AN ALGORITHMIC APPROACH BY LINEAR PROGRAMMING PROBLEMS IN GENERALIZED SEMI-INFINITE OPTIMIZATION*

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Обобщенная полубесконечная задача оптимизации возникает в многочисленных инженерных приложениях и проблемах механики. В настоящей работе эта задача исследуется в следующей формулировке

$$\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v) \qquad \begin{cases} \text{Минимизировать } f(x) \text{ на } M_{\mathcal{SI}}[h,g], \text{ где} \\ M_{\mathcal{SI}}[h,g] := \{x \in I\!R^n \mid h_i(x) = 0 \ (i \in I),\ g(x,y) \geq 0 \ (y \in Y(x))\}. \end{cases}$$

Здесь I — конечно, а возможно бесконечное множество Y задано конечным набором равенств u_k и неравенств v_ℓ. Дополнительно предполагается, что множество $M_{\mathcal{SI}}$ компактно и объединение множеств $Y(x)$ может быть задано конечным набором ограничений. При этих условиях в окрестности каждого элемента множества $M_{\mathcal{SI}}$ задача $\mathcal{P}_{\mathcal{SI}}$ может быть с любой точностью аппроксимирована линейной задачей оптимизации $\mathcal{P}^{\text{lin}}_{\mathcal{F}}$. Такие линейные аппроксимации с конечным числом ограничений порождают итерационную проследовательность, содержащую подпоследовательность, сходящуюся к решению глобальной проблемы $\mathcal{P}_{\mathcal{SI}}$. В будущем на основе этой процедуры может быть разработан алгоритм.

1. Introduction

In the last years, various problems from engineering and mathematics have made *generalized semi-infinite optimization* become an interesting and fruitful field of research. For example, motivating problems of the following kinds may under suitable assumptions be stated as generalized semi-infinite ($\mathcal{GSI}$) optimization problems:

— *optimizing the layout of a special assembly line* (see [14, 17]);
— *maneuverability of a robot* (see [2, 11, 15]);
— *time minimal heating or cooling of a ball of some homogeneous material* (time optimal control; see [15, 18]);
— *reverse Chebychev approximation* (see [7, 11, 15]);

*The authors are responsible for possible misprints and the quality of translation.

— *structure and stability in optimal control of an ordinary differential equation* (see [28]).

Now, our $\mathcal{GSI}$ problems have the following form:

$$\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)\left\{\begin{array}{l}\text{Minimize} \quad f(x) \qquad \text{on } M_{\mathcal{SI}}[h,g], \quad \text{where} \\ \qquad M_{\mathcal{SI}}[h,g] := \{\, x\in I\!R^n \mid\ h_i(x)=0 \ \ (i\in I),\ \ g(x,y)\geq 0 \ \ (y\in Y(x))\,\}.\end{array}\right.$$

The *semi-infinite* character comes from the perhaps infinite number of elements of $Y = Y(x)$, while the *generalized* character is due to the x-dependence of $Y(x)$ $(x\in I\!R^n)$. These latter index sets are supposed to be feasible sets in the sense of finite ($\mathcal{F}$) optimization, i. e., they are defined by *finitely* many inequality constraints, besides the finite number of inequality constraints:

$$Y(x) \;=\; M_{\mathcal{F}}[u(x,\cdot),v(x,\cdot)] := \{\, y\in I\!R^q \mid u_k(x,y)=0 \ (k\in K),\ v_\ell(x,y)\geq 0 \ (\ell\in L)\} \\ (x\in I\!R^n).$$

Let $h=(h_i)_{i\in I}$, $u=(u_k)_{k\in K}$ and $v=(v_\ell)_{\ell\in L}$ comprise the component functions $h_i : I\!R^n\to I\!R$, $i\in I := \{1,\dots,m\}$, $u_k : I\!R^n\times I\!R^q\to I\!R$, $k\in K := \{1,\dots,r\}$, and $v_\ell : I\!R^n\times I\!R^q\to I\!R$, $\ell\in L := \{1,\dots,s\}$, respectively. We assume that $f : I\!R^n\to I\!R$, $g : I\!R^n\times I\!R^q\to I\!R$, $h_i(i\in I)$, $u_k\,(k\in K)$ and $v_\ell\,(\ell\in L)$ are C^1-functions (continuously differentiable). For each C^1-function, e. g. for f, $Df(x)$ denotes the row-vector of the first order partial derivatives $\dfrac{\partial}{\partial x_\kappa}f(x)$ $(\kappa\in\{1,\dots,n\};\ x\in I\!R^n)$, while $D^Tf(x)$ is the correponding notation as a column. Let, e. g., $D_xg(x,y)$, $D_yg(x,y)$, analogously comprise the coordinate functions $\dfrac{\partial}{\partial x_\kappa}g(x,y)$ and $\dfrac{\partial}{\partial y_\sigma}g(x,y)$, respectively.

Provided that the so-called *Reduction Ansatz* holds, meaning some nondegeneracy for the minima of functions $g(x,\cdot)|\,Y(x)$, then our problem $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ can locally be represented as a problem from *finite* ($\mathcal{F}$) optimization (cf. [5, 24, 31]). Hence, under the very strong assumption of that Ansatz (approach), our $\mathcal{GSI}$ problem is well understood from both the qualitative and the iterative or numerical viewpoint. In this paper the Reduction Ansatz is *not supposed*.

For the present more general context, first order necessary or sufficient optimality conditions for a local minimum of $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ were presented in [11, 14, 29]. In the paper [29], *two different approaches* were followed, each of them having its own assumptions. While the first one leads to a local (or global) problem representation of $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ as an *ordinary semi-infinite ($\mathcal{OSI}$) optimization problem* $\mathcal{P}^o_{\mathcal{SI}}(f,h,g^0,u^0,v^0)$, the second approach applies auxiliary $\mathcal{GSI}$ optimization problems which are also representable as $\mathcal{OSI}$ ones. In the $\mathcal{OSI}$ problems, the (index) sets of inequality constraints does no longer depend on x. (For the first of these approaches we shall give a brief sketch of that representation below.)

Based on the problem representations and optimality conditions, different iteration procedures are worked out in [30]. For further numerical approaches in generalized semi-infinite optimization we refer to [7], and to the branch and bound approach given in [19] (related research is given in [22]). Hereby, we extend our result to cases of *one* assumptions *less*, generalizing our approach.

The present paper, however, is founded in a *third approach*, which consists in a problem approximation that will turn out to be very natural. Based on its assumptions, this approach does no longer fully need problem transformations into $\mathcal{OSI}$ problems, but it is based on a convexification and selection technique with respect to $Y(x)$, and on local linearizations of functions. This technique leads to approximative *finite* and *linear optimization problems.* Finally, we shall be able to formulate and prove a *convergence theorem* for this new iteration procedure.

Our first basic assumptions will impose conditions on the sets $Y(x)$. Hereby, we concentrate on elements $x \in \overline{\mathcal{U}^0}$, where $\mathcal{U}^0 \subset I\!R^n$ is a **bounded** open set, $\overline{\mathcal{U}^0}$ denoting the closure of $\mathcal{U}^0$, and where $M_{\mathcal{SI}}[h,g] \cap \mathcal{U}^0 \neq \emptyset$. This set may be a neighbourhood of some given or expected local minimum, i. e., it reflects a *local* study. At the end of this section we shall explain one further assumption which we make for $\overline{\mathcal{U}^0}$, namely on being a **manifold with generalized boundary** [8], where, moreover, the boundary is *piecewise linear* and *transversally intersecting* $M_{\mathcal{SI}}[h,g]$. (Below, we shall come to these properties in greater detail.) In the special case of a *global* study, $\mathcal{U}^0$ could also be a neighbourhood of the whole feasible set $M_{\mathcal{SI}}[h,g]$. (Lateron, we shall also use the notion *local* with respect to much smaller sets.)

ASSUMPTION $A_{\mathcal{U}^0}$ (Boundedness): The set $\cup_{x\in\overline{\mathcal{U}^0}} Y(x)$ is bounded.

Because of the continuity of u, v, and $\overline{\mathcal{U}^0}$ being compact, it easily follows that this boundedness condition is equivalent with the **compactness** of $\cup_{x\in\overline{\mathcal{U}^0}} Y(x)$. Hence, Assumption $A_{\mathcal{U}^0}$ may be regarded as a **compactness assumption**.

For each $\overline{x}^1 \in M_{\mathcal{SI}}[h,g]$, $\overline{x}^2 \in I\!R^n$, $\overline{y} \in M_{\mathcal{F}}[u(\overline{x}^2,\cdot), v(\overline{x}^2,\cdot)]$ we denote the corresponding sets of active inequality constraints as follows:

$$Y_0(\overline{x}^1) := \{\, y \in Y(\overline{x}^1) \mid g(\overline{x}^1, y) = 0 \,\}, \tag{1.1a}$$
$$L_0(\overline{x}^2,\overline{y}) := \{\, \ell \in L \mid v_\ell(\overline{x}^2,\overline{y}) = 0 \,\}. \tag{1.1b}$$

DEFINITION 1.1. *Let points $\overline{x} \in I\!R^n$, $\overline{y} \in Y(\overline{x})$ be given. We say that the* **linear independence constraint qualification**, *in short:* **LICQ**, *holds at $\overline{y}$ as an element of the feasible set $M_{\mathcal{F}}[u(\overline{x},\cdot), v(\overline{x},\cdot)]$, if the vectors*

$$D_y u_k(\overline{x},\overline{y}),\ k \in K, \quad D_y v_\ell(\overline{x},\overline{y}),\ \ell \in L_0(\overline{x},\overline{y})$$

(considered as a family) are linearly independent.

The **linear independence constraint qualification (LICQ)** *is said to hold for $M_{\mathcal{F}}[u(\overline{x},\cdot), v(\overline{x},\cdot)]$, if LICQ is fulfilled for all $y \in Y(\overline{x})$.*

ASSUMPTION $B_{\mathcal{U}^0}$ (LICQ): LICQ holds for all sets $M_{\mathcal{F}}[u(x,\cdot), v(x,\cdot)]$ $(x \in \overline{\mathcal{U}^0})$.

Now, we may state that the set $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$, being representable in the sense of $\mathcal{OSI}$, is also **compact** (cf. also [29, 30]).

In view of our iterative concept with its convergence theorem, the next assumption on *linearity* and *convexity* is made without loss of generality. On the one hand, that assumption will simplify the topological considerations of our iteration procedure. On the other hand, in one part of our explanations it will allow some *exactness* with respect to y, where otherwise a linearization of u_k $(k \in K)$, v_ℓ $(\ell \in L)$ would only lead to set *approximations.*

ASSUMPTION C:

(i) (Affine linearity): There exist functions $a_k \in (C^1(I\!R^n, I\!R))^n$, $b_k \in C^1(I\!R^n, I\!R)$ $(k \in K)$, *such that*

$$u_k(x,y) = a_k^T(x)y + b_k(x) \quad (x \in I\!R^n,\ k \in K).$$

(ii) (Convexity): The functions $v_\ell(x,\cdot): I\!R^q \to I\!R$ $(\ell \in L)$ *are* convex.

Hereby, we understand $a_k^T(x)$ as the row vector corresponding to the column vector $a_k(x)$ $(x \in I\!R^n,\ k \in K)$.

The Assumptions $\mathrm{A}_{\mathcal{U}^0}$, $\mathrm{B}_{\mathcal{U}^0}$ give us the opportunity locally in a smooth (C^1) way to linearize each of the sets $Y(x) = M_{\mathcal{F}}[u(x,\cdot), v(x,\cdot)]$, $x \in \overline{\mathcal{W}^0}$, where $\mathcal{W}^0$ is some small bounded open neighbourhood of $\overline{\mathcal{U}^0}$), by means of a *finite* number of local C^1*-diffeomorphisms* $\phi_x^j : \overline{\mathcal{U}_2^j} \to \overline{\mathcal{C}_2^j}$, $j \in J := \{1, \ldots, s\}$. These diffeomorphisms locally take the variable y to the new variable z. Hence, $Y(x)$ is a compact *manifold with generalized boundary* (cf. [8, 13]). Hereby, the parameter x is an element of $\overline{\mathcal{W}^0} \cap \overline{\mathcal{C}_1^j}$, where $\mathcal{C}_1^j$ is an open cube ($j \in J$), and we have $\overline{\mathcal{W}^0} \subseteq \cup_{j \in J} \overline{\mathcal{C}_1^j}$. Moreover, the sets $\phi_x^j(\overline{\mathcal{U}_2^j}) = \overline{\mathcal{C}_2^j}$ are also closures of open (q-dimensional) cubes $\mathcal{C}_2^j$ ($x \in \overline{\mathcal{W}^0} \cap \overline{\mathcal{C}_1^j}$, $j \in J$). In this way, $Y(x)$ becomes replaced by a finite number of closed (relative) cubes Z^j ($j \in J$) lying in the linear subspace $\{0_r\} \times I\!R^{q-r}$ of $I\!R^q$. These new index sets do no longer locally depend on x. This means that with the help of *local linearizations* we have equivalently expressed our $\mathcal{GSI}$ problem $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ as an $\mathcal{OSI}$ problem $\mathcal{P}^o_{\mathcal{SI}}(f,h,g^0,u^0,v^0)$. Here, $g^0 = (g_j^0)_{j \in J}$ comes from gluing the locally defined function $g(x, (\phi_x^j)^{-1}(z))$ with 0 ($j \in J$), using a partition of unity [6, 9]. Lateron, we shall specify our choice of the diffeomorphisms ϕ_x^j ($j \in J$) a bit. For more details we refer to the paper [29] with its special notations.

DEFINITION 1.2. *Let a point $\overline{x} \in M_{\mathcal{SI}}[h,g]$ and local C^1-linearizations $\phi_{\overline{x}}^j$ of $Y(\overline{x})$ be given. We say that the* **extended Mangasarian-Fromovitz constraint qualification**, *in short:* **EMFCQ**, *holds at $\overline{x}$, if the following two conditions are satisfied:*

EMF1. *The vectors $Dh_i(\overline{x})$, $i \in I$, (considered as a family) are linearly independent.*

EMF2. *There exists a vector $\zeta \in I\!R^n$ such that*

$$D h_i(\overline{x})\,\zeta \;=\; 0 \quad \text{for all } i \in I, \tag{1.2a}$$

$$D_x g_j^0(\overline{x}, z)\,\zeta \;>\; 0 \quad \text{for all } z \in I\!R^q,\ j \in J, \qquad \text{with } (\phi_{\overline{x}}^j)^{-1}(z) \in Y_0(\overline{x}). \tag{1.2b}$$

The **extended Mangasarian-Fromovitz constraint qualification (EMFCQ)** *is said to hold for $M_{\mathcal{SI}}[h,g]$ on $\overline{\mathcal{U}^0}$, if EMFCQ is fulfilled for all $x \in M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$.*

With the help of the chain rule, we see that (1.2b) means

$$\left.\begin{array}{c} \Big(D_x g(\overline{x}, (\phi_{\overline{x}}^j)^{-1}(z)) \;+\; D_y g(\overline{x}, (\phi_{\overline{x}}^j)^{-1}(z)) D_x \hat{y}^j(\overline{x}, z) \Big)\,\zeta \;\;>\;\; 0 \\[2mm] \text{for all } z \in I\!R^q,\ j \in J, \ \ \text{with } \ \hat{y}^j(\overline{x}, z) := (\phi_{\overline{x}}^j)^{-1}(z) \in Y_0(\overline{x}). \end{array}\right\} \tag{1.2b'}$$

For more information on EMFCQ and its versions we refer to [11, 12, 29, 30].

After our assumptions on the (feasible) sets $Y(x)$ on the "lower stage", now we make the following assumption for the feasible set $M_{\mathcal{SI}}[h,g]$ on the "upper stage":

ASSUMPTION $D_{\mathcal{U}^0}$ (EMFCQ): EMFCQ holds for $M_{\mathcal{SI}}[h,g]$ on $\overline{\mathcal{U}^0}$.

Under the basic Assumptions $\mathrm{A}_{\mathcal{U}^0}$, $\mathrm{B}_{\mathcal{U}^0}$, the Assumption $\mathrm{D}_{\mathcal{U}^0}$ guarantees that inside of a small neighbourhood $\mathcal{W}$ of $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ the feasible set $M_{\mathcal{SI}}[h,g]$ is a **topological (Lipschitzian) manifold** (cf. [12, 30]). Without loss of generality, we may say: $\mathcal{W} = \mathcal{W}^0$. Furthermore, let us from now on without loss of generality think that $\overline{\mathcal{U}^0}$ is a *manifold with generalized boundary*, fulfilling LICQ and having **transversal** *intersection* with $M_{\mathcal{SI}}[h,g]$. Here, in the presence of maybe infinitely many active inequality constraints $y \in Y_0(x)$ on $g(x,\cdot)$, this transversality can be accomplished in the following way (and sense).

We denote the relative boundary of $M_{\mathcal{SI}}[h,g]$ in $M[h] := \{x \in I\!\!R^n \mid h_i(x) = 0\ (i \in I)\}$ by $\partial M_{\mathcal{SI}}[h,g]$. Now, the parts of the boundary $\partial\mathcal{U}^0$ of $\overline{\mathcal{U}^0}$ which have nonempty intersection with the $(n-m-1)$-dimensional Lipschitzian manifold $\partial M_{\mathcal{SI}}[h,g]$, may locally be given as the $(m+1)$-dimensional (residual) linear span of the *EMF-vector* ζ and of the set $\{\eta^1,\ldots,\eta^m\}$ being a basis of the orthogonal complement of the tangent space $T_xM[h] := \{\rho \in I\!\!R^n \mid Dh_i(x)\rho = 0\ (i \in I)\}$ at x. Hereby, the point x, where that span is attached, is the necessarily locally unique element in that intersection of the (creased or Lipschitzian) manifolds $\partial\mathcal{U}^0$ and $\partial M_{\mathcal{SI}}[h,g]$ (respectively).

Hence, close to x, $\partial\mathcal{U}^0$ looks like a linear hyperplane. Inwardly, on the relative interior of $M_{\mathcal{SI}}[h,g]$, $\partial\mathcal{U}^0$ can be adapted without becoming tangential with the feasible set (tangentially). For more information on transversality we refer to [6, 9]. In the sequel, the generalized (creased) boundary $\partial\mathcal{U}^0$ may even *globally* be composed by linear faces, shrinking (or in a transversal, affinely linear way perturbing and intersecting) $\overline{\mathcal{U}^0}$ otherwise. Then, $\overline{\mathcal{U}^0}$ is a *compact polyhedron (polytope).* Of course, from the viewpoint of the practice, geometrical insights and linear algebra turn out to be helpful in order to construct $\overline{\mathcal{U}^0}$.

For an illustration see Fig. 1, where we also prepares an impression of a slightly *perturbed* feasible set.

Hence, $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ fulfills EMFCQ, too, namely with EMF-vectors ζ^0 in the tangential and (relatively) "inwardly pointing" sense of Definition 1.2. For an illustration we look at a neighbourhood of the point x' in Fig. 1. Hence, we learn from [12, 30] that $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ is also a *compact topological (Lipschitzian) manifold.*

In the sequel, let $||\cdot||_\infty$ stand for the maximum norm in some Euclidean space, e. g., $I\!\!R^n$ or $I\!\!R^q$. We emphasize that our *local linearizations* of sets given above are *exact* representations while our *local linearizations of functions*, which will be used in the next section, are only *approximations.*

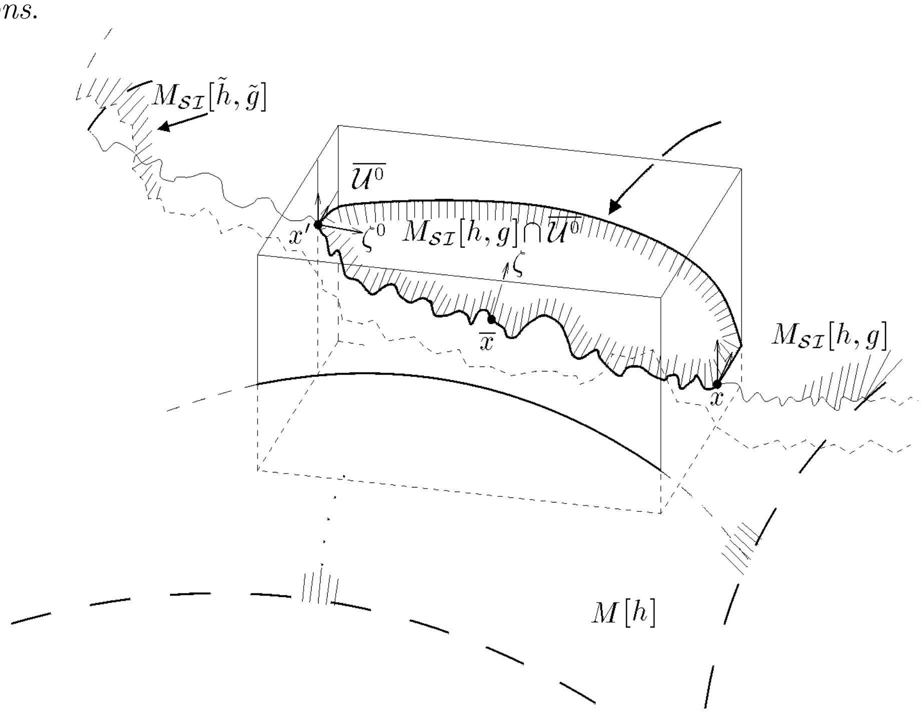

Fig. 1. Transversal intersection of the feasible set $M_{\mathcal{SI}}[h,g]$ with $\overline{\mathcal{U}^0}$ ($m = 1$; an example). The feasible set $M_{\mathcal{SI}}[\tilde{h},\tilde{g}]$, being due to a slight perturbation $(h,g) \to (\tilde{h},\tilde{g})$, is indicated, too.

2. The iteration procedure and its topological background

In order to get a better understanding of our iteration procedure, we present the underlying topology step by step. On the distinguishing of these steps (parts), our (feasible) set approximations and, finally, our convergence proof will be founded.

For our approach, the following *local* consideration plays a central part of motivation.

2.1. Part 1.a: Locally finite approximative problems

Let some open neighbourhood $\mathcal{W}^1$ of $\overline{\mathcal{U}^0}$, points $\overline{x} \in \mathcal{W}^1$, $\overline{y} \in I\!R^q$, open **squares** $\mathcal{S}^1 := \{x \in I\!R^n \mid ||x-\overline{x}||_\infty < \delta^1\}$, $\mathcal{S}^2 := \{y \in I\!R^q \mid ||y-y'||_\infty < \delta^2\}$ $(\delta^1, \delta^2 > 0)$ be given with the properties $\overline{\mathcal{W}^1}, \overline{\mathcal{S}^1} \subseteq \mathcal{W}^0$ and $Y(x) \cap \mathcal{S}^2 \neq \emptyset$ $(x \in \overline{\mathcal{S}^1})$. (We remember that the set $\mathcal{W}^0$ was introduced in Section 1 as some bounded neighbourhood of $\overline{\mathcal{U}^0}$, where for each $x \in \overline{\mathcal{W}^0}$ the set $Y(x)$ can locally be linearized.)

In this *part 1*, let $\overline{y}$ be y'. However, in *part 2*, $\overline{y}$ will be specified in suitable different ways. We replace the inequality constraints $g(x,y) \geq 0$ $(y \in Y(x) \cap \overline{\mathcal{S}^2})$ on $x \in \overline{\mathcal{S}^1}$ by the approximative inequality constraints

$$g^{\mathrm{lin}}_{\overline{y}}(x,y) \;\geq\; 0 \quad (y \in Y(x) \cap \overline{\mathcal{S}^2}), \tag{2.1}$$

where

$$g^{\mathrm{lin}}_{\overline{y}}(x,y) \;:=\; g(\overline{x},\overline{y}) + D_x g(\overline{x},\overline{y})(x-\overline{x}) + D_y g(\overline{x},\overline{y})(y-\overline{y}) \qquad ((x,y) \in I\!R^n \times I\!R^q).$$

Hereby, g has been substituted by its (local) linearization $g^{\mathrm{lin}}_{\overline{y}}$ which is an *affinely linear* function. Now, let us for each $x \in \overline{\mathcal{S}^1}$ introduce the *convex hull* $Co_x := co(Y(x) \cap \overline{\mathcal{S}^2})$ of $Y(x) \cap \overline{\mathcal{S}^2}$, being the smallest convex set $\mathcal{Q} \subseteq I\!R^q$ having $Y(x) \cap \overline{\mathcal{S}^2}$ as a subset. From Carathéodory's theorem (cf. [21, Theorem 17.1] and [16]) we know that Co_x can be represented as follows:

$$Co_x = \{\, \textstyle\sum_{j=1}^{q+1} \lambda_j y^j \mid \lambda_\sigma \in [0,1],\; y^\sigma \in Y(x) \cap \overline{\mathcal{S}^2} \;(\sigma \in \{1,\ldots,q+1\}),\; \sum_{j=1}^{q+1} \lambda_j = 1 \}. \tag{2.2}$$

Of course, all the inequalities from (2.1) remain fulfilled if we replace the (sub)set $Y(x) \cap \overline{\mathcal{S}^2}$ by Co_x. Hence, (2.1) necessarily follows from

$$g^{\mathrm{lin}}_{\overline{y}}(x,y) \;\geq\; 0 \quad (y \in Co_x). \tag{2.3}$$

Reversely, in view of the convex combinations from (2.2) it is easily realized that (2.1) is also sufficient for the inequalities from (2.3). Consequently, (2.1) and (2.3) are *equivalent*.

Now, let us impose one more property on the squares $\mathcal{S}^1$ and $\mathcal{S}^2$. Namely, we assume that their boundaries $\partial\mathcal{S}^1, \partial\mathcal{S}^2$ have **transversal** intersections with the manifolds (with Lipschitzian or generalized boundaries) $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ and $Y(x)$ $(x \in \overline{\mathcal{S}^1})$, respectively. (The intersections $Y(x) \cap \mathcal{S}^2$ $(x \in \overline{\mathcal{S}^1})$ on the "lower stage" were even supposed to be nonempty.)

These transversalities can be achieved by means of shrinking the squares $\overline{\mathcal{S}^\gamma}$ around the center points $\overline{x}$, y' sufficiently much, and by arbitrarily small linear perturbations of the faces of $\overline{\mathcal{S}^\gamma}$ $(\gamma \in \{1,2\})$. Without loss of generality, $\overline{x}$ and y' may remain the centers of $\mathcal{S}^1$ and $\mathcal{S}^2$, respectively. (Here, we also remember the transversal choice of $\overline{\mathcal{U}^0}$ on the "upper stage".)

Of course, a convex hull of some set does not need to reveal a polyhedral structure. However, under our Assumptions $\mathrm{A}_{\mathcal{U}^0}$, $\mathrm{B}_{\mathcal{U}^0}$, C and because of the transversal choice of $\overline{\mathcal{S}^2}$, we have a

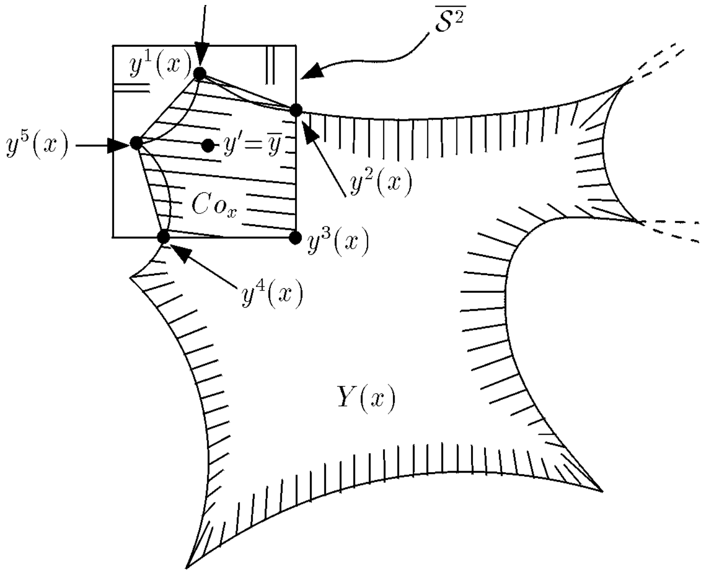

Fig. 2. The index set $Y(x)$, its transversal intersection with the square $\overline{\mathcal{S}^2}$, and Co_x, $x \in \overline{\mathcal{S}^1}$ ($r = 0,\ j^0 = 5;$ an example).

geometrical situation as it is indicated in Fig. 2 where, in particular, Co_x turns out to be *compact* and *polyhedral.* Hence, Co_x is a **polytope**, having a *finite* number j^0 of *vertices* $y^j(x)$ $(j \in \{1, \ldots, j^0\})$.

For Fig. 2 we note that the lower level sets of a convex function is convex [21], and we take into account of the geometrical meaning of LICQ [8, 13]. The notation $y^j(x)$ already indicates the fact that there is a functional dependence of the vertices on x. Indeed, based on Assumption $B_{\mathcal{U}^0}$, on the transversal choice of $\overline{\mathcal{S}^2}$ and with $\overline{\mathcal{S}^1}$ being supposed to be *small* enough and lying in suitable small open neighbourhoods $\mathcal{U}_{\mathcal{S}^1}$, the functions $y^j : x \mapsto y^j(x)$ $(x \in \mathcal{U}_{\mathcal{S}^1};\ j \in \{1, \ldots, j^0\})$ are implicitly given by means of applying the theorem on implicit functions on the following systems of equations. Hereby, these functions turn out to be of class C^1, and we may also state the independence of j^0 on the choice of $x \in \overline{\mathcal{S}^1}$. Now, for each $j \in \{1, \ldots, j^0\}$ our system is

$$\widehat{F}^j(x,y) = 0, \quad \text{given by} \quad \left\{ \begin{array}{rcl} u_1(x,y) &=& 0, \\ &\vdots& \\ u_r(x,y) &=& 0, \\ v_{\ell^1}(x,y) &=& 0, \\ &\vdots& \\ v_{\ell^{p^j}}(x,y) &=& 0, \\ {\xi_1^j}^T(y - {y_1'}^j) &=& 0, \\ &\vdots& \\ {\xi_{q-r-p^j}^j}^T(y - {y_{q-r-p^j}'}^j) &=& 0\,. \end{array} \right. \tag{2.4}$$

Here $L_0(\overline{x}, \overline{y}^j) = \{\ell^1, \ldots, \ell^{p^j}\}$, and $\{y \in I\!R^q \,|\, {\xi_\kappa^j}^T(y - {y_\kappa'}^j) = 0\}$ $(\kappa \in \{1, \ldots, q-r-p^j\})$ stands for the faces of $\overline{\mathcal{S}^2}$ which locally appear around each vertex $\overline{y}^j = y^j(\overline{x})$ $(j \in \{1, \ldots, j^0\})$. The point $(x, y^j(x))$ is the locally unique solution of $\widehat{F}^j(x,y) = 0$. Moreover, the vectors

$\xi_\kappa^j \in \mathbb{R}^p$ $(\kappa \in \{1, \ldots, q-r-p^j\})$ form a basis of the orthogonal complement, and they define the linear half spaces $\mathcal{H}_\kappa^j$ $(\kappa \in \{1, \ldots, q-r-p^j\})$ whose intersection $\cap_{\kappa=1}^{q-r-p^j} \mathcal{H}_\kappa^j$ locally represents the square $\overline{\mathcal{S}^2}$, by

$$\mathcal{H}_\kappa^j := \{y \in \mathbb{R}^q \mid \xi_\kappa^{j\,T} (y - y_\kappa'^{\,j}) \geq 0\}. \tag{2.5}$$

Our implicit C^1-functions $y^j(\cdot)$ $(j \in \{1, \ldots, j^0\})$ with their underlying choice of the vectors $\kappa \in \{1, \ldots, q-r-p^j\}$ may be considered as specifications of the function $\hat{y}^j = \hat{y}$ from Section 1:

$$y^j(x) = \hat{y}^j(x, 0) = (\phi_x^j)^{-1}(0) \quad (x \in \overline{\mathcal{S}^1}). \tag{2.6}$$

Hence, these vertex functions become also involved into the Definition 1.2 of EMFCQ. Now, we may express Co_x $(x \in \overline{\mathcal{S}^1})$ in the following way as a **polytope**:

$$Co_x = \{\textstyle\sum_{j=1}^{j^0} \lambda_j y^j(x) \mid \lambda_\sigma \in [0,1] \qquad (\sigma \in \{1, \ldots, j^0\}), \qquad \sum_{j=1}^{j^0} \lambda_j = 1\} \tag{2.7}$$

(whereby the vertices $y^j(x)$, $j \in \{1, \ldots, j^0\}$, need not to be affinely independent). Hence, we may over $\overline{\mathcal{S}^1}$ (or over the neighbourhood $\mathcal{U}_{\mathcal{S}^1}$) equivalently represent the inequality constraints from (2.3) as

$$g_j(x) := g_{\overline{y}}^{\mathrm{lin}}(x, y^j(x)) \geq 0 \quad (j \in \{1, \ldots, j^0\}). \tag{2.8}$$

We note that the inequality constraint functions can be written as follows:

$$g_j(x) = g(\overline{x}, \overline{y}) + D_x g(\overline{x}, \overline{y})(x - \overline{x}) \qquad + D_y g(\overline{x}, \overline{y})(y^j(x) - \overline{y}) \quad (x \in \mathcal{U}_{\mathcal{S}^1}). \tag{2.8a}$$

In view of the equivalences (2.1) $\Longleftrightarrow$ (2.3) and (2.3) $\Longleftrightarrow$ (2.8) we have arrived at (2.1) $\Longleftrightarrow$ (2.8). This equivalence means a representation (over $\overline{\mathcal{S}^1}$) of the generalized semi-infinite ($\mathcal{GSI}$) inequality constraints from (2.1) by means of the *finite* ($\mathcal{F}$) inequality constraints from (2.8). We underline that this representation is *exact*, not only approximative.

Now, let us turn from the very local to the (in x and y) more global viewpoint.

2.2. Part 1.b: Collecting the locally finite problems

Because of the Assumption $\mathrm{A}_{\mathcal{U}^0}$ on compactness, there exist sufficiently fine and finite open coverings $(\mathcal{S}_\alpha^1)_{\alpha \in \mathsf{A}}$, $\mathsf{A} = \{1, \ldots, \alpha^0\}$, of $\overline{\mathcal{W}^1}$ (hence, of $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$), and $(\mathcal{S}_{\alpha,\beta}^2)_{\beta \in \mathsf{B}^\alpha}$, $\mathsf{B}^\alpha = \{1, \ldots, \beta^\alpha\}$, of $Y(x)$, uniformly in $x \in \overline{\mathcal{S}_\alpha^1}$ $(\alpha \in \mathsf{A})$, consisting of squares as being given in subsection 2.1. Therefore, the shrinking and perturbing of faces may already be performed for $\overline{\mathcal{S}_\alpha^1}$, $\overline{\mathcal{S}_{\alpha,\beta}^2}$ $(\beta \in \mathsf{B}^\alpha,\ \alpha \in \mathsf{A})$ without destroying the covering properties. In particular, we have some *exactness* coming from the equations $Y(x) = \cup_{\beta \in \mathsf{B}^\alpha}(Y(x) \cap \overline{\mathcal{S}_{\alpha,\beta}^2})$ $(x \in \overline{\mathcal{S}_\alpha^1},\ \alpha \in \mathsf{A})$.

With the help of such an underlying covering structure we get (in $\overline{\mathcal{U}^0}$) a (*global*) approximation of $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ by means of (locally) finite optimization problems $\mathcal{P}_{\mathcal{F}}^\alpha(f,h,g_\alpha^\circ)$ on $\overline{\mathcal{S}_\alpha^1}$, where g_α° comprises the constraints $g_{\alpha,\beta,j} := g_j$, $j \in J^{\alpha,\beta} := \{1, \ldots, j_{\alpha,\beta}^0\}$ $(\beta \in \mathsf{B}^\alpha,\ \alpha \in \mathsf{A})$. The corresponding feasible sets are $M_{\mathcal{F}}^\alpha[h, g_\alpha^\circ] := \{x \in \mathbb{R}^n \mid h_i(x) = 0\ (i \in I),\ x \in \overline{\mathcal{S}_\alpha^1},\ g_{\alpha,\beta,j}(x) \geq 0\ (j \in J^{\alpha,\beta},\ \beta \in \mathsf{B}^\alpha)\}$. In other words, that global approximation

$$\mathcal{P}_{\mathcal{F}}^\circ(f,h,g^\circ) \begin{cases} \text{Minimize } \ f(x) \text{ on} \\ M_{\mathcal{F}}^\circ[h,g^\circ] := \{x \in \mathbb{R}^n \mid h_i(x) = 0 \ (i \in I), \\ x \in \overline{\mathcal{S}_\alpha^1}, \quad g_{\alpha,\beta,j}(x) \geq 0 \quad (j \in J^{\alpha,\beta},\ \beta \in \mathsf{B}^\alpha) \quad \text{for some } \alpha \in \mathsf{A}\}, \end{cases}$$

where $g^\circ := (g^\circ_\alpha)_{\alpha \in \mathsf{A}}$ (enumerating all the functions $g_{\alpha,\beta,j}$, $\beta \in \mathsf{B}^\alpha$, $\alpha \in \mathsf{A}$), can be decomposed into the problems $\mathcal{P}^\alpha_{\mathcal{F}}(f,h,g^\circ_\alpha)$ ($\alpha \in \mathsf{A}$). Hereby, we have the representation $M^\circ_{\mathcal{F}}[h,g^\circ] = \cup_{\alpha \in \mathsf{A}} M^\alpha_{\mathcal{F}}[h,g^\circ_\alpha]$. That global problem may also be called a problem from *disjunctive optimization* (cf. [10]). Note that, for simplicity, in the list of functional parameters (f,h,g°), we did not explicitly mention the inequality constraints ${\eta^j_\alpha}^T(x - {x'}^j_\alpha) \geq 0$, $j \in \{1.\ldots,2n\}$, $(\alpha \in \mathsf{A})$ of all these foregoing feasible sets and problems, respectively.

Of course, each of these (only) finitely many problems reveals a much easier structure than our given $\mathcal{GSI}$ problem. Moreover, the approximative character (*perturbations*) is only due to the local linearizations of the form (2.1), while the other changes in modelling were *exact representations*. Hence, in order to make the approximation better, we let the members of the open coverings $(\mathcal{S}^1_\alpha)_{\alpha \in \mathsf{A}}$ and $(\mathcal{S}^2_{\alpha,\beta})_{\beta \in \mathsf{B}^\alpha}$ ($\alpha \in \mathsf{A}$) become arbitrarily small, and the finite cardinalities $|\mathsf{A}|$, $|\mathsf{B}^\alpha|$ ($\alpha \in \mathsf{A}$) become arbitrarily large. Then, for each $\alpha \in \mathsf{A}$, $\beta \in \mathsf{B}^\alpha$, the functions $g^{\mathrm{lin}}_{\overline{y}_{\alpha,\beta}}$ ($y' = \overline{y}_{\alpha,\beta}$ being the center point of $\overline{\mathcal{S}^2_{\alpha,\beta}}$) and g, both being restricted to the closure $\overline{\mathcal{S}_{\alpha,\beta}}$ of the set $\mathcal{S}_{\alpha,\beta} := \mathcal{S}^1_\alpha \times \mathcal{S}^2_{\alpha,\beta}$, become arbitrarily close to each other. This "being close" refers to our (locally uniformly continuous) functions g, $g^{\mathrm{lin}}_{\overline{y}_{\alpha,\beta}}$ and their (locally uniformly continuous) derivatives $Dg(x)$ and $Dg^{\mathrm{lin}}_{\overline{y}_{\alpha,\beta}}(x) \equiv Dg(\overline{x}_\alpha, \overline{y}_{\alpha,\beta})$. To be precise, it is understood in the sense of the corresponding C^1_S**-Whitney topology** on $\overline{\mathcal{S}_{\alpha,\beta}}$. A typical base neighbourhood of g in the sense of that topology is given by the following sets which are due to "controlling" positive valued, continous functions $\varepsilon : \overline{\mathcal{S}_{\alpha,\beta}} \to I\!R$:

$$\mathcal{W}^{\alpha,\beta}_\varepsilon := \{\omega \in C^1(I\!R^{n+p}, I\!R) \mid |\omega(x,y) - g(x,y)| + \sum_{\kappa=1}^{n} |\frac{\partial}{\partial x_\kappa}\omega(x,y) - \frac{\partial}{\partial x_\kappa} g(x,y)| +$$

$$+ \sum_{\sigma=1}^{q} |\frac{\partial}{\partial y_\sigma}\omega(x,y) - \frac{\partial}{\partial y_\sigma} g(x,y)| \; < \; \varepsilon(x,y) \quad ((x,y) \in \overline{\mathcal{S}_{\alpha,\beta}})\}. \tag{2.9}$$

In the sequel, we refer shall always refer to C^1_S-Whitney topologies for spaces of globally defined C^1-functions being restricted to corresponding suitable manifolds, and on the *product-topology* of several such topologies. (For more information see also [6, 9].) From topological investigations in $(\mathcal{G})\mathcal{SI}$ optimization (cf., e.g., [12, 30]) we learn that under the fulfilled condition EMFCQ on $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ and under such close approximations of g by $g^{\mathrm{lin}}_{\overline{y}_{\alpha,\beta}}$ ($\beta \in \mathsf{B}^\alpha$, $\alpha \in \mathsf{A}$), the "perturbed" feasible set $M^\circ_{\mathcal{F}}[h,g^\circ] \cap \overline{\mathcal{U}^0}$ finally lies arbitrarily close to $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$. Therefore, we note that in $\overline{\mathcal{U}^0}$, $M^\circ_{\mathcal{F}}[h,g^\circ]$ is exactly represented by the (finite) union of the $(\mathcal{G})\mathcal{SI}$ feasible sets $M_{\mathcal{SI}}[h, g^{\mathrm{lin},\alpha}] := \{x \in I\!R^n \mid h_i(x) = 0\,(i \in I),\ x \in \overline{\mathcal{S}^1_\alpha},\ g^{\mathrm{lin}}_{\overline{y}_{\alpha,\beta}}(x,y) \geq 0$ $(y \in Y(x) \cap \overline{\mathcal{S}^2_{\alpha,\beta}},\ \beta \in \mathsf{B}^\alpha)\} = M^\alpha_{\mathcal{SI}}[h,g^\circ_\alpha]$ ($\alpha \in \mathsf{A}$). The latter new sets may be considered as coming from slight functional perturbations of our feasible sets $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ with the constraints $g(x,y) \geq 0$ ($y \in Y(x)$) being split up due to different parts $Y(x) \cap \overline{\mathcal{S}^2_{\alpha,\beta}}$ ($\beta \in \mathsf{B}^\alpha$).

This set theoretical approximation means that, finally, $M^\circ_{\mathcal{F}}[h,g^\circ] \cap \overline{\mathcal{U}^0}$ lies in each arbitrarily close neighbourhood $\mathcal{W}'$ of $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$, and that the boundaries of both sets, relatively in $M[h]$, also lie arbitrarily close to each other ([12, 30]).

Of course, for each $x \in \overline{\mathcal{S}^1_\alpha}$ we concentrate on the *discrete* structure of the vertices $y^j_{\alpha,\beta}(x)$ ($j \in J^{\alpha,\beta}$) of $Y(x) \cap \overline{\mathcal{S}^2_\alpha}$, which is of less complexity in comparison with the manifold of points $y\,(= (\phi^j_x)^{-1}(z)) \in Y(x)$. Moreover, we conclude from EMFCQ, necessarily holding for the slighty perturbed topological manifold $M_{\mathcal{SI}}[h, g^{\mathrm{lin},\alpha}] \cap \overline{\mathcal{U}^0}$ with EMF-vectors ζ^0 (cf. Section 1), too, that for each $\alpha \in \mathsf{A}$ also the *finite version* **MFCQ** of EMFCQ is fulfilled at each element $\tilde{x} \in M^{\alpha}_{\mathcal{SI}}[h, g^{\circ}_{\alpha}] \cap \overline{\mathcal{U}^0}$ of the union $M^{\circ}_{\mathcal{F}}[h, g^{\circ}] \cap \overline{\mathcal{U}^0}$ ($\alpha \in \mathsf{A}$ being suitable). In fact, as EMFCQ (being preserved under small perturbations) holds for $M_{\mathcal{SI}}[h, g^{\mathrm{lin},\alpha}] \cap \overline{\mathcal{U}^0}$ ($= M^{\alpha}_{\mathcal{F}}[h, g^{\circ}_{\alpha}] \cap \overline{\mathcal{U}^0}$) (see [12, 30]) and as the corresponding functions $g_{\alpha,\beta,j}$ and $g^0_j(\cdot, 0)$ are (locally) arbitrarily C^1_S-close ($\beta \in \mathsf{B}^{\alpha}$), then *MFCQ* necessarily follows for $M^{\alpha}_{\mathcal{F}}[h, g^{\circ}_{\alpha}] \cap \overline{\mathcal{U}^0}$. (Therefore, a continuity argumentation on a small perturbation is made again, in particular looking at (1.2b,b') and with some simplified enumeration by j.)

Namely, using notations analogously as in Subsection 2.1 and referring to the transversally chosen faces $\{x \in \overline{\mathcal{S}^1_{\alpha}} \mid {\eta^j_{\alpha}}^T (x - {x'_{\alpha}}^j) = 0\} \subseteq \partial \mathcal{S}^1_{\alpha}$ ($j \in \{1, \ldots, 2n\}$) with inwardly pointing normal vectors η^j_{α} ($j \in \{1, \ldots, 2n\}$), here, this condition *MFCQ* means

MF1. *The vectors $Dh_i(\tilde{x})$, $i \in I$, (considered as a family) are linearly independent.*

MF2. *There exists a vector $\zeta \in I\!R^n$ such that*

$$Dh_i(\tilde{x})\,\zeta = 0 \quad \textit{for all } i \in I,$$

$$D_x g_{\alpha,\beta,j}(\tilde{x})\,\zeta = \big(D_x g(\overline{x}_{\alpha}, \overline{y}^j_{\alpha,\beta}) + D_y g(\overline{x}_{\alpha}, \overline{y}^j_{\alpha,\beta}) D y^j_{\alpha,\beta}(\tilde{x})\big)\,\zeta > 0$$

$$\textit{for all } j \in J^{\alpha,\beta}_0(\tilde{x}), \quad \beta \in \mathsf{B}^{\alpha},$$

$${\eta^j_{\alpha}}^T \zeta > 0 \qquad \textit{for all } j \in {J'_0}^{\alpha}(\tilde{x}),$$

where

$$J^{\alpha,\beta}_0(\tilde{x}) := \{j \in J^{\alpha,\beta} \mid g_{\alpha,\beta,j}(\tilde{x}) = 0\},$$

$${J'_0}^{\alpha}(\tilde{x}) := \{j \in \{1, \ldots, 2n\} \mid {\eta^j_{\alpha}}^T (x - {x'_{\alpha}}^j) = 0\}.$$

Hence, $M^{\alpha}_{\mathcal{F}}[h, g^{\circ}_{\alpha}] \cap \overline{\mathcal{U}^0}$ is a **topological manifold** ($\alpha \in \mathsf{A}$). This fulfillment of MFCQ will be valuable in next Subsection (part 2).

However, our implicit *functions* $y^j_{\alpha,\beta}(\cdot)$ would still remain to be determined, and by inserting them into $g^{\mathrm{lin}}_{\overline{y}_{\alpha,\beta}}$ (cf. (2.8)) the affine linearity of our inequality constraints (see (2.3)) gets lost ($\beta \in \mathsf{B}^{\alpha}$, $\alpha \in \mathsf{A}$). Therefore, in the next subsection we shall overcome these disadvantages with the help of further approximative (local) linearizations.

2.3. Part 2: Locally linear approximative problems

Based on the first (in $\overline{\mathcal{U}^0}$ global) approximation $\mathcal{P}^{\circ}_{\mathcal{F}}(f, h, g^{\circ})$ with its finite subproblems $\mathcal{P}^{\alpha}_{\mathcal{F}}(f, h, g^{\circ}_{\alpha})$ and underlying square structure, we perform one more perturbation by means of local linearizations in the variable x. Therefore, we specify the parameter $\overline{y}$ (cf. (2.1)) by means of the vertices $\overline{y}^j_{\alpha,\beta} = y^j_{\alpha,\beta}(\overline{x}_{\alpha})$ ($\beta \in \mathrm{B}^{\alpha}$, $\alpha \in \mathsf{A}$) and we set for all $x \in I\!R^n$, $i \in I$, $j \in J^{\alpha,\beta}$, $\beta \in \mathsf{B}^{\alpha}$, $\alpha \in \mathsf{A}$:

$$\hat{f}_{\alpha}(x) := f(\overline{x}_{\alpha}) + Df(\overline{x}_{\alpha}) \quad (x - \overline{x}_{\alpha}), \tag{2.10a}$$

$$\hat{h}_{\alpha,i}(x) := h_i(\overline{x}_{\alpha}) + Dh_i(\overline{x}_{\alpha}) \quad (x - \overline{x}_{\alpha}), \tag{2.10b}$$

$$\hat{y}^j_{\alpha,\beta}(x) := y^j_{\alpha,\beta}(\overline{x}_{\alpha}) + Dy^j_{\alpha,\beta}(\overline{x}_{\alpha})(x - \overline{x}_{\alpha}) = \overline{y}^j_{\alpha,\beta} + Dy^j_{\alpha,\beta}(\overline{x}_{\alpha})(x - \overline{x}_{\alpha}), \tag{2.10c}$$

and, herewith,

$$\hat{g}_{\alpha,\beta,j}(x) := g^{\mathrm{lin}}_{\overline{y}^j_{\alpha,\beta}}(x, \quad \hat{y}^j_{\alpha,\beta}(x)) = $$

$$= g(\overline{x}_{\alpha}, \overline{y}^j_{\alpha,\beta}) + \big(D_x g(\overline{x}_{\alpha}, \overline{y}^j_{\alpha,\beta}) + D_y g(\overline{x}_{\alpha}, \overline{y}^j_{\alpha,\beta}) \quad Dy^j_{\alpha,\beta}(\overline{x}_{\alpha})\big) \quad (x - \overline{x}_{\alpha}). \tag{2.10d}$$

We note that for the definition of $\hat{g}_{\alpha,\beta,j}$ $(j \in J^{\alpha,\beta},\ \beta \in \mathsf{B}^\alpha,\ \alpha \in \mathsf{A})$ we do no longer need explicitly to know $y^j_{\alpha,\beta}(x)$, $Dy^j_{\alpha,\beta}(x)$ as functions, but *only* their (special) values $y^j_{\alpha,\beta}(\overline{x}_\alpha) = \overline{y}^j_{\alpha,\beta}$ and $Dy^j_{\alpha,\beta}(\overline{x}_\alpha)$ at the one point $\overline{x}_\alpha$. Moreover, from the proof of the implicit function theorem (cf., e. g., [1]) we see the following representation of $Dy^j_{\alpha,\beta}(\overline{x}_\alpha)$ (cf. [29]):

$$Dy^j_{\alpha,\beta}(\overline{x}_\alpha) \;=\; -(D_y\widehat{F}^j_{\alpha,\beta}(\overline{x}_\alpha, \overline{y}^j_{\alpha,\beta}))^{-1} D_x\widehat{F}^j_{\alpha,\beta}(\overline{x}_\alpha, \overline{y}^j_{\alpha,\beta}), \tag{2.11}$$

which can further be evaluated by means of (2.4), where $\widehat{F}^j = \widehat{F}^j_{\alpha,\beta}$. In this way we get the (locally) *linear* finite optimization problems

$$\mathcal{P}^\alpha_{\mathcal{F},\text{lin}}(\hat{f}_\alpha, \hat{h}_\alpha, \hat{g}_\alpha): \quad \text{Minimize } \hat{f}_\alpha(x) \text{ on } M^\alpha_{\mathcal{F},\text{lin}}[\hat{h}_\alpha, \hat{g}_\alpha],$$

being located in $\overline{\mathcal{S}^1_\alpha}$, where $\hat{h}_\alpha$, $\hat{g}_\alpha$ comprise the constraints $\hat{h}_{\alpha,i}$, $i \in I$, $\hat{g}_{\alpha,\beta,j}$, $j \in J^{\alpha,\beta}$ $(\beta \in \mathsf{B}^\alpha$, $\alpha \in \mathsf{A})$. In view of $\overline{\mathcal{S}^1_\alpha}$ being a square, the corresponding feasible set

$$M^\alpha_{\mathcal{F},\text{lin}}[\hat{h}_\alpha, \hat{g}_\alpha] := \{x \in I\!R^n \mid \hat{h}_{\alpha,i}(x) = 0\,(i \in I),\ x \in \overline{\mathcal{S}^1_\alpha},\ \hat{g}_{\alpha,\beta,j}(x) \geq 0\,(j \in J^{\alpha,\beta},\ \beta \in \mathsf{B}^\alpha)\},$$

on which $\hat{f}_\alpha$ has to be minimized, are *compact* and *completely* defined by *affinely linear* constraints $(\alpha \in \mathsf{A})$. These problems may be regarded as the *(linear) subproblems* of the following collected global approximation of our $\mathcal{GSI}$ problem:

$$\mathcal{P}^{\triangleright}_{\mathcal{F},\text{lin}}(\hat{f}^{\triangleright}, \hat{h}^{\triangleright}, \hat{g}^{\triangleright}): \quad \text{Minimize } \hat{f}_\alpha(x), \quad \text{where } x \in M^\alpha_{\mathcal{F},\text{lin}}[\hat{h}_\alpha, \hat{g}_\alpha],\ \alpha \in \mathsf{A},$$

over the collected feasible set

$$\begin{aligned} M^{\triangleright}_{\mathcal{F},\text{lin}}[\hat{h}^{\triangleright}, \hat{g}^{\triangleright}] := \{\, & x \in I\!R^n \mid \hat{h}_{\alpha,i}(x) = 0 \ (i \in I), \\ & x \in \overline{\mathcal{S}^1_\alpha}, \quad \hat{g}_{\alpha,\beta,j}(x) \geq 0 \quad (j \in J^{\alpha,\beta},\ \beta \in \mathsf{B}^\alpha) \quad \text{for some } \alpha \in \mathsf{A}\} = \\ = {} & \cup_{\alpha \in \mathsf{A}} M^\alpha_{\mathcal{F},\text{lin}}[\hat{h}_\alpha, \hat{g}_\alpha]. \end{aligned}$$

Here

$$\hat{f}^{\triangleright} = (\hat{f}_\alpha)_{\alpha \in \mathsf{A}},\ \hat{h}^{\triangleright} = (\ldots, \hat{h}_{\alpha,i}\,(i \in I,\ \alpha \in \mathsf{A}), \ldots),\ \hat{g}^{\triangleright} = (\ldots, \hat{g}_{\alpha,\beta,j}\,(j \in J^{\alpha,\beta},\ \beta \in \mathsf{B}^\alpha,\ \alpha \in \mathsf{A}), \ldots)$$

(with suitable enumerations), and the affinely linear inequality constraints defining $(x \in)\,\overline{\mathcal{S}^1_\alpha}$ have (for simplicity) again been suppressed in the list of functional parameters. As for each given $\alpha \in \mathsf{A}$ the open coverings (by means of squares) may again be thought to be fine enough, the functions $\hat{f}_\alpha$, $\hat{h}_\alpha$, $\hat{g}_\alpha$ are (locally) arbitrarily close perturbations of the functions f, h, g°_α. Now, under the guaranteed condition MFCQ for $M^\circ_{\mathcal{F}}[h, g^\circ] \cap \overline{\mathcal{U}^0}$ (cf. Subsection 2.2), $M^{\triangleright}_{\mathcal{F},\text{lin}}[\hat{h}^{\triangleright}, \hat{g}^{\triangleright}] \cap \overline{\mathcal{U}^0}$ is an arbitrarily good approximation of $M^\circ_{\mathcal{F}}[h, g^\circ] \cap \overline{\mathcal{U}^0}$. From the considerations in Subsection 2.2, moreover, we know that the latter set may lie arbitrarily close to $M_{\mathcal{SI}}[h, g] \cap \overline{\mathcal{U}^0}$.

Altogether, after these two approximations in $(\mathcal{G})\mathcal{SI}$ and $\mathcal{F}$ optimization, respectively, we state: *in* $\overline{\mathcal{U}^0}$, $M_{\mathcal{SI}}[h,g]$ *can arbitrarily well be described by means of the compact approximative set* $M^{\triangleright}_{\mathcal{F},\text{lin}}[\hat{h}^{\triangleright},\hat{g}^{\triangleright}]$.

Moreover, the components of $\hat{f}^{\triangleright}$ locally approximate f, such that, together with the previous reflections on set approximations, $\mathcal{P}^{\triangleright}_{\mathcal{F},\text{lin}}(\hat{f}^{\triangleright},\hat{h}^{\triangleright},\hat{g}^{\triangleright})$ *may serve as a very fine approximative description of our problem* $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$. This fact will be exploited in the proof of the convergence theorem (Section 3). Both the feasible set $M^{\triangleright}_{\mathcal{F},\text{lin}}[\hat{h}^{\triangleright},\hat{g}^{\triangleright}]$ and the problem $\mathcal{P}^{\triangleright}_{\mathcal{F},\text{lin}}(\hat{f}^{\triangleright},\hat{h}^{\triangleright},\hat{g}^{\triangleright})$ can in $\overline{\mathcal{U}^0}$ be considered as a "**mosaic**" consisting of the *linearly* defined feasible sets $M^{\alpha}_{\mathcal{F},\text{lin}}[\hat{h}_\alpha,\hat{g}_\alpha]$ (see Fig. 3) and the *linear* subproblems $\mathcal{P}^{\alpha}_{\mathcal{F},\text{lin}}(\hat{f}_\alpha,\hat{h}_\alpha,\hat{g}_\alpha)$ $(\alpha \in \mathsf{A})$. Hereby, each of the latter subproblems can be solved by means of **linear programming**, e. g., using the *simplex algorithm* for *finite* optimization; cf. [17, 25, 27]. (We also mention that there is a simplex algorithm in the case of semi-infinite optimization; see [23].)

Note, that those (compact) "simplices" are given as the *intersection* between the polytopes, which piece together $M^{\alpha}_{\mathcal{F},\text{lin}}[\hat{h}_\alpha,\hat{g}_\alpha]$, and the polytope $\overline{\mathcal{U}^0}$. In this way, for our approximative problems we also have a *polytope* structure in the x-space $I\!R^n$. As under our approximation the simplicial structure becomes finer and finer and $M_{\mathcal{SI}}[h,g]$ transversally meets $\overline{\mathcal{U}^0}$, that intersection will finally also be *transversal.*

We emphasize that up to approximation, the (structural) complexity of our $\mathcal{GSI}$ problem has been *reduced to the complexity of a* <u>*linear*</u> $\mathcal{F}$ *problem.*

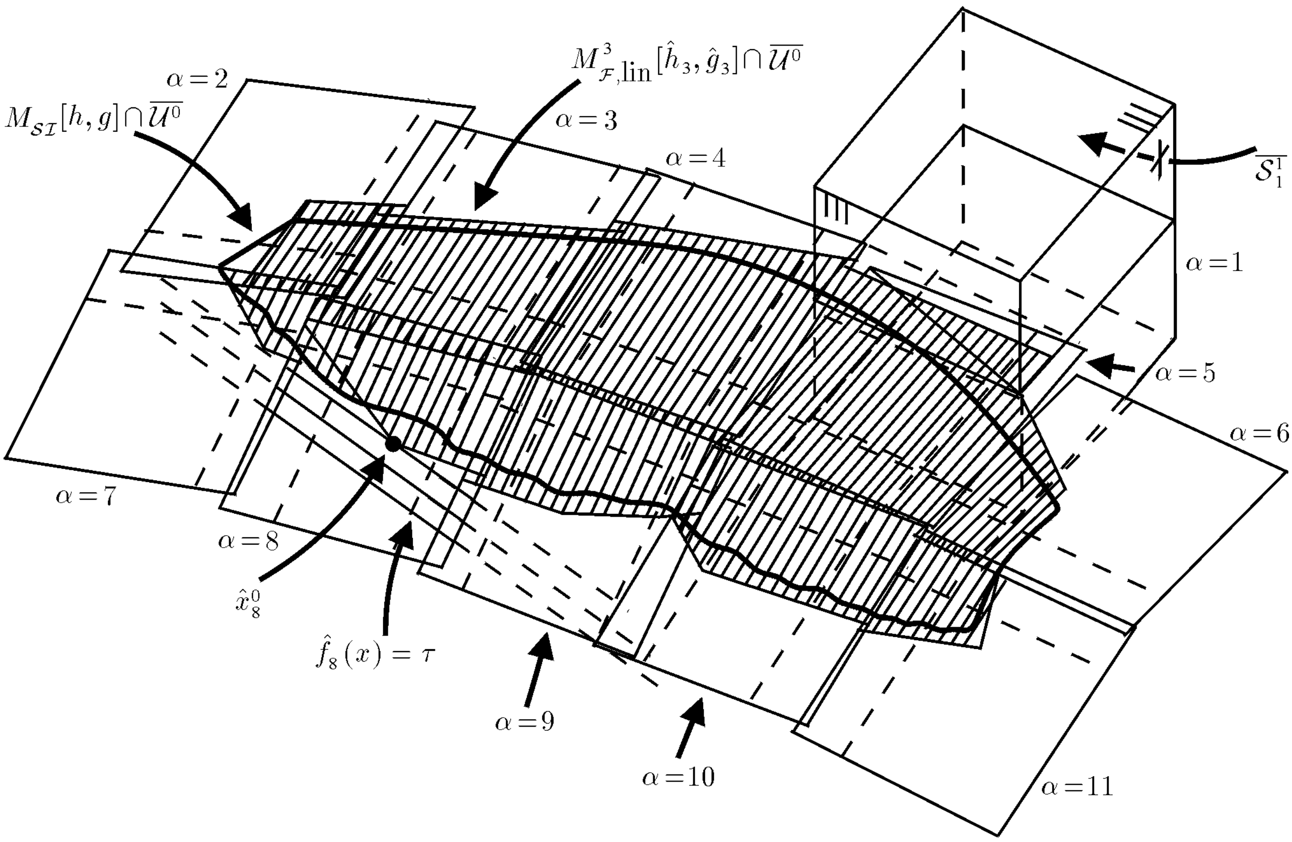

Fig. 3. The *mosaic* $M^{\triangleright}_{\mathcal{F},\text{lin}}[\hat{h}^{\triangleright},\hat{g}^{\triangleright}]$ in $\overline{\mathcal{U}^0}$ (indicaded in a hatched way), and lower level sets $\{x\in I\!R^n \mid \hat{f}_\alpha(x)=\tau\}$ $(\tau\in I\!R,\ \alpha=8)$ $(m=1,\ \alpha^0=11;$ cf. Fig. 1; an example).

2.4. Part 3: Completion of the iteration procedure

After all the preparations made in Subsections 2.1 − 2.3, we are in a position to summarize our iteration procedure in the following way. We start with the initialization step given at the index $\nu = 0$. Here, some open coverings consisting of squares are given, namely $(\mathcal{S}_\alpha^{1,0})_{\alpha\in\mathsf{A}^0}$, $(\mathcal{S}_{\alpha,\beta}^{2,0})_{\beta\in\mathsf{B}^{\alpha,0}}$ $(\alpha \in \mathsf{A}^0)$, whereby for the first covering the family $(\overline{x}_\alpha)_{\alpha\in\mathsf{A}^0}$ consists of corresponding elements, say, center points. For instance, the edges of the (in pairs perhaps overlapping) squares $\overline{\mathcal{S}_\alpha^{\gamma,0}}$ $(\alpha \in \mathsf{A}^0)$ may be taken from some grid structure and with $(||\cdot||_\infty$-) radius $\delta_\alpha^{\gamma,0} > 0$, e. g., in the way of Fig. 4.

Up to slightly (transversally) perturbing and shrinking $\mathcal{S}_{\alpha,\beta}^{2,0}$ $(\beta \in \mathsf{B}^{\alpha,0})$, we get the *mosaic* problem $\mathcal{P}_{\mathcal{F},\text{lin}}^{\triangleright}(\hat{f}^{\triangleright,0}, \hat{h}^{\triangleright,0}, \hat{g}^{\triangleright,0})$. The squares $\mathcal{S}_\alpha^{1,0}$ $(\alpha \in \mathsf{A}^0)$ may also be sufficiently small in order to come (as sub-domains) from applications of the implicit function theorem (see part 1.a); otherwise, they shall become sufficiently small in a later step (where $\nu > 0$). Of course, if it is desired for our implicit vertex functions, the finite index set A^0 is also allowed immediately to be enlarged due to a subdivision of some square $\overline{\mathcal{S}_1^{1,0}}$ into smaller squares.

Now, with the help of linear programming we choose (global) minima $\hat{x}_\alpha^0$ $(\alpha \in \mathsf{A}^0 = \{1,\ldots,\alpha^{0,0}\})$ of the subproblems $\mathcal{P}_{\mathcal{F},\text{lin}}^{\alpha}(\hat{f}_\alpha, \hat{h}_\alpha, \hat{g}_\alpha)$ (see also Fig. 3). These points need not to be uniquely defined. Let $\hat{x}^0 = \hat{x}_{\alpha'^0}^0$ be some element of $\{\hat{x}_\alpha^0 \mid \alpha \in \mathsf{A}^0\}$ which is minimal for $\hat{f}^{\triangleright,0}$ in the following sense:

$$\hat{f}_{\alpha'^0}^0(\hat{x}^0) = \min\{\hat{f}_\alpha^0(\hat{x}_\alpha^0) \mid \alpha \in \{1,\ldots,\alpha^{0,0}\}\} =$$

$$= \min\{\hat{f}_\alpha^0(x) \mid x \in M_{\mathcal{F},\text{lin}}^{\alpha}[\hat{h}_\alpha^0, \hat{g}_\alpha^0] \cap \overline{\mathcal{U}^0},\ \alpha \in \mathsf{A}^0\}. \tag{2.12^0}$$

In the case of a tangential effect between $\partial\mathcal{S}_{\alpha,\beta}^{2,0}$ and $Y(x)$ for some $\beta \in \mathsf{B}^{\alpha,0}$, $\alpha \in \mathsf{A}^0$, $x \in$

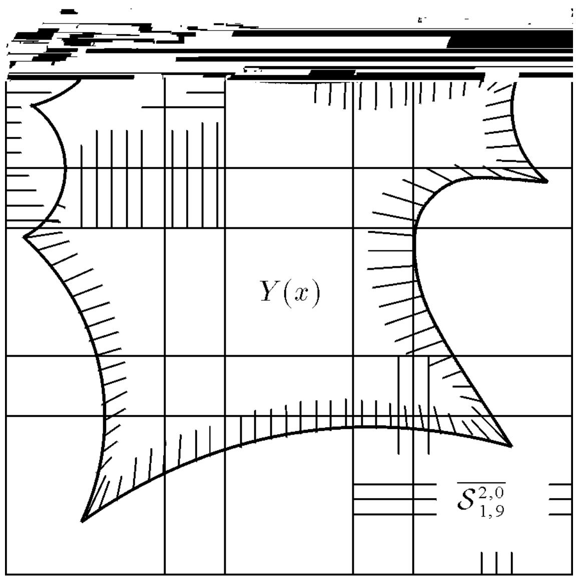

Fig. 4. Initial step $\nu = 0$: decomposition of $Y(x)$ $(r = 0,\ \alpha = 1,\ \beta^{1,0} = 9;$ cf. Fig. 2; an example).
Here transversality between $Y(x)$ and $\partial\mathcal{S}_{1,\beta}^{2,0}$ is already established $(\beta \in \{1,\ldots,\beta^{1,0}\};$ $x \in \overline{\mathcal{S}_1^{1,0}})$.

$\overline{\mathcal{S}_1^{1,0}}$, a small linear transversal perturbation preserves the open covering by means of squares. Hereby, the ($\|\cdot\|_\infty$-) radius $\delta_{\alpha,\beta}^{2,0}$ may decrease a bit.

Let for some $\nu \in I\!N_0 := \{0,1,2,3,\dots\}$ a global minimum $\hat{x}^\nu \in M_{\mathcal{F},\mathrm{lin}}^{\triangleright}[\hat{h}^{\triangleright,\nu},\hat{g}^{\triangleright,\nu}] \cap \overline{\mathcal{U}^0}$ of the mosaic $\mathcal{P}_{\mathcal{F},\mathrm{lin}}^{\triangleright}(\hat{f}^{\triangleright,\nu},\hat{h}^{\triangleright,\nu},\hat{g}^{\triangleright,\nu})$ (over $\overline{\mathcal{U}^0}$) with underlying open coverings $(\mathcal{S}_\alpha^{1,\nu})_{\alpha\in\mathsf{A}^\nu}$, $(\mathcal{S}_{\alpha,\beta}^{2,\nu})_{\beta\in\mathsf{B}^{\alpha,\nu}}$ ($\alpha \in \mathsf{A}^\nu$) already be given. Then, for the index $\nu+1$ we decompose $\overline{\mathcal{S}_\alpha^{\gamma,\nu}}$ by means of a further grid which is given by dividing the radius by some natural integer $\chi_\alpha^{\nu+1} \in I\!N$, $\chi_\alpha^{\nu+1} \geq 2$. Then, the new radii are $\delta_\alpha^{\gamma,\nu+1} := \dfrac{\delta_\alpha^{\gamma,\nu}}{\chi_\alpha^{\nu+1}} > 0$. We have again to guarantee some overlapping of the new squares (e. g.: two or more inner parts vs. two outer part at each side, as it was done in the Figures 3, 4). After small transversal perturbations and restrictions we get open coverings $(\mathcal{S}_\alpha^{1,\nu+1})_{\alpha\in\mathsf{A}^{\nu+1}}$, $(\mathcal{S}_{\alpha,\beta}^{2,\nu+1})_{\beta\in\mathsf{B}^{\alpha,\nu+1}}$ ($\alpha\in\mathsf{A}^{\nu+1}$).

Now, we can again select global minima $\hat{x}_\alpha^{\nu+1}$, $\hat{x}^{\nu+1}$ of the subproblems $\mathcal{P}_{\mathcal{F},\mathrm{lin}}^{\alpha}(\hat{f}_\alpha^{\nu+1},\hat{h}_\alpha^{\nu+1}, \hat{f}_\alpha^{\nu+1})$ ($\alpha\in\mathsf{A}^{\nu+1}$) and of the mosaic $\mathcal{P}_{\mathcal{F},\mathrm{lin}}^{\triangleright}(\hat{f}^{\triangleright,\nu+1},\hat{h}^{\triangleright,\nu+1},\hat{g}^{\triangleright,\nu+1})$, all of them being restricted to $\overline{\mathcal{U}^0}$. Hence, for some $\alpha'^{\nu+1} \in \mathsf{A}^{\nu+1} = \{1,\dots,\alpha^{0,\nu+1}\}$ we have: $\hat{x}^{\nu+1} = \hat{x}_{\alpha'^{\nu+1}}^{\nu+1}$ and

$$\begin{aligned} \hat{f}_{\alpha'^{\nu+1}}^{\nu+1}(\hat{x}^{\nu+1}) &= \min\{\hat{f}_\alpha^{\nu+1}(\hat{x}_\alpha^{\nu+1}) \,|\, \alpha\in\{1,\dots,\alpha^{0,\nu+1}\}\} = \\ &= \min\{\hat{f}_\alpha^{\nu+1}(x) \,|\, x \in M_{\mathcal{F},\mathrm{lin}}^{\alpha}[\hat{h}_\alpha^{\nu+1},\hat{g}_\alpha^{\nu+1}] \cap \overline{\mathcal{U}^0},\ \alpha\in\mathsf{A}^{\nu+1}\}. \end{aligned} \tag{2.12$^{\nu+1}$}$$

In this way, we have arrived at *a sequence* $(\hat{x}^\nu)_{\nu\in I\!N_0}$ *of global minimizers of our approximative mosaics of linear problems.*

2.5. On practical treatments and a generalization

We note that when the transversal squares $\overline{\mathcal{S}_\alpha^{1,\nu}}$, $\overline{\mathcal{S}_{\alpha,\beta}^{2,\nu}}$ have become sufficiently small, they need furthermore no longer transversally to be perturbed, but translations are already sufficient. Hence, their inwardly pointing normal vectors ξ_κ^ν $(= \xi_\kappa^{j,\alpha,\beta,\nu},\ \kappa\in\{1,\dots,q-r-p^{j,\alpha,\beta,\nu}\},\ j \in J^{\alpha,\beta},\ \beta\in\mathsf{B}^{\alpha,\nu}, \alpha\in\mathsf{A}^\nu)$ (cf. (2.4)) may remain unchanged. From this fact we conclude that our specification of the condition EMFCQ, on which our approximation is based, can be made independently from the choice of the iteration step $\nu\in I\!N_0$.

Hereby, we may also use $\overline{x}_\alpha$ as some controlling parameter for center points in order to guarantee open coverings based on the implicit function theorem, if otherwise there might arise a problem based on a tapering of $\overline{\mathcal{S}_\alpha^{1,\nu}}$.

Looking at our iteration procedure, we note that the squares do not explicitly appear in the approximative linear problems. Moreover, each vertex function $y_{\alpha,\beta}^{j,\nu}(x)$ and its derivative $Dy_{\alpha,\beta}^{j,\nu}(x)$ need only *one* single evaluation, namely at $\overline{x}_\alpha^\nu$ ($\nu\in I\!N_0$). Hence, we do only need to determine the vertices $y_{\alpha,\beta}^{j,\nu}(\overline{x}_\alpha^\nu)$ locally being extremal points for $Y(\overline{x}_\alpha^\nu)\cap\overline{\mathcal{S}_\alpha^{2,\nu}}$ and its convex hull, with the locally minimal (total) number (in short: p_0) of active inequality constraints $v_\ell(\overline{x}_\alpha^\nu,\cdot)$ ($\ell\in L$) or $\xi_\kappa^{j,\alpha,\beta,\nu^T}(y - y_\kappa'^{\,j,\alpha,\beta,\nu})$ ($\kappa\in\{1,\dots,q-p^{j,\alpha,\beta,\nu}-r\}$). Here, $p^{j,\alpha,\beta,\nu}(\leq p_0)$ is the number of active constraints $v_\ell(\overline{x}_\alpha^\nu,\cdot)$. Moreover, in these points some (one dimensional) lines in $\partial(Y(\overline{x}_\alpha^\nu)\cap\overline{\mathcal{S}_{\alpha,\beta}^{2,\nu}})$ meet, along which that total number p_0 locally is $q-r$. Sometimes it may be helpful to follow these lines, "paths", in order finally to detect their intersection point being the vertex $\overline{y}_{\alpha,\beta}^{j,\nu} = y_{\alpha,\beta}^{j,\nu}(\overline{x}_\alpha^\nu)$ (for related techniques see [4]; cf. also [26]). We only remark that following of $y_{\alpha,\beta}^{j,\nu}(x)$ in the n parameters x_σ ($\sigma\in\{1,\dots,n\}$) around the point $\overline{x}_\alpha^\nu$ may lead to the boundary of that neighbourhood $\mathcal{U}_{\mathcal{S}_\alpha^{1,\nu}}$ of $\overline{x}_\alpha^\nu$, where the theorem on implicit functions applies.

After a phase of adjustment (being due to small ν) of our procedure, the existence of our coverings consisting of squares in transversal position around given center points, is automatically satisfied in the following sense. Namely, if for some $\nu' \in I\!N_0$ at $x = \overline{x}_\alpha^{\nu'} (\in \overline{\mathcal{S}_\alpha^{1,\nu'}}, \ \alpha \in \mathsf{A}^{\nu'})$ the squares $\overline{\mathcal{S}_\alpha^{1,\nu'}}$, $\overline{\mathcal{S}_{\alpha,\beta}^{2,\nu'}}$ $(\beta \in \mathsf{B}^{\alpha,\nu'})$, *transversally* intersect $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ and $Y(\overline{x}_\alpha^{\nu'})$, respectively, and if ν' is large enough, then the same *transversality* conditions hold *for all* $x \in \overline{\mathcal{S}_\alpha^{1,\nu}}$, $\nu \geq \nu'$. This comes from the sufficienctly great fineness of our antitonely shrinking squares, with their faces finally (from one ν to the other, $\nu + 1$, and in pairs) remaining parallel.

Hence, we may concentrate on the center points $\overline{x}_\alpha^\nu$ and the vertices $\overline{y}_{\alpha,\beta}^{j,\nu}$, where after this adjustment tranversality is locally fulfilled. We would only intervene, if we have some more geometrical insight how the sets $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$, $Y(x)$ and their intersections with the corresponding squares look. Then, we could accelerate the getting tranversal of squares by means of suitable perturbations.

When ν increases more and more, then we may with the help of geometrical observations become convinced that our minimum does not lie in a certain part of $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$. In such a case, we can my means of transversal hyperplanes *excise* a smaller subset of $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$, which may also be expressed as a shrinking of $\overline{\mathcal{U}^0}$. Then, the *increase* of the number of squares $\overline{\mathcal{S}_\alpha^{1,\nu}}$ and, hence, $\overline{\mathcal{S}_{\alpha,\beta}^{2,\nu}}$ $(\beta \in \mathsf{B}^{\alpha,\nu}, \ \alpha \in \mathsf{A}^\nu)$ becomes *weakened.*

Fig. 3 indicates that often those auxiliary linear subproblems $\mathcal{P}_{\mathcal{F},\mathrm{lin}}^\alpha(\hat{f}_\alpha^\nu, \hat{h}_\alpha^\nu, \hat{g}_\alpha^\nu)$, which are located nearby the relative interior of $M_{\mathcal{SI}}[h,g]$, can *"extremely" easy* be evaluated. (We note the simple structure of $M_{\mathcal{F},\mathrm{lin}}^\alpha[\hat{h}_\alpha^\nu, \hat{g}_\alpha^\nu] \cap \overline{\mathcal{U}^0})$ there.)

In the following section we shall see that there is a **subsequence** $(\hat{x}^{\nu_\kappa})_{\kappa \in I\!N_0}$ for our iteration procedure **converging to a (global) minimizer** $\hat{x}$ for $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ in $\overline{\mathcal{U}^0}$. The points $\hat{x}^{\nu_\kappa}$ $(\kappa \in I\!N_0)$ do not need to be feasible for $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$. However, each sufficiently good approximation $\hat{x}^{\nu_{\kappa_0}}$ of $\hat{x}$ can be made feasible by means of a slight shift $\hat{x}^{\nu_{\kappa_0}} \to \hat{x}^* \in \partial(M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0})$ in the direction of an EMF-vector ζ^0. Hereby, the number κ_0 may be chosen sufficiently large, or depending on our desire, how close to get to the minimizer $\hat{x}$ of our problem. Of course, as a foregoing task, in practice we have to look for converging subsequences, and always to exploit all the structural and geometrical features of the given problem under consideration.

As we *typically* use *linear approximations* of our functional data, our problem need *only* to be of class C^1. We mention that in different parts of our approximations, *higher* differentiability (if it is given) could be exploited by means of Taylor polynomials of degree ≥ 2.

Now, let us come back to the general situation, where the **Assumption C** on affine linearity and convexity is **not made**. Hereby, for simplicity, at first we suppress the index ν. Then, we would replace the defining functions u_k, v_ℓ by their linearizations $u_{\alpha,\beta,k}^{\mathrm{lin}}$, $v_{\alpha,\beta,\ell}^{\mathrm{lin}}$ $(k \in K, \ \ell \in L)$, respectively $(\beta \in \mathsf{B}^\alpha, \ \alpha \in \mathsf{A})$, which are given by

$$u_{\alpha,\beta,k}^{\mathrm{lin}}(x,y) := u_k(\overline{x}_\alpha, \overline{y}_{\alpha,\beta}) + D_x u_k(\overline{x}_\alpha, \overline{y}_{\alpha,\beta})(x - \overline{x}_\alpha) + D_y u_k(\overline{x}_\alpha, \overline{y}_{\alpha,\beta})(y - \overline{y}_{\alpha,\beta}) \ (k \in K), \tag{2.13a}$$

$$v_{\alpha,\beta,\ell}^{\mathrm{lin}}(x,y) := v_\ell(\overline{x}_\alpha, \overline{y}_{\alpha,\beta}) + D_x v_\ell(\overline{x}_\alpha, \overline{y}_{\alpha,\beta})(x - \overline{x}_\alpha) + D_y v_\ell(\overline{x}_\alpha, \overline{y}_{\alpha,\beta})(y - \overline{y}_{\alpha,\beta}) \ (\ell \in L). \tag{2.13b}$$

(We remember, that $\overline{y}_{\alpha,\beta}$ is the center point of $\overline{\mathcal{S}_{\alpha,\beta}^2}$, $\beta \in \mathsf{B}^\alpha$, $\alpha \in \mathsf{A}$.) Let us set $\widetilde{Y}_{\alpha,\beta}(x) := M_{\mathcal{F}}[u_{\alpha,\beta}^{\mathrm{lin}}(x,\cdot), v_{\alpha,\beta}^{\mathrm{lin}}(x,\cdot)]$ $(x \in I\!R^n)$. Firstly, for each $x \in \overline{\mathcal{S}_\alpha^1}$, $\widetilde{Y}_{\alpha,\beta}(x)$ is allowed to be a *rough* approximation for $Y(x)$.

Now, we have again to look for a square $\mathcal{S}^2_{\alpha,\beta}$ in such a (nonempty) way that $\partial\mathcal{S}^2_{\alpha}$ transversally meets $\widetilde{Y}_{\alpha,\beta}(x)$. Then, $\widetilde{Y}_{\alpha,\beta}(x)$ is *already* a polytope with the vertices $\tilde{y}^j_{\alpha,\beta}(x)$ $(j \in \{1,\ldots,\tilde{j}^0_{\alpha,\beta}\})$, which are computable by means of linear algebra. In the case of Assumption C, these vertices play the part of $y^j_{\alpha,\beta}(x)$, while in general the points $\tilde{y}^j_{\alpha,\beta}(x)$ need no longer to lie in $Y(x)$. However, if with increasing ν our square structure becomes finer and finer, then $((u^{\text{lin}}_{\alpha,\beta}, v^{\text{lin}}_{\alpha,\beta}) =)(u^{\text{lin},\nu}_{\alpha,\beta}, v^{\text{lin},\nu}_{\alpha,\beta})$ locally approaches (u,v) such that in virtue of the Assumption $\mathrm{B}_{\mathcal{U}^0}$ on LICQ (hence, MFCQ), for each $x \in \overline{\mathcal{S}^{1,\nu}_{\alpha}}$, the union $\cup_{\beta\in\mathsf{B}^{\alpha,\nu}}\widetilde{Y}^{\nu}_{\alpha,\beta}(x) \cap \overline{\mathcal{S}^{2,\nu}_{\alpha,\beta}}$ gets arbitrarily close to $Y(x)$ (*approximation*; [3, 30]). Hereby, $\widetilde{Y}^{\nu}_{\alpha,\beta}(x)$ is understood in the sense of $\widetilde{Y}_{\alpha,\beta}(x)$ and being due to $\nu \in I\!N_0$. Again, we distinguish two approximation steps (parts), one in the $(\mathcal{G})\mathcal{SI}$ sense and a further one in the $\mathcal{F}$ sense.

Then, under our local perturbations from above, we finally arrive at *mosaics* in $\overline{\mathcal{U}^0}$, namely $M^{\triangleright}_{\mathcal{F},\text{lin}}[\tilde{h}^{\triangleright,\nu}, \tilde{g}^{\triangleright,\nu}]$, and $\mathcal{P}^{\triangleright}_{\mathcal{F},\text{lin}}(\tilde{f}^{\triangleright,\nu}, \tilde{h}^{\triangleright,\nu}, \tilde{g}^{\triangleright,\nu})$ consisting of linear subproblems $\mathcal{P}^{\alpha}_{\mathcal{F},\text{lin}}(\tilde{f}^{\nu}_{\alpha}, \tilde{h}^{\nu}_{\alpha}, \tilde{g}^{\nu}_{\alpha})$, $\alpha \in \mathsf{A}^{\nu}$ $(\nu \in I\!N_0)$. These mosaics approximate $M_{\mathcal{SI}}[h,g]$ and $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ (in $\overline{\mathcal{U}^0}$), respectively.

Let us remark, that we could also comprise the functions $u^{\text{lin},\nu}_{\alpha,\beta}$, $v^{\text{lin},\nu}_{\alpha,\beta}$ $(\beta \in \mathsf{B}^{\alpha}$, $\alpha \in \mathsf{A})$ in the vector notation $\tilde{u}^{\triangleright,\nu}$, $\tilde{v}^{\triangleright,\nu}$, respectively $(\nu \in I\!N_0)$.

As there is one more step of perturbation (see (2.13a,b)) instead of exactness involved, this approximation is less close than the one under Assumption C.

Finally, our mosaics yield us a sequence $(\tilde{x}^{\nu})_{\nu\in I\!N_0}$ consisting of (global) minimizers of $\mathcal{P}^{\triangleright}_{\mathcal{F},\text{lin}}(\tilde{f}^{\triangleright,\nu}, \tilde{h}^{\triangleright,\nu}, \tilde{g}^{\triangleright,\nu})$, restricted to $\overline{\mathcal{U}^0}$, respectively. Because of the less close approximation we cannot expect that this sequence, or some subsequence, converges stronger than it is accomplished in the case of Assumption C.

We underline that by means of our referring to polytopes, we *did not* explicitly need a change of our coordinates $y \mapsto z$. For further treatments on polytopes we refer to [32].

Taking account of both all the preceding explanations of our iterative approach and the special features of some concretely given problem, an algorithm which solves our given $\mathcal{GSI}$ problem, can be developped.

3. On the convergence of the iteration procedure

3.1. The convergence theorem and its proof

Based on the preparations given in Sections 1 and 2, we may formulate our main result as follows.

THEOREM 3.1 (Theorem on Convergence).

Let the Assumptions $\mathrm{A}_{\mathcal{U}^0}$, $\mathrm{B}_{\mathcal{U}^0}$, C *and* $\mathrm{D}_{\mathcal{U}^0}$ *be satisfied due to a bounded open set* $\mathcal{U}^0 \subseteq I\!R^n$, *fulfilling* $M_{\mathcal{SI}}[h,g] \cap \mathcal{U}^0 \neq \emptyset$, $\mathcal{U}^0$ *being a manifold with piecewise linear boundary, and* $\partial\mathcal{U}^0$ *being in transversal position with* $M_{\mathcal{SI}}[h,g]$.

Then, on the one hand, there exists a sequence $(\hat{x}^{\nu})_{\nu\in I\!N_0}$ *of global minimizers of topologically approximative* mosaics $\mathcal{P}^{\triangleright}_{\mathcal{F},lin}(\hat{f}^{\triangleright,\nu}, \hat{h}^{\triangleright,\nu}, \hat{g}^{\triangleright,\nu})$ $(\nu \in I\!N_0)$, *being restricted to* $\overline{\mathcal{U}^0}$, *respectively.*

On the other hand, there is a **convergent subsequence** $(\hat{x}^{\nu_\kappa})_{\kappa\in I\!N_0}$ of $(\hat{x}^{\nu})_{\nu\in I\!N_0}$ such that its limit point $\hat{x} = \lim_{\kappa\to\infty} \hat{x}^{\nu_\kappa}$ is a **global minimizer** *for the generalized semi-infinite optimization problem* $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$ *being restricted on* $\overline{\mathcal{U}^0}$. *(Hence, it is also a* candidate *for a* local *minimum of* $\mathcal{P}_{\mathcal{SI}}(f,h,g,u,v)$.*)*

If the Assumption C *is* violated, *then the same conclusion holds, too. However, the approximation* *by means of mosaics* $\mathcal{P}^{\triangleright}_{\mathcal{F},\mathrm{lin}}(\tilde{f}^{\triangleright,\nu},\tilde{h}^{\triangleright,\nu},\tilde{g}^{\triangleright,\nu})$ $(\nu \in I\!N_0)$ *on* $\overline{\mathcal{U}^0}$ *and the corresponding (sub)sequence* $(\tilde{x}^{\nu_\kappa})_{\kappa\in I\!N_0}$ *of minimizers can not in general be expected to be as fast approximating and converging, respectively, as it can be accomplished under* Assumption C.

Proof:

Let us first of all under *all* four assumptions reflect the approximation of $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ by the sequence $(M^{\triangleright}_{\mathcal{F},\mathrm{lin}}[\tilde{h}^{\triangleright,\nu},\tilde{g}^{\triangleright,\nu}] \cap \overline{\mathcal{U}^0})_{\nu\in I\!N_0}$. There are *two effects of linearization* which come together. Namely, as a first effect we have linearizations of our defining functions and of our vertex functions (see (2.1), (2.10a – d), and, for the more general case, (2.13a – b), too). As a second effect, we have the getting arbitrarily fine of our covering squares' structure. The common virtue of both effects is very comparable with approximations of functions by means of arbitrarily small perturbations in the sense of the C^1_S-Whitney topology. The only differences firstly consist in the local *splitting* of the constraints $g(x,y) \geq 0$ $(y \in Y(x);$ we remember the square structure underlying $M_{\mathcal{SI}}[h,g^{\mathrm{lin},\alpha}])$ and, then, in the *approximations* in the sense of both $(\mathcal{G})\mathcal{SI}$ and $\mathcal{F}$ optimization. For the purpose of our set theoretical approximation, these differences mean no problem.

Indeed, we remember that there are *two parts* which contribute to our functional approximations. *Part 1* is based on a local linearization of g; here, the approximation happens in $\mathcal{GSI}$ optimization (see, firstly, Subsections 2.1 – 2.2 and, lateron, 2.5). *Part 2* is based on further linearizations, which give rise to approximations of defining functions in $\mathcal{F}$ optimization (see Subsections 2.3 – 2.4).

Based on our Assumptions $\mathrm{B}_{\mathcal{U}^0}$ on LICQ (for $Y(x)$, $x \in \overline{\mathcal{U}^0}$) and $\mathrm{D}_{\mathcal{U}^0}$ (for $M_{\mathcal{SI}}[h,g]$ on $\overline{\mathcal{U}^0}$), respectively, and on our transversal choice of covering squares, we may translate these approximations of functions into the language of set approximations. Namely, for part 1 we take account of the $\mathcal{GSI}$ investigation from [30], while for part 2 we utilize the $\mathcal{F}$ investigation [3]. Moreover, if we also (locally) perturb u, v by means of their linearizations $u^{\mathrm{lin}}_{\alpha,\beta}$, $v^{\mathrm{lin}}_{\alpha,\beta}$, then $Y(x)$ gets in the same way locally approximated by $Y^{u^{\mathrm{lin},\nu}_{\alpha,\beta},v^{\mathrm{lin},\nu}_{\alpha,\beta}}(x) := \widetilde{Y}^{\nu}_{\alpha,\beta}(x)$. Hereby, we always exploit suitable (by transversal configurations enriched) versions of EMFCQ, MFCQ and LICQ, respectively.

Because of the compactness of $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$, the minimum of f on this set is also attained. Let us demonstrate that *the minima* $\min\{\hat{f}^{\nu}_{\alpha}(x) \mid x \in M^{\alpha}_{\mathcal{F},\mathrm{lin}}[\hat{h}^{\nu}_{\alpha},\hat{g}^{\nu}_{\alpha}] \cap \overline{\mathcal{U}^0},\ \alpha \in \mathsf{A}^{\nu}\}$ $(\nu \in I\!N_0;$ see $(2.12^0 - 12^{\nu+1}))$ *tend to* $\min\{f(x) \mid x \in M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}\}$ *when* ν *tends to infinity.*

Let some $\epsilon > 0$ be given. As f is continuous and $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ is compact, we know that there is a finite open covering $(\mathcal{U}^{\sigma})_{\sigma\in\{1,\ldots,\sigma^0\}}$ of $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ such that

$$\left.\begin{array}{c} M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0} \subseteq \cup_{\sigma=1}^{\sigma^0}\mathcal{U}^{\sigma} \subseteq \mathcal{W}^0, \\ M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0} \cap\ \mathcal{U}^{\sigma} \neq \emptyset \quad (\sigma \in \{1,\ldots,\sigma^0\}), \end{array}\right\} \tag{3.1a}$$

$$f(x) - f(\tilde{x})| \leq \frac{\epsilon}{2} \quad \text{for all} \quad x,\tilde{x} \in \mathcal{U}^{\sigma},\ \sigma \in \{1,\ldots,\sigma^0\}. \tag{3.2}$$

Moreover, as we demonstrated in Section 2, the sets $M^{\triangleright}_{\mathcal{F},\mathrm{lin}}[\hat{h}^{\triangleright,\nu},\hat{g}^{\triangleright,\nu}] \cap \overline{\mathcal{U}^0} = \cup_{\alpha\in\mathsf{A}^{\nu}} M^{\alpha}_{\mathcal{F},\mathrm{lin}}[\hat{h}^{\nu}_{\alpha},\hat{g}^{\nu}_{\alpha}] \cap \overline{\mathcal{U}^0}$ $(\nu \in I\!N_0)$ approximate $M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$ whereby the (relative) boundaries $\partial(M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0})$ and $\partial(M^{\triangleright}_{\mathcal{F},\mathrm{lin}}[\hat{h}^{\triangleright,\nu},\hat{g}^{\triangleright,\nu}] \cap \overline{\mathcal{U}^0})$ (in $M[h]$) get arbitrarily close together (see [3, 12, 30]). Hence, there is some $\nu'_{\frac{\epsilon}{2}} \in I\!N_0$ such that

$$\left.\begin{array}{rcl} M^{\triangleright}_{\mathcal{F},\mathrm{lin}}[\hat{h}^{\triangleright,\nu},\hat{g}^{\triangleright,\nu}] \cap \overline{\mathcal{U}^0} & \subseteq & \cup_{\sigma=1}^{\sigma^0}\mathcal{U}^{\sigma}, \\ M^{\triangleright}_{\mathcal{F},\mathrm{lin}}[\hat{h}^{\triangleright,\nu},\hat{g}^{\triangleright,\nu}] \cap \overline{\mathcal{U}^0} \cap\ \mathcal{U}^{\sigma} & \neq & \emptyset \quad (\sigma \in \{1,\ldots,\sigma^0\}) \end{array}\right\} \quad \text{for all} \ \nu \geq \nu'_{\frac{\epsilon}{2}}. \tag{3.1b}$$

Now, we may on the one hand (e. g., indirectly, by contradiction) conclude from (3.1a-b) and (3.2) that it holds

$$\Big|\min\{f(x)\,|\,x\in M_{\mathcal{SI}}[h,g]\cap\overline{\mathcal{U}^0}\}\;-\;\min\{f(x)\,|\,x\in M^{\triangleright}_{\mathcal{F},\mathrm{lin}}[\hat h^{\triangleright,\nu},\hat g^{\triangleright,\nu}]\cap\overline{\mathcal{U}^0}\}\Big|\le\frac{\epsilon}{2}$$
$$\text{for all }\ \nu\ge\nu'_{\frac{\epsilon}{2}}. \tag{3.3}$$

On the other hand, as everywhere on our approximating mosaics the collected functions $\hat f^{\nu}_{\alpha}$ $(\alpha\in\mathsf{A}^{\nu})$ locally approaches f $(\nu\to\infty)$, there is some $\nu''_{\frac{\epsilon}{2}}$ such that it holds

$$|f(x)-\hat f^{\nu}_{\alpha}(x)|\;\le\;\tfrac{\epsilon}{2}\quad\text{for all }\ x\in M^{\alpha}_{\mathcal{F},\mathrm{lin}}[\hat h^{\nu}_{\alpha},\hat g^{\nu}]\cap\overline{\mathcal{U}^0},\ \ \alpha\in\mathsf{A}^{\nu},\ \ \nu\ge\nu''_{\frac{\epsilon}{2}}. \tag{3.4}$$

From the pointwise given inequalities (3.4) we may also (e. g., indirectly) conclude:

$$\Big|\min\{f(x)\,|\,x\in M^{\triangleright}_{\mathcal{F},\mathrm{lin}}[\hat h^{\triangleright,\nu},\hat g^{\triangleright,\nu}]\cap\overline{\mathcal{U}^0}\}\;-\;\min\{\hat f^{\nu}_{\alpha}(x)\,|\,x\in M^{\alpha}_{\mathcal{F},\mathrm{lin}}[\hat h^{\nu}_{\alpha},\hat g^{\nu}_{\alpha}]\cap\overline{\mathcal{U}^0},\ \alpha\in\mathsf{A}^{\nu}\}\Big|\le\frac{\epsilon}{2}$$
$$\text{for all }\ \nu\ge\nu''_{\frac{\epsilon}{2}}. \tag{3.5}$$

Altogether, a simple estimation, based on (3.3) and (3.5), delivers

$$\Big|\min\{f(x)\,|\,x\in M_{\mathcal{SI}}[h,g]\cap\overline{\mathcal{U}^0}\}\;-\;\min\{\hat f^{\nu}_{\alpha}(x)\,|\,x\in M^{\alpha}_{\mathcal{F},\mathrm{lin}}[\hat h^{\nu}_{\alpha},\hat g^{\nu}_{\alpha}]\cap\overline{\mathcal{U}^0},\ \alpha\in\mathsf{A}^{\nu}\}\Big|\;\le\;\epsilon$$
$$\text{for all }\ \nu\ge\nu_{\frac{\epsilon}{2}}, \tag{3.6}$$

where $\nu_{\epsilon}:=\max\{\nu'_{\frac{\epsilon}{2}},\nu''_{\frac{\epsilon}{2}}\}$. So, we have given the proof of the relation(s)

$$\min\{f(x)\,|\,x\in M_{\mathcal{SI}}[h,g]\cap\overline{\mathcal{U}^0}\}\;=\;\lim_{\nu\to\infty}\big(\min\{\hat f^{\nu}_{\alpha}(x)\,|\,x\in M^{\alpha}_{\mathcal{F},\mathrm{lin}}[\hat h^{\nu}_{\alpha},\hat g^{\nu}_{\alpha}]\cap\overline{\mathcal{U}^0},\ \alpha\in\mathsf{A}^{\nu}\}\big)=$$
$$=\lim_{\nu\to\infty}\hat f^{\nu}_{\alpha'^{\nu}}(\hat x^{\nu}), \tag{3.7}$$

which was asserted above.

That sequence $(\hat x^{\nu})_{\nu\in\mathbb{N}_0}$ consisting of minimizers of our mosaic problems $\mathcal{P}^{\triangleright}_{\mathcal{SI}}(\hat f^{\triangleright,\nu},\hat h^{\triangleright,\nu},\hat g^{\triangleright,\nu})$ $(\nu\in\mathbb{N}_0)$, being restricted to $\overline{\mathcal{U}^0}$, however, is bounded. Hence, for our iteration procedure there is a *subsequence* $(\hat x^{\nu_\kappa})_{\kappa\in\mathbb{N}_0}$ of $(\hat x^{\nu})_{\nu\in\mathbb{N}_0}$ which converges to some point $\hat x\in\mathbb{R}^n$:

$$\hat x\;=\;\lim_{\kappa\to\infty}\hat x^{\nu_\kappa}. \tag{3.8}$$

Because of $\hat x^{\nu_\kappa}$ being elements of the sets $M^{\triangleright}_{\mathcal{F},\mathrm{lin}}[\hat h^{\triangleright,\nu_\kappa},\hat g^{\triangleright,\nu_\kappa}]\cap\overline{\mathcal{U}^0}$ $(\kappa\in\mathbb{N}_0)$ which approximate the *closed* set $M_{\mathcal{SI}}[h,g]\cap\overline{\mathcal{U}^0}$, the limit point $\hat x$ is an element of $M_{\mathcal{SI}}[h,g]\cap\overline{\mathcal{U}^0}$. As, moreover, $\hat f^{\nu_\kappa}$ $(\alpha\in\mathsf{A}^{\nu_\kappa},\ \kappa\in\mathbb{N}_0)$ in a collected way locally approaches the continuous function f, there are numbers $\kappa'_{\frac{\epsilon}{2}},\kappa''_{\frac{\epsilon}{2}}\in\mathbb{N}_0$ such that it holds

$$\big|\hat f^{\nu_\kappa}_{\alpha'^{\nu_\kappa}}(\hat x^{\nu_\kappa})\quad-f(\hat x^{\nu_\kappa})\big|\le\tfrac{\epsilon}{2}\quad\text{for all }\ \kappa\ge\qquad\kappa'_{\frac{\epsilon}{2}}, \tag{3.9a}$$

$$\big|f(\hat x^{\nu_\kappa})\quad-f(\hat x)\big|\le\tfrac{\epsilon}{2}\quad\text{for all }\ \kappa\ge\qquad\kappa''_{\frac{\epsilon}{2}}. \tag{3.9b}$$

Altogether, from (3.9a−b) we conclude

$$\big|\hat f^{\nu_\kappa}_{\alpha'^{\nu_\kappa}}(\hat x^{\nu_\kappa})\qquad-f(\hat x)\big|\le\epsilon\quad\text{for all }\ \kappa\ge\qquad\kappa''_{\epsilon}, \tag{3.10}$$

where $\kappa_{\epsilon}:=\max\{\kappa'_{\frac{\epsilon}{2}},\kappa''_{\frac{\epsilon}{2}}\}$. From (3.10) we learn

$$\lim_{\kappa\to\infty}\hat f^{\nu_\kappa}_{\alpha'^{\nu_\kappa}}(\hat x^{\nu_\kappa})=\;f(\hat x), \tag{3.11}$$

and, hence, in view of (3.7):

$$\min\{f(x) \mid x \in M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}\} \quad = \quad f(\hat{x}). \tag{3.12}$$

As the limit point $\hat{x}$ is feasible, i. e., $\hat{x} \in M_{\mathcal{SI}}[h,g] \cap \overline{\mathcal{U}^0}$, our proof is *finished* under *all* our assumptions.

We already indicated that for the more general situation where **Assumption C** on affine linearity and convexity may be is **violated**, our topological argumentations remain true. However, then the process of approximation and, hence, the corresponding convergence of some *minimizing sequence* are usually less fast.

Let us only remember that the new vertices $\tilde{y}^j_{\alpha,\beta}(x)$ may have become infeasible in the sense of $\tilde{y}^j_{\alpha,\beta}(x) \notin Y(x)$, and that the stability theory on the $\mathcal{GSI}$ feasible set also allows local perturbations, e. g., $u^{\mathrm{lin},\nu}_{\alpha,\beta}$ and $v^{\mathrm{lin},\nu}_{\alpha,\beta}$ $(\nu \in I\!N_0)$, of u and v, respectively ([30]).

3.2. Conclusion

In this paper we presented a concept of an iteration procedure for a wide class of generalized semi-infinite optimization problems under assumptions on boundedness and constraint qualifications, for both the feasible sets and the index sets of inequality constraints. We worked out the topological background and gave a proof of our convergence theorem. Hereby, the subclass of problems, where the defining functions of the index sets fulfill conditions of affine linearity and convexity, allowed special insights. Moreover, aspects of local-global modelling, of practical treatments and of comparisons with former approaches were also given. For a concrete given generalized semi-infinite optimization problem fulfilling our assumptions, the development of a solution algorithm can be performed, based on the problem's structural or geometrical features and on our iterative approach.

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