OPTIMIZATION OF A TIME-DISCRETE NONLINEAR DYNAMICAL SYSTEM FROM A PROBLEM OF ECOLOGY. AN ANALYTICAL AND NUMERICAL APPROACH

St. Pickl
Department of Mathematics, University of Cologne, Germany
G.-W. Weber
Department of Mathematics, Darmstadt University of Technology, Germany
e-mail: pickl@zpr.uni-koeln.de

Работа посвящена математическому анализу нелинейной дискретной по времени модели промышленных выбросов (TEM) для решения проблем защиты окружающей среды. Введение управляющих параметров в динамику модели TEM дает возможность получить новые результаты в области дискретных по времени систем управления. С другой стороны, эти результаты позволяют усовершенствовать важный экономический инструмент управления состоянием окружающей среды. Устойчивые состояния модели TEM можно рассматривать как предельные значения, указанные в Киотском протоколе. Приводятся результаты расчетов.

1. Introduction

The conferences of Rio de Janeiro 1992 and Kyoto 1997 demand for new economic instruments which focus on environmental protection in both macro and micro economy. An important economic tool in that area is *Joint-Implementation (JI)* which is mentioned explicitly in Kyoto Protocol [1]. It is an international program which intends to strengthen international cooperations between enterprises on reducing CO_2-emissions. A sustainable development can only be guaranteed if the instrument is embedded in an optimal energy management. In this context, optimal energy management according to *JI* means that it must work on the micro level with minimal costs and it should be protected against misuse on the macro level.

For that reason, the TEM model (Technology-Emissions-Means model) was developed, giving the possibility to simulate such an economic behaviour. The realization of *JI* is determined subject to technical and financial constraints. In a *JI* program emissions reduced by technical cooperations are registrated at the *clearing house* whose establishment is also a demand of Kyoto Protocol. The associated cost reductions should then be allocated in an optimal way. This approach is as well integrated in the TEM-Model as the possibility to regard the influence of several cost allocations on the feasible set of control parameters. Furthermore, in the played cost-game a special solution, called the $\tau-$value is examined in [2]. This value stands for a rational allocation process. Now, the question arises in which situations the $\tau-$value is equivalent to the necessary control-parameters to reach the regions, mentioned in the treaty of Kyoto. This question is answered by the *Equivalence Theorem* [2]. Numerical results are based on a qualitative analysis of the TEM model in order to simulate such a *JI* program. The results may lead to new insights in *JI* and improve that important management tool.

2. The model

The presented TEM model describes the economic interaction between several actors (players) which intend to minimize their emissions E_i caused by technologies T_i using expenditures of money M_i or financial means, respectively. The index $i \in \{1, \ldots, n\}$ stands for the i-th player. The players are linked by technical cooperations and the market, which expresses itself in the nonlinear time-discrete dynamics of the Technology-Emissions-Means model, in short: TEM model. The TEM model is based on a general model which was introduced in [3] and refined in [4] and [2]:

$$\Delta E_i(t) = \sum_{j=1}^{n} em_{ij}(t) M_j(t),$$

$$\Delta M_i(t) = -\lambda_i M_i(t)(M_i^* - M_i(t))(E_i(t) + \varphi_i \Delta E_i(t)). \tag{2. 1}$$

The first equation describes the time-dependent behaviour of the emissions reduced so far by each player. These levels are influenced by financial investigations which are determined by the second equation. The em_{ij}-parameters determine the effect on the emissions of the i-th actor, if the j-th actor invests money. We can say that it expresses how *effective* technology cooperations are, which is the kernel of a *JI* program. Besides that, the integration into the TEM model of the memory parameter φ and the growth parameter λ guarantees a realistic economic market behaviour while the $M_i^*(t)$ $(i \in \{1, \ldots, n\}\ t \in \mathbb{N})$ are upper bounds for the financial investigations.

3. Numerical results

The TEM model can be regarded as a mathematical model which supports the development of a management tool in the area of the international climate change convention, namely in the creation of a *JI* program.

In the following we regard the TEM model as a routine with which we are able to compare several scenarios. Let us begin with the following example to get a first insight into the dynamics.

Table 1

Data of the TEM model

Player i	$E_i(0)$	$M_i(0)$	M_i^*	λ	$1/60 * em$-matrix		
1	-0.1	30	60	1/60	1	-0.525	-0.475
2	-0.1	20	60	1/60	-0.475	1	0.525
3	-0.1	10	60	1/60	-0.1	-0.1	0.2

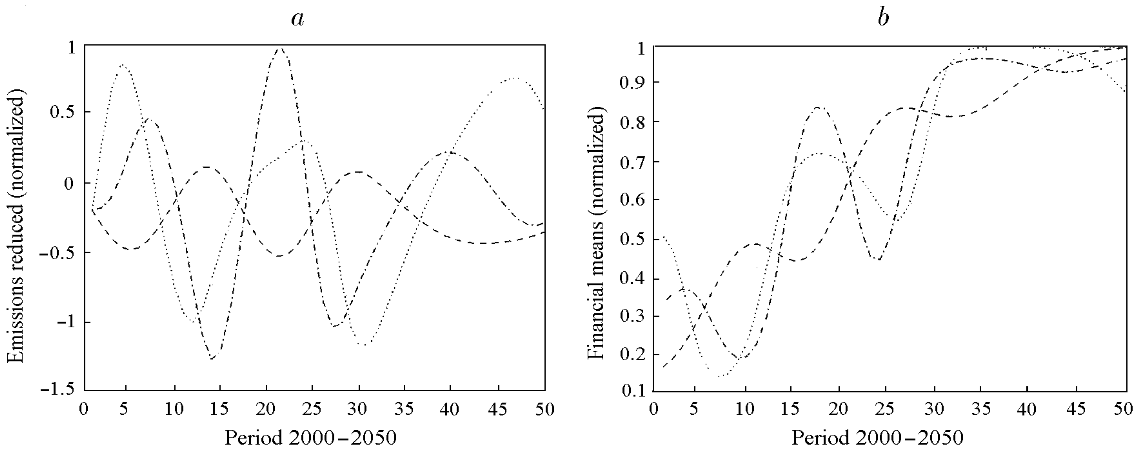

Figure 1. Influence of the memory parameter on the emissions reduced (a) and on the financial means (b), data of table 1, $\varphi = (1, 1, 1)^T$.

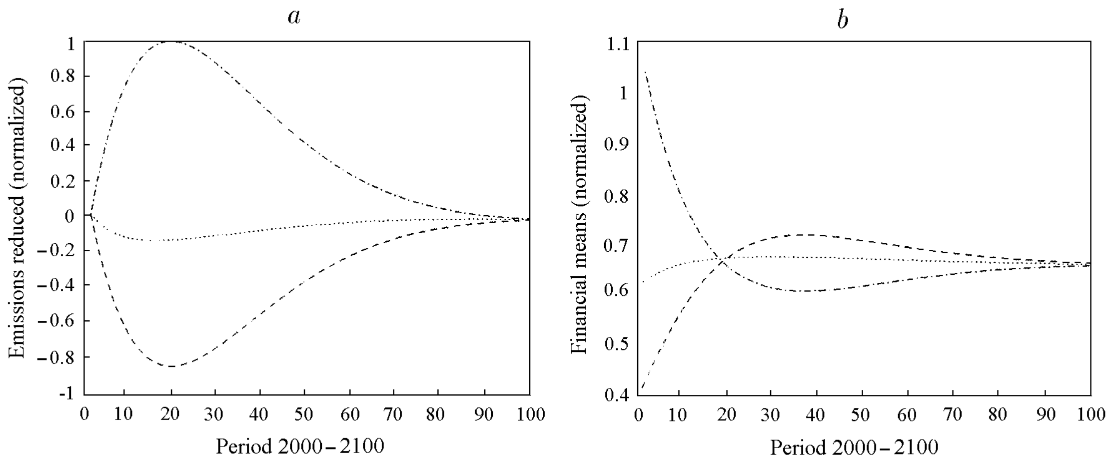

Figure 2. Influence of the memory parameter on the emissions reduced (a) and on the financial means (b), data of table 2, $\varphi = (30, 30, 30)^T$.

We regard the behaviour between three actors who have not yet reached the limiting value of Kyoto at the beginning of the time period, indicated by a normalized value of $(-0.1, -0.1, -0.1)$ for each actor (actor 1: $\cdot$, actor 2: $-\cdot$ and player 3: $--$, in the figures 1, 2). The em-matrix has positive and negative entries which means that we deal with *cooperative* and *competitive* behaviour. The memory parameter might be 1, the growth parameter $1/60$ for each actor. Every actor starts with financial means of 30, 20 and 10 financial units, respectively. Each of the actors has a budget of 60 financial means at his disposal. We observe the strong oscillation of the curves. For example, with his emissions reduced the second actor has reached the value fixed in the treaty of Kyoto. After a reduction of the financial means the curve will even fall under the baseline.

The extraordinary case with a growth parameter vector $(30, 30, 30)$ stands for a scenario where every actor reaches the demanded value.

Nevertheless, because of the nonlinear structure of the dynamics we can not guarantee that such a value exists in general.

Table 2

Data of the TEM model

Player i	$E_i(0)$	$M_i(0)$	M_i^*	φ	λ	em-matrix		
1	0	0.3	1	30	0.01	1	−0.5	−0.5
2	0	0.5	1	30	0.01	−0.5	1	−0.5
3	0	0.2	1	30	0.01	−0.5	−0.5	1

With the numerical results, we can get an intuition for the situation in which developing countries are expanding their energy consumption.

4. A short remark about controllability

Under the simplifying conditions

$$\sum_{j=1}^{n} em_{ij}(t)\widehat{M_j}(t) = 0, \quad \widehat{M_i}(t)[M_i^* - \widehat{M_i}(t)] = 0, \quad i \in \{1, \dots, n\}$$

we are able to determine the fixed points $\widehat{E}, \widehat{M}$ of the dynamical system given by (2. 1). The fixed points are steady states and have no time-dependence.

Additionally, regarding the Jacobi-matrix of the right-hand side for the special case $em_{ij}(t) = em_{ij}^*$, where the economic relationship is constant over a long period, we get

$$\begin{pmatrix}
1 & 0 & \dots & 0 & em_{11} & em_{12} & \dots & em_{1n} \\
0 & 1 & & \vdots & em_{21} & em_{22} & \dots & \dots \\
\vdots & & \ddots & & \vdots & & & \vdots \\
0 & 0 & 0 & 1 & em_{n1} & em_{n2} & \dots & em_{nn} \\
0 & 0 & 0 & 0 & 1 - \lambda_1 M_1^* \widehat{E}_1 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 1 - \lambda_2 M_2^* \widehat{E}_2 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\
0 & \dots & \dots & 0 & 0 & \dots & & 1 - \lambda_n M_n^* \widehat{E}_n
\end{pmatrix}$$

and the *eigenvalues*: $\lambda_1^* = \dots = \lambda_n^* = 1$, $\lambda_{n+j}^* = 1 - \lambda_j^* M_j \widehat{E}_j(t)$ for $j \in \{1, \dots, n\}$. We see that the fixed points under the simplifying conditions mentioned above are not attractive. For further studies to this field see the fundamental contributions of [5] and [6].

5. The problem of controllability

In order to formulate the abstract controllability problem, we prefer the following notation of a system of difference equations:

$$x_i(t+1) = x_i(t) + f_i(x(t), u(t)) \tag{5. 1}$$

for $i \in \{1, \dots, n\}$ and $t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here $x_i : \mathbb{N}_0 \to \mathbb{R}^{l_i}$ and $u_i : \mathbb{N}_0 \to \mathbb{R}^{m_i}$ ($i \in \{1, \dots, n\}$) are state and control vector functions, respectively, which are the components of vector functions

$$x(t) = (x_1(t), \ldots, x_n(t)),$$
$$u(t) = (u_1(t), \ldots, u_n(t)), \quad t \in \mathbb{N}_0.$$

Furthermore,

$$f_i : \prod_{j=1}^{n} \mathbb{R}^{l_j} \times \prod_{j=1}^{n} \mathbb{R}^{m_j} \to \mathbb{R}^{l_j}$$

are given vector functions for $i \in \{1, \ldots, n\}$. Additionally, we assume for every $i \in \{1, \ldots, n\}$ non-empty sets $X_i \subseteq \mathbb{R}^{l_i}$ and $U_i \subseteq \mathbb{R}^{m_i}$ to be given, and require control conditions

$$u_i(t) \in U_i \quad \text{for all} \quad i \in \{1, \ldots, n\} \quad \text{and} \quad t \in \mathbb{N}_0 \tag{5. 2}$$

as well as state constraints

$$x_i(t) \in X_i \quad \text{for all} \quad i \in \{1, \ldots, n\} \quad \text{and} \quad t \in \mathbb{N}_0. \tag{5. 3}$$

Finally, we require initial conditions of the form

$$x_i(0) = x_{0i} \quad \text{for} \quad i \in \{1, \ldots, n\} \tag{5. 4}$$

where $x_{0i} \in X_i$ $(i \in \{1, \ldots, n\})$ are given.

If one chooses n control functions $u_i : \mathbb{N}_0 \to U_i$ $(i \in \{1, \ldots, n\})$, then, by (5. 1) and (5. 4), n state vector functions $x_i : \mathbb{N}_0 \to \mathbb{R}^{l_i}$ are uniquely determined, i. e., we are able to compare several scenarios. Now we make the following assumptions:

1. For every $i \in \{1, \ldots, n\}$ the zero vector Θ_{m_i} of $\mathbb{R}^{m_i}$ belongs to U_i.
2. The nonlinear system

$$f_i(\widehat{x}_1, \ldots, \widehat{x}_n, \Theta_{m_1}, \ldots, \Theta_{m_n}) = \Theta_{l_i} \quad \text{for} \quad i \in \{1, \ldots, n\} \tag{5. 5}$$

has at least one solution

$$\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n) \in \prod_{j=1}^{n} X_j.$$

Under these assumptions we formulate the *problem of controllability* as follows:
Let $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n) \in \prod_{j=1}^{n} X_j$ be a solution of (5. 5). We are looking for control functions $u_i : \mathbb{N}_0 \to \mathbb{R}^{m_i}$ $(i \in \{1, \ldots, n\})$ and some $N \in \mathbb{N}_0$ such that the conditions (5.1) – (5.4) are satisfied as well as

$$u_i(t) = \Theta_{m_i} \quad \text{and} \quad x_i(t) = \widehat{x}_i, \quad i \in \{1, \ldots, n\},\ t \in \mathbb{N},\ t \geq N. \tag{5. 6}$$

In other words: Given an initial state $x_0 \in \prod_{j=1}^{n} X_j$ of the dynamical system under consideration, find control functions $u_i : \mathbb{N}_0 \to \mathbb{R}^{m_i}$ $(i \in \{1, \ldots, n\})$ which satisfy (5. 2) and steer the system, under the conditions (5. 3), into a steady state of the uncontrolled system whose dynamics is described by

$$x_i(t+1) = x_i(t) + f_i(x(t), \Theta), \quad i \in \{1, \ldots, n\},\ t \in \mathbb{N}_0, \tag{5. 7}$$

where $\Theta = (\Theta_{m_1}, \ldots, \Theta_{m_n})$.

6. An iterative solution

For some $t \in \mathbb{N}_0$ we assume vectors

$$x_i(t) \in X_i, \quad i \in \{1, \ldots, n\}$$

to be given. For $t = 0$ we choose

$$x_i(0) = x_{0i}, \quad i \in \{1, \ldots, n\}$$

with $x_{0i} \in X_i$ being the initial values in (5. 4).

For every vector $u \in \prod_{j=1}^{n} \mathbb{R}^{m_j}$ we define

$$x_i(u)(t+1) := x_i(t) + f_i(x(t), u), \quad i \in \{1, \ldots, n\}, \tag{6. 1}$$

where $x(t) = (x_1(t), \ldots, x_n(t))$, and

$$a_i^t(u) := \|x_i(u)(t+1) - \widehat{x}_i\|_2^2 + \|u_i\|_2^2, \quad i \in \{1, \ldots, n\}. \tag{6. 2}$$

Here, $\|\cdot\|_2$ denotes the Euclidean norm. For every $i \in \{1, \ldots, n\}$ we consider the function

$$a_i^t : \prod_{j=1}^{n} \mathbb{R}^{m_j} \to \mathbb{R}_+ \tag{6. 3}$$

as payoff function of the i-th actor. This actor is regarded as the i-th player of a game in which he has the set U_i at his disposal as the set of strategies by which he tries to control the game. However, the players are linked by the set

$$Z_t := \{u \in \prod_{j=1}^{n} U_j \,|\, x_i(u)(t+1) \in X_i \quad \text{for all} \quad i \in \{1, \ldots, n\}\} \tag{6. 4}$$

of feasible controls. Every player intends to minimize the value $a_i^t(u)$ of his own payoff function. This value depends on *all* controls $u_1, \ldots, u_n$ and therefore cannot be determined by the i-th player alone. Let us comprise the individual payoff functions by putting

$$\varphi_t(u) := \sum_{i=1}^{n} a_i^t(u), \quad u \in Z_t. \tag{6. 5}$$

Then, the players have to solve the following
Problem: Find a global minimizer $u^t \in Z_t$ of φ_t, i. e.,

$$\varphi_t(u^t) \le \varphi_t(u) \quad \text{for all} \quad u \in Z_t. \tag{6. 6}$$

We have seen that the fixed points are not attractive. Thus the steady states $(\widehat{E}, \Theta_n) \in \mathbb{R}^{2n}$ are not even attractive. Therefore, we assume the dynamics (2. 1) to be controlled in the following way

$$\begin{aligned} E_i(t+1) &= E_i(t) + \sum_{j=1}^{n} em_{ij}(M_j(t) + u_j(t)), \\ M_i(t+1) &= M_i(t) + u_i(t) - \lambda_i(M_i(t) + u_i(t))(M_i^* - M_i(t) - u_i(t))E_i(t) \end{aligned} \tag{6. 7}$$

for $i \in \{1, \ldots, n\}$ and $t \in \mathbb{N}_0$. This means that the i-th nation has at its disposal a control function $u_i : \mathbb{N}_0 \to \mathbb{R}$ which we assume to satisfy

$$u_i(t) \in U_i, \quad i \in \{1, \ldots, n\}, \quad t \in \mathbb{N}_0, \tag{6. 8}$$

where every U_i is some subset of $\mathbb{R}$ with $0 \in U_i$.

Puting $n_i = 2$, $m_i = 1$, $x_i = (E_i, M_i)$ for $i \in \{1, \ldots, n\}$ and defining functions $f_i : \mathbb{R}^{2n} \times \mathbb{R}^n \to \mathbb{R}^2$ by the right-hand sides of (6. 7), then (6. 7) is of the form (5. 1). Furthermore (6. 8) is of the form (5. 2).

We put

$$X_i = \{(E_i, M_i) \in \mathbb{R}^2 \mid 0 \leq M_i \leq M_i^*\}$$

for $i \in \{1, \ldots, n\}$, then (2. 1) can be rewritten in the form (5. 1). Finally, we put $x_{0i} = (E_{0i}, M_{0i})$ for $i \in \{1, \ldots, n\}$. The assumptions 1) and 2) of Section 5 are also satisfied.

Now, the problem of controllability reads as follows:

Given $(\widehat{E}, \Theta_n) \in \mathbb{R}^{2n}$, find control functions $u_i^t : \mathbb{N}_0 \to \mathbb{R}$ ($i \in \{1, \ldots, n\}$) and some $N \in \mathbb{N}_0$ such that the conditions (6.2), (6.3), (6.5), (6.6) and

$$u_i^t(t) = 0, \quad (E_i(t), M_i(t)) = (\widehat{E}_i, 0),$$

for all $i \in \{1, \ldots, n\}$ and $t \geq N$, are fulfilled.

Several algorithms to solve this problem are investigated in [2].

7. An application of the gradient method

Let us consider the noncooperative case where each actor tries to minimize the value $a_i^t(u)$ from (6. 3). Then we have to solve iteratively at each time step the following problem

$$\frac{\partial a_i^t}{\partial u_i} = 0, \quad i \in \{1, \ldots, n\}.$$

This can be done by an algorithm which was implemented by the first author. A detailed description is presented in [2]. For further numerical techniques, e. g., of second order, see [7] and [8].

Here, we only want to present the numerical results (see Fig. 3) which show that the insertion of the calculated control parameters might be successful. The column entitled by *Kyoto* indicates the emission targets mentioned in Kyoto Protocol.

Table 3

Data for applying the gradient method

Player i	$E_i(0)$	$M_i(0)$	M_i^*	φ	λ	em-matrix			Kyoto
1	-1	20	40	0	0.01	1	-0.8	0.1	1
2	-1	10	40	0	0.02	0.2	1	-0.8	1
3	-1	30	40	0	0.08	-0.1	-0.5	1	1

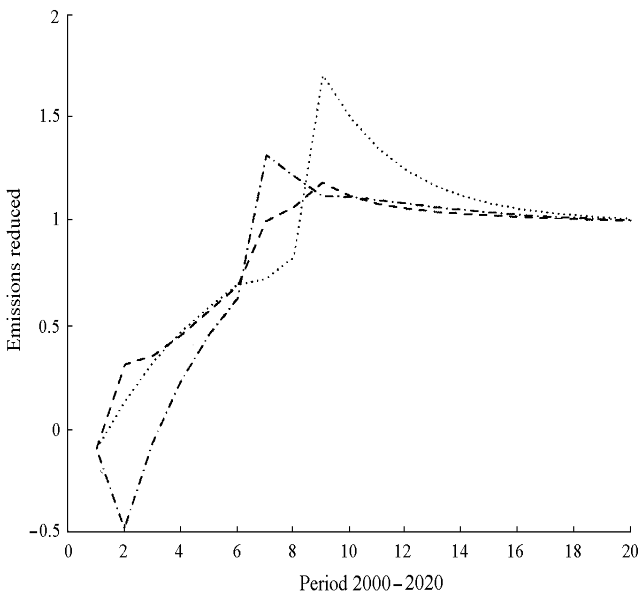

Figure 3. Application of the gradient method.

8. Conclusion

The Framework Convention on Climate Change (FCCC) of Kyoto Protocol demands for reductions in greenhouse gas emissions by the industrialized countries. On the other hand, developing countries are expanding their energy consumption, leading to increased levels of greenhouse gas emissions. The preparation of an optimal management tool in that field requires the possibility to identify, assess and compare several technological options. For that reason, the mathematical TEM model presented in this paper was elaborated. According to the FCCC (Article 4, paragraph 2(a)), control parameters were incorporated which have to be determined iteratively, according to a negotiation process.

9. Acknowledgements

The authors want to acknowledge Werner Krabs and Yurii Shokin for their critical remarks and encouraged motivations.

References

[1] Oberthür S. The Kyoto Protocol. Berlin: Springer Verlag, 1999.

[2] Pickl S. Der τ-value als Kontrollparameter. Modellierung und Analyse eines Joint-Implementation Programmes mithilfe der kooperativen dynamischen Spieltheorie und der diskreten Optimierung: Dissertation TU Darmstadt, 1998. Aachen: Shaker Verlag, 1999.

[3] Scheffran J. Strategic Defense, Disarmament and Stability. Dissertation. Univ. Marburg. Marburg, 1989.

[4] Krabs W. Mathematische Modellierung. Stuttgart: B. G. Teubner, 1997.

[5] Amann H. Gewöhnliche Differentialgleichungen. Berlin, N. Y.: de Gruyter, 1983.

[6] Krause U., Nesemann T. Differentialgleichungen und diskrete dynamische Systeme. Stuttgart: B. G. Teubner, 1999.

[7] Gaffke N., Heiligers B. Algorithms for optimal design with applications to multiple polynomial regression // Metrika. 1995. Vol. 42. P. 173–190.

[8] Spellucci P. Numerische Verfahren der nichtlinearen Optimierung. Basel, Boston, Berlin: Birkhäuser Publ. House, 1993.

[9] Domschke W., Drexl A. Einführung ins Operations Research. Berlin: Springer Verlag, 1998.

[10] Feichtinger G., Hartl R. F. Optimale Kontrolle ökonomischer Prozesse. Berlin, N. Y.: de Gruyter, 1986.

[11] Haasis H.-D. Betriebliche Umweltökonomie, Bewerten, Optimieren, Entscheiden. Berlin: Springer Verlag, 1996.

[12] Ipsen D., Rösch R., Scheffran J. Cooperation and Global Climate Policy — Potentialities and Limitations // Interdisziplinäre Arbeitsgruppe Naturwissenschaft, Technik und Sicherheit, Technical Rep. No. 5. Darmstadt, 1999.

[13] Krabs W., Pickl S. Controllability of a time-discrete dynamical system with the aid of the solution of an approximation problem // J. of Control Theory and Cybernetics (to appear).

[14] La Salle J. P. The Stability and Control of Discrete Processes. Berlin: Springer Verlag, 1986.

[15] Pickl S. The τ-value as a control parameter in a Joint-Implementation program: Lecture // Intern. Kolloquium über Anwendung der Mathematik zum Gedächtnis an Lothar Collatz, 1997.

[16] Joint-Implementation: Stand und Perspektiven aus Sicht von Politik, Industrie und Forschung / O. Rentz, M. Wietschel, W. Fichtner, A. Ardone (Eds). Frankfurt a. M. et. al.: Verlag Peter Lang, 1996.

[17] Pickl S., Scheffran J. Control and game theoretic assessment of climate change — options for Joint-Implementation // Proc. of the Intern. Conf. on Transition to Advanced Market Institutions and Economies. Warschau, 1997.

[18] Pezzey J. Analysis of unilateral CO_2 control in the European Community and OECD // Energy J. 1992. No. 13.

[19] Tijs S. H., Driessen T. S. H. Game theory and cost allocation problems // Management Sci. 1986. No. 32.

[20] Weber G.-W. Generalized Semi-Infinite Optimization and Related Topics: Habilitation thesis. TU Darmstadt, 1999.

Received for publication August 15, 2000