position (relative to the boundary) of cones or balls, together with the growth behaviours of f, $\tilde{f}$ there. We have two conical types and one radial type, governed by strong stability (under EMFCQ; [27, 53, 59]. See, e.g., Figure 4(I). We arrive back in *case 1* (interior position) by means of fictive interior problems, **extrapolating** the "characteristic" of $\hat{x}^u$, $\hat{x}^d$ and implanting fictive stationary points $\hat{x}^u_{fic}$, $\hat{x}^d_{fic}$ with their local level structures. Herewith, for all $\tau \in I\!R$ the mapping task is fulfilled in case 2, too.

The delicate dynamical and topological techniques (and substeps) exhibited in Fig. 4(I) are due to the local construction in case 2. They can be elaborated, e.g., in terms of *boundary displacement*, *positioning*, *sharpening* or *tapering flows* [27, 53].

Necessity Part: Let $\mathcal{P}_{\mathcal{GSI}}(f, h, g, u, v)$ be structurally stable. Our proof of $\mathcal{C}_{\mathcal{GSI}\,1,2,3}$ is indirect. Assuming one of the first two regularity conditions or the third technical condition to be violated always contradicts structural stability (see Figure 5). Based on our assumptions, we carry over the proof the $\mathcal{OSI}$ necessity part from [23] into our $\mathcal{GSI}$ setting. Many details of argumentations are Morse theoretical [11, 26, 27, 53, 59]. To avoid loss of differentiability, we assume that all data are C^∞ [11]. This smoothness can be achieved by fine perturbations of all $\mathcal{OSI}$ data and, by tracing them back, of all $\mathcal{GSI}$ ones.

Here, we make the inequalities of different indices $\overline{z}^{\varrho^1} \neq \overline{z}^{\varrho^2}$ independent from each other (by small shifts).

$\mathcal{C}_{\mathcal{GSI}\ 1}$. As $M_{\mathcal{GSI}}[h, g]$ is compact, there exists the finite number $\tau^{\max} := \max\{f(x) | x \in M_{\mathcal{GSI}}[h, g]\}$. Herewith, $M_{\mathcal{GSI}}[h, g] = L^{\tau}_{\mathcal{GSI}}(f, h, g, u, v)$ $(\tau \in [\tau^{\max}, \infty))$. Moreover, we can choose perturbations slight enough such that $M_{\mathcal{GSI}}[\tilde{h}, \tilde{g}]$ remains compact. Let $\tilde{\tau}^{\max}$ for each sufficiently slight perturbation $(\tilde{f}, \tilde{h}, \tilde{g}, \tilde{u}, \tilde{v})$ denote the maximal (feasible) value of $\tilde{f}$. Taking $\tau^* := \max\{\tau^{\max}, \psi^{-1}(\tilde{\tau}^{\max})\}$, the homeomorphism $\varphi_{\mathcal{P},\tau^*}$ gives topological equivalence between $M_{\mathcal{GSI}}[h, g, u, v] = L^{\tau^*}_{\mathcal{GSI}}(f, h, g, u, v)$ and $M_{\mathcal{GSI}}[\tilde{h}, \tilde{g}, \tilde{u}, \tilde{v}] = L^{\psi(\tau^*)}_{\mathcal{GSI}}(\tilde{f}, \tilde{h}, \tilde{g}, \tilde{u}, \tilde{v})$. By *Stability Theorem*, topological stability implies *EMFCQ*. In fact, by suitable perturbations any violation of EMFCQ at a feasible point leads to compact sets $M_{\mathcal{GSI}}[\tilde{h}, \tilde{g}]$, $M_{\mathcal{GSI}}[\tilde{\tilde{h}}, \tilde{\tilde{g}}]$, satisfying ELICQ but being *not* of the same homotopy type [12, 26, 53, 59]. When, e.g., the two sets have a *different* finite number of connected components, this must contradict topological equivalence [19].

$\mathcal{C}_{\mathcal{GSI}\ 2}$. Suppose EMFCQ, but $\mathrm{C}_{\mathcal{GSI}\,2}$ not fulfilled: some $\mathcal{G}$-$\mathcal{O}$ point $\hat{x}^u$ be *not* ($\mathcal{G}$-$\mathcal{O}$) strongly stable.

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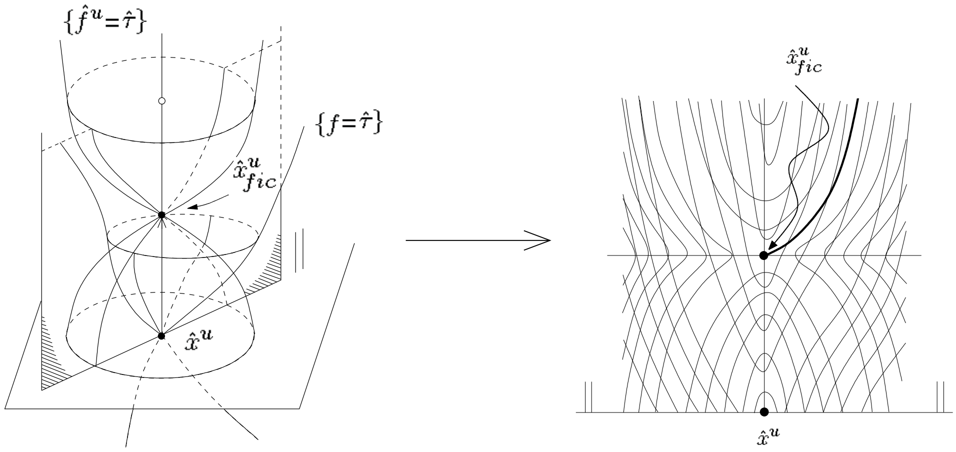

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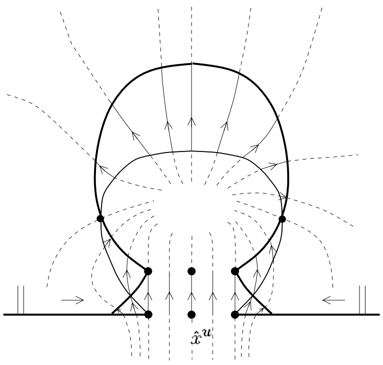

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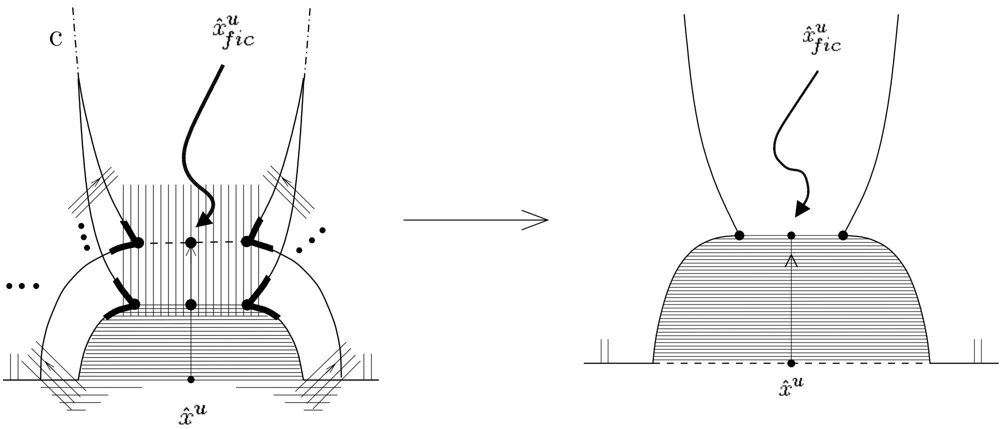

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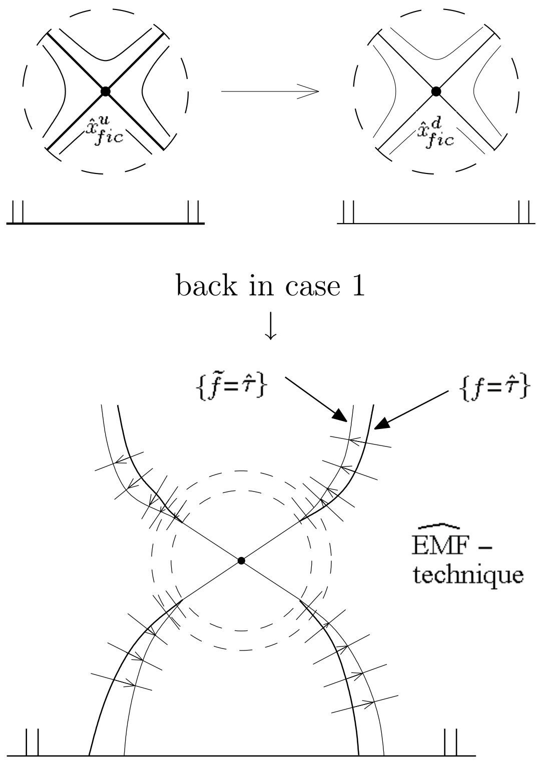

Fig. 4. Proof of sufficiency part, Characterization Theorem

Perturbation Lemma [59]. *Let a $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker point $\hat{x}^u$ of $\mathcal{P}_{\mathcal{GSI}}(f,h,g,u,v)$ be given, where EMFCQ is fulfilled, but ($\mathcal{G}$-$\mathcal{O}$) strong stability violated. Then, for each open C^2-neighbourhood $\mathcal{O}'$ of (f,h,g,u,v) there are $(\tilde{f},\tilde{h},\tilde{g},\tilde{u},\tilde{v}),(\tilde{\tilde{f}},\tilde{\tilde{h}},\tilde{\tilde{g}},\tilde{\tilde{u}},\tilde{\tilde{v}}) \in \mathcal{O}'$ and a $k' \in I\!N$ such that:*

(i) $\mathcal{P}_{\mathcal{GSI}}(\tilde{f},\tilde{h},\tilde{g},\tilde{u},\tilde{v})$ has k' $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points, all being ($\mathcal{G}$-$\mathcal{O}$) strongly stable, except one (namely, $\hat{x}$).

(ii) $\mathcal{P}_{\mathcal{GSI}}(\tilde{\tilde{f}},\tilde{\tilde{h}},\tilde{\tilde{g}},\tilde{\tilde{u}},\tilde{\tilde{v}})$ has at least $k'+1$ $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points, all being ($\mathcal{G}$-$\mathcal{O}$) strongly stable.

(iii) In both $\mathcal{P}_{\mathcal{GSI}}(\tilde{f},\tilde{h},\tilde{g},\tilde{u},\tilde{v})$ and $\mathcal{P}_{\mathcal{GSI}}(\tilde{\tilde{f}},\tilde{\tilde{h}},\tilde{\tilde{g}},\tilde{\tilde{u}},\tilde{\tilde{v}})$, EMFCQ is satisfied everywhere, and different $\mathcal{G}$-$\mathcal{O}$ Kuhn-Tucker points have different critical ($\tilde{f}$ - or $\tilde{\tilde{f}}$) values.

In $\mathcal{F}$ or $\mathcal{OSI}$ necessity parts of [11, 53] (cf. also [23]), these perturbations are realized by three steps. *Step 1* yields local isolatedness of $\hat{x}^u$ as a stationary point where, additionally, (E)LICQ is guaranteed but unstability preserved. In *step 2*, outside of the local situation, (E)MFCQ and strong stability of all (other) stationary points are established. Finally, in *step 3*, the unstable Kuhn-Tucker point $\hat{x}^u$ "splits": By this bi- (or tri-) furcation we locally get two new stationary points; they have *strongly stability.* In this $\mathcal{GSI}$ situation, we use the algebraical characterization from our preparations. Now, we introduce a topological idea: For $L^{\tau}_{\mathcal{GSI}}(\tilde{f},\tilde{h},\tilde{g},\tilde{u},\tilde{v})$, $L^{\tau}_{\mathcal{GSI}}(\tilde{\tilde{f}},\tilde{\tilde{h}},\tilde{\tilde{g}},\tilde{\tilde{u}},\tilde{\tilde{v}})$ we have to take into account each change of the homeomorphy type of a lower level set, when τ traverses $(-\infty,\infty)$. Based on the perturbations from above, we apply the following items on $\mathcal{P}_{\mathcal{GSI}}(\tilde{f},\tilde{h},\tilde{g},\tilde{u},\tilde{v})$, and $\mathcal{P}_{\mathcal{GSI}}(\tilde{\tilde{f}},\tilde{\tilde{h}},\tilde{\tilde{g}},\tilde{\tilde{u}},\tilde{\tilde{v}})$. We look at a C^2-problem $\mathcal{P}_{\mathcal{GSI}}(\hat{f}, \hat{h}, \hat{g}, \hat{u}, \hat{v})$ having a compact feasible set and fulfilling EMFCQ, and we put $L^b_{\mathcal{GSI}\,a}(\hat{f}, \hat{h}, \hat{g}, \hat{u}, \hat{v}) := \{x \in M_{\mathcal{GSI}}[\hat{h}, \hat{g}] \,|\, a \leq \hat{f}(x) \leq b\}$ for some $a, b \in I\!R,\ a < b$.

Item 1. If $L^b_{\mathcal{GSI}\,a}(\hat{f}, \hat{h}, \hat{g}, \hat{u}, \hat{v})$ does not contain a stationary point, then $L^a_{\mathcal{GSI}}(\hat{f}, \hat{h}, \hat{g}, \hat{u}, \hat{v})$ and $L^b_{\mathcal{GSI}}(\hat{f}, \hat{h}, \hat{g}, \hat{u}, \hat{v})$ are homeomorphic.

Item 2. Let $L^b_{\mathcal{GSI}\,a}(\hat{f}, \hat{h}, \hat{g}, \hat{u}, \hat{v})$ contain exactly one stationary point $\hat{x}'$. Moreover, let $a < f(\hat{x}') < b$ and $\hat{x}'$ be $(\mathcal{G}$-$\mathcal{O})$ strongly stable. Then, $L^a_{\mathcal{GSI}}(\hat{f}, \hat{h}, \hat{g}, \hat{u}, \hat{v})$ and $L^b_{\mathcal{GSI}}(\hat{f}, \hat{h}, \hat{g}, \hat{u}, \hat{v})$ are *not* homeomorphic.

These two items immediately result from corresponding facts on $\mathcal{P}_{\mathcal{OSI}}(\tilde{f}, \tilde{h}, \tilde{g}^0, \tilde{u}^0, \tilde{v}^0)$, $\mathcal{P}_{\mathcal{OSI}}(\tilde{\tilde{f}}, \tilde{\tilde{h}}, \tilde{\tilde{g}}^0, \tilde{\tilde{u}}^0, \tilde{\tilde{v}}^0)$ stated in [48]. Here, *Item 2* can be expressed with attaching κ-cells ($\kappa =$ stationary index at $\hat{x}'$; [59]). By Manifold Theorem and Lemma (Sections 1–2) we conclude for all noncritical levels τ: $L^\tau_{\mathcal{GSI}}(\hat{f}, \hat{h}, \hat{g}, \hat{u}, \hat{v}) = M_{\mathcal{GSI}}[\hat{h}, (\hat{g}, -\hat{f} + \tau)]$ is a *compact* topological manifold (with boundary). So, their homology spaces (over $I\!R$) are of different *finite* dimensions [51]. As these spaces are topological invariants, the two considered lower level sets cannot be homeomorphic [19].

Now, we can make the following "discrete" statement on numbers of topological changes for the lower level sets: The homeomorphy type of $L^\tau_{\mathcal{GSI}}(\tilde{\tilde{f}}, \tilde{\tilde{h}}, \tilde{\tilde{g}}, \tilde{\tilde{u}}, \tilde{\tilde{v}})$ changes (at least) at $k' + 1$ times, while the homeomorphy type of $L^\tau_{\mathcal{GSI}}(\tilde{f}, \tilde{h}, \tilde{g}, \tilde{u}, \tilde{v})$ changes (at least) at $k' - 1$ times, but at most at k' times. This difference contradicts structural stability of $\mathcal{P}_{\mathcal{GSI}}(f, h, g, u, v)$ (cf. [59], or see Fig. 5).

$\mathcal{C}_{\mathcal{GSI}\ 3}$: Let $\mathrm{C}_{\mathcal{GSI}\ 3}$ be violated, but the former two properties on EMFCQ and strong stability be satisfied. By local addition of arbitrarily small constant functions on f, we get a problem $\mathcal{P}_{\mathcal{GSI}}(f^*, h, g, u, v)$ satisfying $\mathrm{C}_{\mathcal{GSI}\ 3}$. Let k^* stand for the number of critical points of $\mathcal{P}_{\mathcal{GSI}}(f^*, h, g, u, v)$. Then the homeomorphy type of $L^\tau_{\mathcal{GSI}}(f^*, h, g, u, v)$ changes k^* times, while the number of changes of the homeomorphy type of $L^\tau_{\mathcal{GSI}}(f, h, g, u, v)$ is less than k^*. Hence, we are faced again with a situation which is incompatible with structural stability of $\mathcal{P}_{\mathcal{GSI}}(f, h, g, u, v)$ (Figure 5).

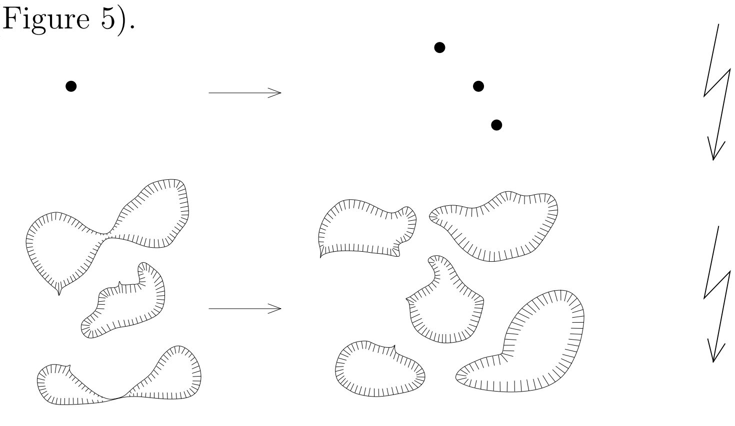

Fig. 5. Proof of necessity part, Characterization Theorem

4. Generalizations, Optimal Control and Conclusion.

4.1. Generalizations.

There are two lines for generalizing our topological results:

(i) $M_{\mathcal{GSI}}[h, g]$ is *unbounded* (noncompactness),

(ii) f is of the *nondifferentiable* $\mathcal{GSI}$ maximum-type $f(x) = \max_{\gamma \in \Upsilon(x)} w(x, \gamma)$.

On (i): We overcome noncompactness by turning to the entity of *excised* subsets of $M_{\mathcal{GSI}}[h, g]$. Roughly speaking, the effect of intersection is performed by subtracting lower semi-continuous functions from $h_i\,(i \in I)$ and g [49, 53, 59]. Herewith, we can express cuts, e.g., by cylinders or balls, by $\mathbb{R}^n$ itself or by bizarre sets. Referring to all excised sets, we get the condition of **excisional topological stability** which can actually be characterized by the overall validity of EMFCQ in the unbounded set $M_{\mathcal{GSI}}[h, g]$. For that *(Excisional) Stability Theorem* see [59].

On (ii): Nonsmoothness is overcome by expressing $\mathcal{P}_{\mathcal{GSI}}(f, h, g, u, v)$ as minimization of x_{n+1} over the *epigraph* $E_{\mathcal{GSI}}(f) := \{(x, x_{n+1}) | \, x \in M_{\mathcal{GSI}}[h, g], \; f(x) \leq x_{n+1}\}$. From this problem in $\mathbb{R}^{n+1}$ we obtain our stationary points of $\mathcal{P}_{\mathcal{GSI}}(f, h, g, u, v)$ and the appropriate condition of strong stability [53, 54, 59]. Now, *(max-) structural stability* of our nondifferentiable problem can be characterized by EMFCQ, strong stability and the technical separateness condition again. This *Characterization Theorem* and the one for the *case combination* of *(i)* and *(ii)* are demonstrated in [59].

4.2. Optimal Control of Ordinary Differential Equations.

We turn to infinite dimensions by studying the following minimization problem in (x, u) [13, 36, 44]:

$$\mathcal{P}(\ell, L, F, H, G) \quad \begin{cases} \text{Min } \mathcal{I}(x, u) := \ell(x(a), x(b)) + \int_a^b L(t, x(t), u(t))\, dt \\ (\, x \in (C^0_{\mathrm{pw}\,2}([a, b], \mathbb{R}))^n, \;\; u \in (F_{\mathrm{pw}\,2}([a, b], \mathbb{R}))^q\,), \\ \text{such that} \\ \dot{x}(t) = F(t, x(t), u(t)) \quad (\text{for almost every } t \in [a, b]), \\ (x(a), x(b)) \in M[H], \\ x(t) \in M_{\mathcal{F}}[G(t, \cdot, u(t))] \quad (\text{for almost every } t \in [a, b]), \end{cases}$$

where (L, F, G), (ℓ, H) are C^3- and C^2-functions (vector notation), respectively. Instead of referring to the larger classes of Sobolev or Lebesgue spaces, we concentrate on spaces of continuous and piecewise C^2 *states* x, and piecewise C^2 *controls*, called $C^0_{\mathrm{pw}\,2}$ and $F_{\mathrm{pw}\,2}$. For these spaces, strong topologies in Whitney's sense can be generally introduced [59].

Assumption (BOUND). *$M[H] \subseteq \mathbb{R}^n \times \mathbb{R}^n$ and $M_{\mathcal{F}}[G] \subseteq [a, b] \times \mathbb{R}^n \times \mathbb{R}^q$, defined by the equality and inequality contraints, are bounded.*

Assumption (LB). *There exist positive functions $\alpha_0, \beta_0 \in C(\mathbb{R}^{q+1}, \mathbb{R})$ such that (under $|| \cdot ||_\infty =$ maximum norm) we have linear boundedness of F:*

$$||F(t, \mathbf{x}, \mathbf{u})||_\infty \leq \alpha_0(t, \mathbf{u}) ||\mathbf{x}||_\infty + \beta_0(t, \mathbf{u}) \quad ((t, \mathbf{x}, \mathbf{u}) \in \mathbb{R}^{n+q+1}).$$

We briefly present two approaches to global structure and stability of $\mathcal{P}(\ell, L, F, H, G)$ (cf. [59], where a third one can also be found). While our main Approach II is refined, Approach I is given for a better understanding.

Approach I: **Particular Structure.** Let u be considered as C^2 and a parameter. Then, for each fixed $u = u^*$ the optimal control problem $\mathcal{P}(\ell, L, F, H, G)$ becomes a problem $\mathcal{P}^{u^*}(\ell, L, F, H, G)$ from *calculus of variations.* The corresponding system of differential equations (on x) generates a *flow* (in $\mathbb{R}^{n+1}$; [1, 20]). Under this flow, we *trace back* the equality and inequality constraints, and the objective functional as well (cf. [55, 56, 59]). So we obtain an

$\mathcal{OSI}$ problem $\mathcal{P}^{u^*}_{\mathcal{OSI}}(f^*, h^*, g^*)$ (where $Y^j = [a,b]$). Then, referring to the family of all u and to perturbations of (f^*, h^*, g^*), we get the condition of **(particular) structural stability** with its **Characterization Theorem** again (cf. Section 3; [56, 59]). The C^2-property and parametrical treatment of u, however, are not sufficient for optimal control. That is why we turn to the

Approach II: **Composite Structure.** We evaluate the necessary optimality condition *Pontryagin's minimum principle* [13, 44] in the way of "Kuhn-Tucker" for almost every $t \in [a,b]$. Here, we have suitable multiplier vectors, (adjoint) variables, and $\mathcal{H}(t, \mathbf{x}, \mathbf{u}, \lambda, \mu) := L(t, \mathbf{x}, \mathbf{u}) - \lambda^T F(t, \mathbf{x}, \mathbf{u}) - \mu^T G(t, \mathbf{x}, \mathbf{u})$. Then, our evaluation, called **minimum principle** here [6, 38, 39, 41], reads

$$\begin{aligned} D^T_{\mathbf{u}}\mathcal{H}(t, x^0(t), u^0(t), \lambda^0(t), \mu^0(t)) \;&=\; \mathbf{0}_q, \\ \mu^0_j(t) \;\geq\; 0 \;\; (j \in J) \quad \text{and} \quad \mu^{0^T}(t)\, G(t, x^0(t), u^0(t)) \;&=\; 0, \\ \lambda^0(a) \;&=\; -\, D^T_{\mathbf{x}_1}(\ell - \rho^{\mathbf{0}^T} H)(x^0(a), x^0(b)), \\ \lambda^0(b) \;&=\; D^T_{\mathbf{x}_2}(\ell - \rho^{\mathbf{0}^T} H)(x^0(a), x^0(b)), \\ \dot{\lambda}^0(t) \;&=\; -D^T_{\mathbf{x}}\mathcal{H}(t, x^0(t), u^0(t), \lambda^0(t), \mu^0(t)). \end{aligned}$$

For our *causal* (composite) structure we need a condition like strong stability [59]:

Assumption (CONT). *All the ($C^0_{pw2} \times F_{pw2}$-) solution components (x^0, u^0) of the minimum principle depend continuously on $C^3_S \times C^2_S$-perturbations $((L,F,G),(\ell,H)) \to ((\widetilde{L}, \widetilde{F}, \widetilde{G}), (\tilde{\ell}, \widetilde{H}))$.*

We interpret the first four lines of the minimum principle as Kuhn—Tucker conditions of two families of optimization problems: $(*)$ $\mathcal{P}^{t,\mathbf{x},\mathbf{w}}_{\mathcal{F}}(L, F - \mathbf{w}, G)$ and $(**)$ $\mathcal{P}_{\mathcal{F}}(\lambda^0(a), \lambda^0(b), \ell, H)$, an index set $M^{\eta_0}_{pr}[F,G]$ of $(t, \mathbf{x}, \mathbf{w})$ being appropriately chosen in view of $\mathcal{P}(\ell, L, F, H, G)$. For each of these problems we introduce *(composite) structural stability* and characterize it essentially by (E)MFCQ and strong stability (see Section 3). Analyzing $(*)$ so, we locally get implicit C^2-control functions $u_\vee(t, \mathbf{x}, \mathbf{w})$, which are Kuhn-Tucker point-valued and fulfill $u^0(t) = u_\vee(t, x^0(t), \dot{x}(t))$. Substituting $\mathbf{w} := \dot{x}(t)$ for any trajectory x of some auxiliary flow, adapted to our system of differential equations, we locally receive **core functions** $u^0_\vee(t, \mathbf{x})$. The choices of these auxiliary or *test flows* etablish a *structural frontier* of our theory [59]. In order to globalize a core such that its domain covers $[a,b]$, we admit **jumps** in $I\!R^{n+1}$ (see Figure 6). These jumps shall be compatible with the jumps of our variables u^0. Again we say that the globalized core functions $(\heartsuit)$ $u^0_\vee$ are of class $F_{\mathrm{pw}\,2}$. Let B, B be ("boundary") sets where the jumps may or really do happen, respectively. When these sets exist as *Lipschitzian manifolds* of dimension q, and if they define (by decomposition) *piecewise structures* before or after jumps, which quantitatively remain preserved under small perturbation of (ℓ, L, F, H, G), then the core $(\heartsuit)$ is called *(composite) structurally stable* [59]. A further regularity condition, called **structural transversality**, in short: **ST**, analytically determines the boundary sets (up to a finite number of choices) and guarantees this (composite) structural stability of a core. (See also [21, 39, 41].) The refined condition ST essentially means *transversal* intersection of $u^0_\vee(\cdot, x(\cdot))$ (along trajectories x) at the boundary of the corresponding feasible set in $I\!R^q$. This implies transversality of x at the manifolds B, B.

Now, inserting $u(t) = u^0_\vee(t, x(t))$ in $\mathcal{P}(\ell, L, F, H, G)$ delivers again a problem $\mathcal{P}^{u^0_\vee}(\ell, L, F, H, G)$ from calculus of variations, which we also trace back under its flow. In this way we get an optimization problem with a complex underlying piecewise structure. Up to the *structural frontiers* given by combinatorially more complicate index sets $Y(\mathbf{x})$ and objective functions f of *continuous selection type* [22], we arrive at a $\mathcal{GSI}$ problem $(***)$ $\mathcal{P}_{\mathcal{GSI}}(f, h, g, v)$ with f of

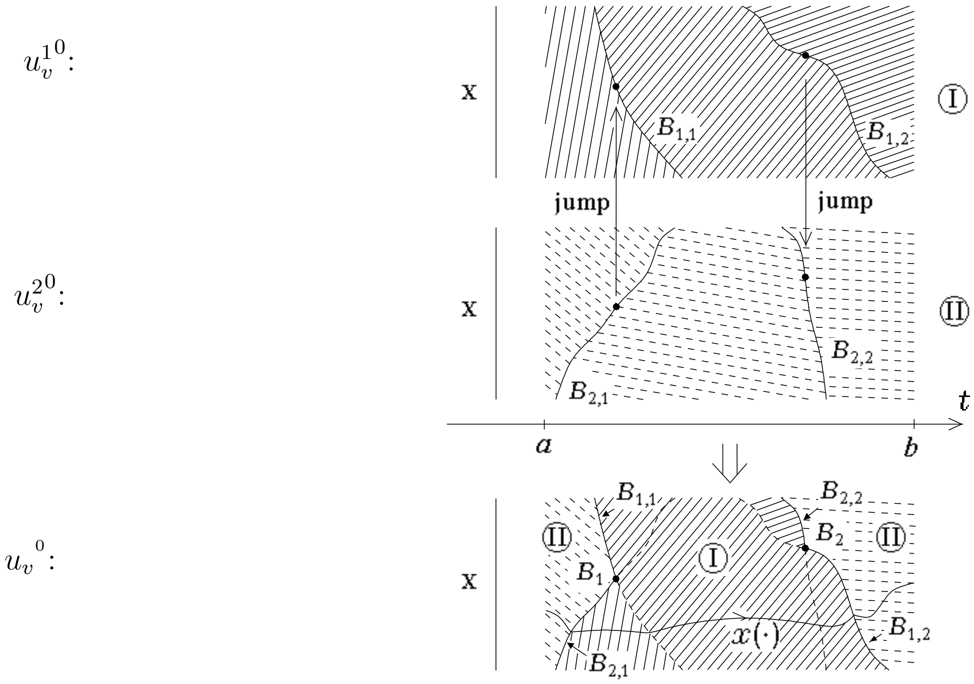

Fig. 6. Piecewise structure and jumps of cores

maximum-type (cf. Subsection 4.1). Then, we introduce this optimization problem's condition of *(composite) structural stability* referring to perturbations of the original data (ℓ, L, F, H, G).

In that sense, we call $\mathcal{P}(\ell, L, F, H, G)$ **composite structurally stable** if *all* the structural elements $(*)$, $(**)$, $(***)$, $(\heartsuit)$ are (composite) structurally stable. Under our basic Assumptions (BOUND), (LB) and up to those more complex problems we state (with simplified presentation):

Characterization Theorem on Composite Structural Stability [59].
The problem $\mathcal{P}(\ell, L, F, H, G)$ is composite structurally stable, if and only if the conditions $\mathcal{C}_{1,2,3,4}$ are satisfied:

C$_1$. *(E)MFCQ holds for all the feasible sets underlying $(*)$, $(**)$, $(***)$, $(\heartsuit)$.*

C$_2$. *All the Kuhn-Tucker points $\overline{\mathbf{u}}, \overline{\mathbf{x}}$ of the problems represented in $(*)$, $(**)$, $(***)$ are strongly stable (in $\mathcal{F}$ or $\mathcal{G}$-$\mathcal{O}$ sense).*

C$_3$. *For all optimization problems represented in $(*)$, $(**)$, $(***)$ each two different Kuhn-Tucker points have different (separate) critical values.*

C$_4$. *For all core functions $(\heartsuit)$ ST is fulfilled.*

Sketch of Proof: The main lines are the same as in Subsection 3.3. The new item, given in the necessity part, "$\Longrightarrow$ C$_4$," concerns the undisturbed or disturbed piecewise structures, and it is illustrated in Figure 7.

For *controllability*, i.e., to come from time a to time b under given constraints of $\mathcal{P}(\ell, L, F, H, G)$, **discrete** mathematics [5] often turns out to be a tool of investigation as follows. (For underlying finiteness and genericity considerations see [59].) Our control problem asks for a domain of the core $u_\vee^0$ (compatible with u^0) that is sufficiently large, say: tending to maximality. Provided a careful choice of the set of jumps, this *maximal domain* problem can be represented as a **maximal matching** problem in a partite graph (see, e.g., Figure 8). In a subset of arcs called *matching*, different elements are disjoint. Here, each partition stands for a locally defined continuous core, the directedness of the ars reflects orientation by time t. This matching problem can be solved by *Edmond's algorithm* [25].

Inserting the global cores, arriving at an x-depending problem, we may, for example, consider the objective function as the arc length of our piecewise structured solutions $x = x^0 \in C^0_{\mathrm{pw}\,2}$ of minimum principle. Therefore, we take into account arcs between neighbouring vertices (manifolds $\mathsf{B}_1, \mathsf{B}_2$) of the same former partition such that the partite character gets lost (see, e.g, Figure 9, periodic constraint $x^0(a) - x^0(b) = 0$ implied). The corresponding minimization problem can be regarded as a **shortest path** problem, solvable by *Dijkstra's algorithm* [25].

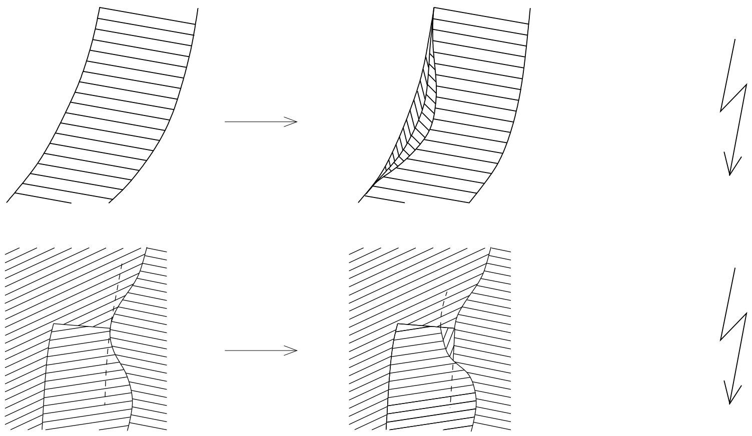

Fig. 7. Proof of necessity part (composite structural stability)

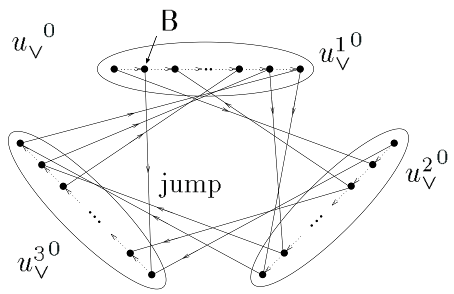

Fig. 8. Tripartite directed graph featuring controllability problem

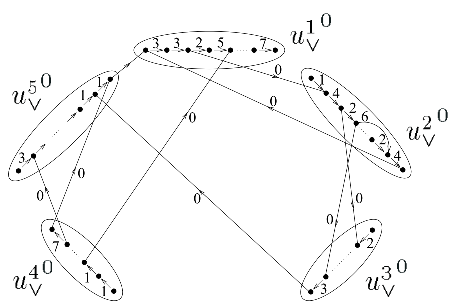

Fig. 9. Directed graph featuring minimization of arc length

4.3. Conclusion.

Besides the "*discrete*" (stationary) index, piecewise structures and optimization problems mentioned above, there is a variety of further theoretical and methodical connections between $\mathcal{GSI}$ optimization, optimal control and discrete mathematics. Concerning discrete Morse theory, topology and systems analysis we mention [2, 8] and [42]; many other examples can be found in [35, 59, 60]. We noted that structural frontiers can often be understood in a combinatorial manner. Just the same is true with respect to solution *algorithms* for a given $\mathcal{GSI}$ problem. The more complex the problem is, the more important becomes discrete intrinsic information of the problem for transparency, convergence and stability (cf. [52], see also [34] on time-minimal control). Here, we refer to the research [43, 58, 59] that is based on our optimality conditions, analytical techniques and stability results. Often, there is little knowledge about the geometry and topology of the feasible $\mathcal{GSI}$ set. As a "manifold" and stability condition, our version of EMFCQ, that bases on Assumption $B_{\mathcal{U}^0}$ of LICQ, was of central importance for a rigorous study of optimization and control. In future, there may be both a weakening in the theoretical field of assumptions (e.g., in the way of [24]) and a systematic look for combinatorial and geometrical treatments from reverse engineering, discrete tomography with its inverse problems, or randomization [59].

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