Вычислительные технологии, 2024, том 29, № 6, с. 108–124. © ФИЦ ИВТ, 2024
Computational Technologies, 2024, vol. 29, no. 6, pp. 108–124. © FRC ICT, 2024

ISSN 1560-7534
eISSN 2313-691X

COMPUTATIONAL TECHNOLOGIES

DOI:10.25743/ICT.2024.29.6.007

Theoretical and numerical analysis of semi-linear problem with two small parameters and turning points

V. D. Liseikin
Federal Research Center for Information and Computational Technologies SB RAS, 630090, Novosibirsk, Russia
Corresponding author: Vladimir D. Liseikin, e-mail: liseikin.v@gmail.com
Received March 29, 2024, revised June 04, 2024, accepted July 01, 2024.

The paper addresses a semi-linear problem with two small parameters and turning points. Estimates of solution derivatives and construction of layer-resolving grids based on layer-eliminating coordinate transformations and analyzes the convergence of numerical solutions obtained are given using an upwind scheme on the layer-resolving grids. Numerical experiments confirm that the upwind scheme on the proposed layer-resolving grids is uniformly convergent.

Keywords: adaptive grid, power-of-type layer, hybrid-boundary layer, upwind scheme.

Citation: Liseikin V.D. Theoretical and numerical analysis of semi-linear problem with two small parameters and turning points. Computational Technologies. 2024; 29(6):108–124. DOI:10.25743/ICT.2024.29.6.007.

Introduction

Problems with interior and boundary layers of various types are widely encountered in various fields of science and engineering, in particular in gas- and hydrodynamics, elasticity theory, chemistry and biology. Due to this layer phenomenon, developing uniformly convergent algorithms for solving such problems is a challenging task. Numerical grids provide resources which can significantly reduce the adverse effects of layers on the accuracy of numerical experiments. Efficient application of these resources requires detailed knowledge of the layers themselves — their types and structures; situations in which they occur; and the means to combat them, in particular, the rules for grid clustering in layers (see [1]).

Turning-point problems are considered important in practical applications. Some examples of turning-point problems and analytical and numerical aspects of their study are discussed in reviews [1–3], and books [4–7]. Analytical and numerical treatment of turning-point problems is more complicated than that problems without turning-point. In particular, it is not always possible to decompose a solution into regular and singular components to find estimates of the solution derivatives, since the reduced problem may be ill-posed. Thus, even though the reduced problem does not depend on a small parameter, the values of the regular component and/or its derivatives may be unbounded. Some special techniques for obtaining estimates of the derivatives of solutions to turning-point problems are shown in [1, 4, 5, 8–11], and in this paper.

The present paper discusses the following problem with two small parameters ε and μ:

$$\begin{array}{rcl} L[u] & \equiv & -\varepsilon u'' + \mu a(x) u' + f(x,u) = 0, \quad 0 < x < 1, \\ \Gamma[u] & \equiv & [u(0,\varepsilon,\mu), u(1,\varepsilon,\mu)] = (A_0, A_1), \end{array} \tag{1}$$

where $1 > \varepsilon > 0$, $1 > \mu > 0$, $a(x) \in C^n[0,1]$, $f(x,u) \in C^{n,n+1}([0,1] \times R)$, $f_u(x,u) \geq c_1 > 0$, $(x,u) \in [0,1] \times R$.

The linear case of this problem with two small parameters ε and μ, but without turning points ($a(x) \neq 0$), was initially analysed theoretically using the expansion technique in [12]. A quasi-linear problem with two small parameters ε and $\mu = \varepsilon^p$, $p > 1$ was analysed theoretically and numerically in [11, 13]. These papers also provide an introduction to areas of application of such problems. Linear problems without turning points with arbitrarily small parameters ε and μ have been analysed in [14–16], and in many others.

The present paper discusses the semi-linear problem (1), with an arbitrary coefficient $a(x)$ and without any relation between the parameters ε and μ.

1. Estimates of derivatives

This section describes estimates of solution derivatives for a two-point boundary-value problem (1).

1.1. Preliminary estimates

It is well known that the pair (L,Γ) in (1) is inverse-monotone, i. e., if for two functions $u(x)$ and $v(x)$, $0 \leq x \leq 1$,

$$(L,\Gamma)[u] \leq (L,\Gamma)[v], \quad 0 \leq x \leq 1, \qquad \text{then} \qquad u(x) \leq v(x), \quad 0 \leq x \leq 1.$$

This results in ε-uniform bounds on a solution $u(x,\varepsilon,\mu)$ to (1):

$$|u(x,\varepsilon,\mu)| \leq M, \quad 0 \leq x \leq 1. \tag{2}$$

In this equation and hereafter, by m, M, m_i, M_j we designate positive constants independent of ε and μ.

If μ is small enough, namely,

$$n\mu a'(x) + c(x) \geq c > 0, \quad 0 \leq x \leq 1, \tag{3}$$

for some $c > 0$, then according to [8] for a linear case and [4] for a semi-linear case, we have that

$$|u^{(i)}(x,\varepsilon,\mu)| \leq M, \quad 0 < m \leq x \leq 1 - m < 1, \quad i \leq n+1, \tag{4}$$

for any constant $0.5 > m > 0$. On the other hand, if x_0 is an interior turning point and $k\mu a'(x_0) + c \leq 0$, $k \leq n+1$, i. e., μ is not a small parameter, then in the vicinity of x_0 we have the following estimate (see [8] and [4, p. 90]):

$$|u^{(k)}(x,\varepsilon,\mu)| \leq M \left[(\varepsilon^{1/2} + |x - x_0|)^{\alpha - k} + 1 \right], \quad |x - x_0| \leq m,$$

for $k \leq n+1$, $0 < \alpha < \mu |a'(x_0)| / c(x_0)$, and some $m > 0$. Detailed estimates of the solution derivatives of problem (1) when (3) does not hold are given in [17].

We further assume that μ obeys constraint (3), and therefore solutions to (1) with any $a(x)$ can have only boundary layers. It is likely that a solution outside the boundary points is close to the solution of the reduced equation $f(x,u) = 0$, while near the boundaries it changes abruptly to satisfy the boundary conditions.

1.2. Estimates of solution derivatives near boundary turning points

This section uses an approach based on barrier functions to obtain estimates of solution derivatives near turning points.

Near the turning point $x = 0$, i. e., when $a(0) = 0$, we have

$$|u^{(i)}(x, \varepsilon)| \le M\varepsilon^{-i/2}, \quad 0 \le x \le m\varepsilon^{1/2}, \quad i \le n + 1. \tag{5}$$

To prove (5), we use estimate (2) to conclude that in the interval $[0, \delta]$ there exists a point x_0 such that $|u'(x_0, \varepsilon, \mu)| \le M/\delta$, where $u(x, \varepsilon, \mu)$ is a solution to (1). By integrating the equation (1) from x_0 to $x \in [0, \delta]$, we get

$$\begin{gathered}|u'(x, \varepsilon, \mu)| \le |u'(x_0, \varepsilon, \mu)| + \frac{1}{\varepsilon}\left|\int_{x_0}^{x}[\mu a(x)u'(x, \varepsilon, \mu) + f(x, u(x, \varepsilon, \mu))]dx\right| \le \\ \le \frac{M}{\delta} + \frac{1}{\varepsilon}\left|\mu a(x)u(x, \varepsilon, \mu)|_{x_0}^{x}\right| + \frac{1}{\varepsilon}\left|\int_{x_0}^{x} -\mu a'(x)u(x, \varepsilon, \mu) + f(x, u(x, \varepsilon, \mu))dx\right| \le \\ \le M(1/\delta + \mu\delta/\varepsilon + \delta/\varepsilon) \le M\varepsilon^{-1/2},\end{gathered} \tag{6}$$

when $\delta = m\varepsilon^{1/2}$, where m is an arbitrary positive constant. By using (1), we get estimate (5) for $i > 1$.

Similarly, in the vicinity of the boundary turning point $x = 1$, we have

$$|u^{(i)}(x, \varepsilon)| \le M\varepsilon^{-i/2}, \quad 0 \le 1 - x \le m\varepsilon^{1/2}, \quad i \le n + 1.$$

To estimate the derivatives of $u(x, \varepsilon, \mu)$ on the interval $[0, m]$, $0 < m < 1$, we introduce an operator

$$L_i[v] \equiv -\varepsilon v'' + \mu a(x)v' + [f_v(x, v) + i\mu a'(x)]\, v, \quad i \le n + 1. \tag{7}$$

Since (3), the pair (L_i, Γ) is inverse monotone on the interval $0 \le x \le 1$.

1.2.1. Estimates of the first derivative

Case $a(0) = 0$, $a'(0) \le 0$. In this case, to estimate the derivatives of $u(x, \varepsilon, \mu)$ on the interval $[0, m]$, $0 < m < 1$, we use the barrier function

$$d_i(x, \varepsilon, \mu) = M_1\varepsilon^{-i/2}\exp(-bx/\varepsilon^{1/2}) + M_2, \quad 0 < b < \sqrt{c}, \quad 0 \le x \le m < 1. \tag{8}$$

We have

$$\begin{aligned} L_1[d_1](x, \varepsilon, \mu) \;&=\; M_1\varepsilon^{-1/2}\exp(-bx/\varepsilon^{1/2})[-b^2 - b\mu\varepsilon^{-1/2}a(x) + f_u(x, u) + \mu a'(x)]+ \\ &+\; M_2[f_u(x, u) + \mu a'(x)]. \end{aligned} \tag{9}$$

In the case $a(0) = 0$, $a'(0) = 0$, we have $|a(x)| \le m_1 x^2$, $0 \le x \le 1$, for some $m_1 > 0$, and so

$$\varepsilon^{-1}\exp(-bx/\varepsilon^{1/2})b\mu|a(x)| \le m_2\mu, \quad 0 \le x \le 1, \tag{10}$$

for some $m_2 > 0$. Therefore, taking into account (3), (4), (6), and (9), we conclude that there exist constants $M_1 > 0$ and $M_2 > 0$ in (8) such that

$$\begin{gathered} d_1(0, \varepsilon, \mu) \ge |u'(0, \varepsilon, \mu)|, \quad d_1(m, \varepsilon, \mu) \ge |u'(m, \varepsilon, \mu)|, \quad 0 < m < 1, \quad 0 < b < \sqrt{c}, \\ L_1[d_1](x, \varepsilon, \mu) \ge L_1[u'](x, \varepsilon, \mu) = -f_x(x, u) \ge L_1[-d_1](x, \varepsilon, \mu),\; 0 \le x \le m < 1,\; 0 < b < \sqrt{c}. \end{gathered} \tag{11}$$

These relations yield the estimate

$$|u'(x,\varepsilon,\mu)| \le M[\varepsilon^{-1/2}\exp(-bx/\varepsilon^{1/2})+1], \quad 0 \le x \le m < 1, \tag{12}$$

when $a(0)=0$, $a'(0)=0$, where b is an arbitrary constant independent of ε and μ satisfying $0 < b < \sqrt{c}$.

In the case $a(0)=0$, $a'(0)<0$, we have $a'(x)<0$, $0 \le x \le m_3$, for some $m_3>0$, and so relations (11), and consequently estimates (12), are valid for $0 \le x \le m_3$. Taking into account (4), the estimates are valid for $0 \le x \le m < 1$.

Case $a(0)=0$, $a'(0)>0$, $\mu \le \varepsilon^{1/2}$. In this case we also have estimate (10) as well as the relations (11), and consequently estimate (12).

Case $a(0)=0$, $a'(0)>0$, $\mu \ge \varepsilon^{1/2}$. To estimate the solution derivatives in this case, we introduce the barrier function

$$v_i(x,\varepsilon,\mu) = M_1\frac{\varepsilon^{\alpha/2}}{(\varepsilon^{1/2}+x)^{\alpha+i}} + M_2, \quad \alpha > 0. \tag{13}$$

We have

$$L_1[v_1](x,\varepsilon,\mu) = M_1\frac{\varepsilon^{\alpha/2}}{(\varepsilon^{1/2}+x)^{\alpha+1}}\left(-\frac{\varepsilon(\alpha+1)(\alpha+2)}{(\varepsilon^{1/2}+x)^2} - \frac{(\alpha+1)\mu a(x)}{\varepsilon^{1/2}+x} + f_u(x,u) + \mu a'(x)\right) + $$
$$+M_2[f_u(x,u)+\mu a'(x)].$$

Since

$$-\frac{\varepsilon(\alpha+1)(\alpha+2)}{(\varepsilon^{1/2}+x)^2} - \frac{(\alpha+1)\mu a(x)}{\varepsilon^{1/2}+x} + f_u(x,u) + \mu a'(x) > 0, \quad x \ge M_0\varepsilon^{1/2}$$

for some $M_0 > 0$, therefore, for sufficiently large M_1 and M_2 in (13), we obtain

$$\begin{aligned} L_1[v_1](x,\varepsilon,\mu) &\ge L_1[u'](x,\varepsilon,\mu) \ge L_1[-v_1](x,\varepsilon,\mu), \quad M_0\varepsilon^{1/2} \le x \le m < 1,\\ v_1(M_0\varepsilon^{1/2},\varepsilon,\mu) &\ge |u'(M_0\varepsilon^{1/2},\varepsilon,\mu)|, \quad v_1(m,\varepsilon,\mu) \ge |u'(m,\varepsilon,\mu)|, \quad 0 < m < 1. \end{aligned}$$

Thus,

$$|u'(x,\varepsilon,\mu)| \le M\left(\frac{\varepsilon^{\alpha/2}}{(\varepsilon^{1/2}+x)^{\alpha+1}} + 1\right), \quad \alpha > 0, \quad M_0\varepsilon^{1/2} \le x \le m < 1,$$

and from (5) we have that this estimate is valid for $0 \le x < m < 1$, i. e.,

$$|u'(x,\varepsilon,\mu)| \le M\left(\frac{\varepsilon^{\alpha/2}}{(\varepsilon^{1/2}+x)^{\alpha+1}} + 1\right), \quad \alpha > 0, \quad 0 \le x \le m < 1,$$

for any $\alpha > 0$.

1.2.2. Estimates of higher derivatives

Estimates near the boundary turning point $x=0$. Extending the approach discussed above for $i=1$ by using for $i>1$ the operators L_i, the estimate (5), the corresponding barrier functions $d_i(x,\varepsilon,\mu)$ and $v_i(x,\varepsilon,\mu)$, and the estimates of the solution derivatives for $j<i$, we easily obtain the following estimates near the boundary turning point $x=0$:

$$|u^{(i)}(x,\varepsilon,\mu)| \le M[\varepsilon^{-i/2}\exp(-bx/\varepsilon^{1/2})+1], \quad 0 \le x \le m < 1, \quad 0 < b < \sqrt{c}, \tag{14}$$

if $1: a(0) = 0,\ a'(0) \le 0$, or $2: a(0) = 0,\ a'(0) > 0,\ \mu \le \varepsilon^{1/2}$, and also

$$|u^{(i)}(x,\varepsilon,\mu)| \le M\left(\frac{\varepsilon^{\alpha/2}}{(\varepsilon^{1/2}+x)^{\alpha+i}}+1\right), \quad \alpha>0, \quad 0\le x\le m<1, \tag{15}$$

for any $\alpha > 0$, if $a(0) = 0,\ a'(0) > 0,\ \mu \ge \varepsilon^{1/2}$, and so in the last case the solution has a power-of-type-1 boundary layer.

Note that if $\eta > 0$ is a small parameter, then

$$\eta^{-i}\exp(-bx/\eta) \le M\left(\frac{\eta^{\alpha}}{(\eta+x)^{\alpha+i}}+1\right), \quad 0\le x\le 1, \tag{16}$$

for an arbitrary $\alpha > 0$, $b > 0$ and some $M > 0$, where b, α, and M are independent of η. Thus, from (14)–(16) we get a compact formula in the case $a(0) = 0$, assuming in (16) that $\eta = \varepsilon^{1/2}$,

$$|u^{(i)}(x,\varepsilon,\mu)| \le M\left(\frac{\varepsilon^{\alpha/2}}{(\varepsilon^{1/2}+x)^{\alpha+i}}+1\right), \quad 0\le x\le m<1, \tag{17}$$

for an arbitrary $\alpha > 0$.

Estimates near the boundary turning point $x = 1$. It is obvious that the solution derivatives near the boundary turning point $x = 1$ ($a(1) = 0$) are estimated through the formulas obtained from (14) and (15) by substituting $1 - x$ for x, namely,

$$|u^{(i)}(x,\varepsilon,\mu)| \le M[\varepsilon^{-i/2}\exp(-b(1-x)/\varepsilon^{1/2})+1], \quad 0\le(1-x)\le m<1, \quad 0<b<\sqrt{c},$$

if $1: a(1) = 0,\ a'(1) \le 0$, or $2: a(1) = 0,\ a'(1) > 0,\ \mu \le \varepsilon^{1/2}$;

$$|u^{(i)}(x,\varepsilon,\mu)| \le M\left(\frac{\varepsilon^{\alpha/2}}{(\varepsilon^{1/2}+1-x)^{\alpha+i}}+1\right), \quad \alpha>0, \quad 0\le(1-x)<m<1,$$

for any $\alpha > 0$ if $a(1) = 0,\ a'(1) > 0,\ \mu \ge \varepsilon^{1/2}$.

Similarly to (17), we have the following compact formula in the case $a(1) = 0$:

$$|u^{(i)}(x,\varepsilon,\mu)| \le M\left(\frac{\varepsilon^{\alpha/2}}{(\varepsilon^{1/2}+1-x)^{\alpha+i}}+1\right), \quad 0\le(1-x)\le m<1, \tag{18}$$

for an arbitrary $\alpha > 0$.

1.3. Estimates near non-turning boundary points

Estimates of solution derivatives near non-turning boundary points have been obtained in many papers, e. g., for nonlinear equations with $\mu = \varepsilon^{1+p}$ in [13, 18]; for linear equations with $a(x) < 0$ in [19]; while with $a(x) > 0$ in [20]. A popular approach to obtaining estimates of solution derivatives for linear equations without turning points is based on certain characteristic equations (see for example [20]), but it is questionable whether this approach is appropriate in the case of semi-linear equations or interior turning points. We present here estimates near a non-turning boundary point x_0 in the most general semi-linear case and without any restrictions on $a(x)$, $x \ne x_0$, obtained by the barrier-function approach.

In the neighbourhood of the non-turning boundary point $x = 0$ ($a(0) \ne 0$) we get, similarly to (6),

$$|u'(x,\varepsilon,\mu)| \le M(1/\delta + \mu/\varepsilon + \delta/\varepsilon),$$

yielding

$$|u'(x,\varepsilon,\mu)| \le M \begin{cases} \mu/\varepsilon, & \mu \ge \varepsilon^{1/2}, \quad 0 \le x \le m\varepsilon/\mu, \\ \varepsilon^{-1/2}, & \mu \le \varepsilon^{1/2}, \quad 0 \le x \le m\varepsilon^{1/2}. \end{cases}$$

This estimate is readily generalized to $i \le n+1$:

$$|u^{(i)}(x,\varepsilon,\mu)| \le M \begin{cases} (\mu/\varepsilon)^i, & \mu \ge \varepsilon^{1/2}, \quad 0 \le x \le m\varepsilon/\mu, \\ \varepsilon^{-i/2}, & \mu \le \varepsilon^{1/2}, \quad 0 \le x \le m\varepsilon^{1/2}, \end{cases} \tag{19}$$

when $a(0) \ne 0$.

1.3.1. Case $a(0) > 0$, $\mu \ge \varepsilon^{1/2}$

We will use here the following resolution of (1) with respect to $u'(x,\varepsilon,\mu)$:

$$u'(x,\varepsilon,\mu) = \exp[\phi(x_0,x,\varepsilon,\mu)] \left(u'(x_0,\varepsilon,\mu) + \varepsilon^{-1} \int_{x_0}^{x} \exp[-\phi(x_0,\xi,\varepsilon,\mu)] f[\xi, u(\xi,\varepsilon,\mu)] \mathrm{d}\xi \right), \tag{20}$$

where

$$\phi(\xi,x,\varepsilon,\mu) = \varepsilon^{-1} \int_{\xi}^{x} \mu a(\eta) \mathrm{d}\eta.$$

Note that (20) is held for an arbitrary point x_0 in $[0,1]$.

As $a(0) > 0$, so $a(x) \ge m$, $0 \le x \le m$ for some $m > 0$, and therefore,

$$\phi(\xi,x,\varepsilon,\mu) \ge m\mu(x-\xi)/\varepsilon, \quad 0 \le \xi \le x \le m.$$

Thus, taking into account estimates (4), (5), and (19) in (20) for $i = 1$, we obtain

$$|u'(x,\varepsilon,\mu)| \le |u'(m,\varepsilon,\mu)| \exp[\phi(m,x,\varepsilon,\mu)] + M\varepsilon^{-1} \int_{m}^{x} \exp[m\mu(\xi - x)/\varepsilon] \mathrm{d}\xi \le M\mu^{-1}, \quad 0 \le x \le m.$$

Sequentially differentiating i times the equation in (1), resolving the corresponding $(i+1)^{\text{th}}$ derivative in the form (20), and using estimates of derivatives for $j < i$, we obtain the estimates

$$|u^{(i)}(x,\varepsilon,\mu)| \le M\mu^{-i}, \quad 0 \le x \le m < 1, \tag{21}$$

which are more accurate than estimate (19) for $\mu \ge \varepsilon^{1/2}$.

For estimating solution derivatives in the case $a(0) > 0$, $\mu \ge \varepsilon^{1/2}$, we introduce the barrier function

$$z_i(x,\varepsilon,\mu) = M_1 \mu^{-i} \exp(-bx/\mu) + M_2, \quad 0 \le x \le m < 1.$$

We get

$$L_1[z_1](x,\varepsilon,\mu) = M_1\mu^{-1} \exp(-bx/\mu)[-b^2\varepsilon/\mu^2 - ba(x) + f_u(x,u) + \mu a'(x)] + M_2[f_u(x,u) + \mu a'(x)],$$

thus, if $c - b(b(\varepsilon/\mu^2) + a(0)) > 0$, then

$$L_1[z_1](x,\varepsilon,\mu) \ge |L_1[u'](x,\varepsilon,\mu)|, \quad 0 \le x \le m_3$$

and

$$z_1(0,\varepsilon,\mu) \geq |u'(0,\varepsilon,\mu)|, \quad z_1(m_3,\varepsilon,\mu) \geq |u'(m_3,\varepsilon,\mu)|,$$

for some $m_3 > 0$, $M_1 > 0$, and $M_2 > 0$ and, consequently, we have

$$|u'(x,\varepsilon,\mu)| \leq M[\mu^{-1}\exp(-bx/\mu) + 1], \quad 0 \leq x \leq m_3,$$

where b is an arbitrary positive constant satisfying $c - b((\varepsilon/\mu^2)b + a(0)) > 0$, in particular when $c - b(b + a(0)) > 0$; and from (5) we have that this estimate is valid for $0 \leq x < m < 1$.

Extending this process for $i > 1$ by using the operator L_i from (7), barrier function z_i, and applying estimates (4) and (21), we obtain, in the same manner,

$$|u^{(i)}(x,\varepsilon,\mu)| \leq M(\mu^{-i}\exp(-bx/\mu) + 1), \quad 0 \leq x < m < 1, \tag{22}$$

for b satisfying $c - b(b + a(0)) > 0$.

1.3.2. Case $a(0) < 0$, $\mu \geq \varepsilon^{1/2}$

To estimate the derivatives of the solution in this case, we introduce the barrier function

$$w_i(x,\varepsilon,\mu) = M_1(\mu/\varepsilon)^i \exp(-\mu bx/\varepsilon) + M_2, \quad 0 \leq x \leq m < 1.$$

Using the operator (7) for $i = 1$, we get

$$\begin{aligned} L_1[w_1](x,\varepsilon,\mu) &= M_1(\mu/\varepsilon)\exp(-\mu bx/\varepsilon)[-b^2(\mu^2/\varepsilon) - b(\mu^2/\varepsilon)a(x) + f_u(x,u) + \mu a'(x)] + \\ &+ M_2[f_u(x,u) + \mu a'(x)]. \end{aligned}$$

Then, for b satisfying $c - (\mu^2/\varepsilon)b(b + a(0)) > 0$, in particular since $a(0) < 0$ for $0 < b < -a(0)$, we have, using (2) and (19) for $i = 1$,

$$L_1[w_1](x,\varepsilon,\mu) \geq |L_1[u'](x,\varepsilon,\mu)|, \quad 0 \leq x \leq m_0,$$

and

$$w_1(0,\varepsilon,\mu) \geq |u'(0,\varepsilon,\mu)|, \quad w_1(m_0,\varepsilon,\mu) \geq |u'(m_0,\varepsilon,\mu)|,$$

for sufficiently large M_1 and M_2 when $0 \leq x \leq m_0$ for some $m_0 > 0$. These relations and (4) yield the estimate

$$|u'(x,\varepsilon,\mu)| \leq M(\mu/\varepsilon\exp(-\mu bx/\varepsilon) + 1), \quad 0 \leq x \leq m < 1,$$

when $a(0) < 0$, $\mu \geq \varepsilon^{1/2}$, $0 < b < -a(0)$.

Further, using sequentially the barrier function w_i, operator L_i, and estimates (19) and (4), we obtain

$$|u^{(i)}(x,\varepsilon,\mu)| \leq M((\mu/\varepsilon)^i \exp(-\mu bx/\varepsilon) + 1), \quad 0 \leq x \leq m < 1, \tag{23}$$

when $a(0) < 0$, $\mu \geq \varepsilon^{1/2}$, $0 < b < -a(0)$.

1.3.3. Case $a(0) \neq 0$, $\mu \leq \varepsilon^{1/2}$

To estimate the derivatives of the solution in this case, we introduce the barrier function

$$\psi_i(x,\varepsilon,\mu) = M_1\varepsilon^{-i/2}\exp(-bx/\varepsilon^{1/2}) + M_2, \quad 0 \leq x \leq m < 1.$$

We get

$$L_1[\psi_1](x,\varepsilon,\mu) = M_1\varepsilon^{-1/2}\exp(-bx/\varepsilon^{1/2})[-b^2 - b(\mu/\varepsilon^{1/2})a(x) + f_u(x,u) + \mu a'(x)] + \\ + M_2[f_u(x,u) + \mu a'(x)];$$

thus, if $c - b(b + (\mu/\varepsilon^{1/2})a(0)) > 0$, in particular $c - b(b + a(0)) > 0$ when $a(0) > 0$ and $c - b^2 > 0$ when $a(0) < 0$, then

$$L_1[\psi_1](x,\varepsilon,\mu) \geq |L_1[u'](x,\varepsilon,\mu)|, \quad 0 \leq x \leq m_4,$$

for some $m_4 > 0$; and from (19) and (4) for $\mu \leq \varepsilon^{1/2}$, we have

$$\psi_1(0,\varepsilon,\mu) \geq |u'(0,\varepsilon,\mu)|, \quad \psi_1(m_4,\varepsilon,\mu) \geq |u'(m_4,\varepsilon,\mu)|,$$

for sufficiently large M_1 and M_2. Consequently, we obtain

$$|u'(x,\varepsilon,\mu)| \leq M[\varepsilon^{-1/2}\exp(-bx/\varepsilon^{1/2}) + 1], \quad 0 \leq x \leq m_4,$$

where b is an arbitrary positive constant satisfying $c - b(b + a(0)) > 0$ when $a(0) > 0$, and $c - b^2 > 0$ when $a(0) < 0$.

Extending this process for $i > 1$ by using ψ_i, L_i, and applying (19) and (4), we obtain

$$|u^{(i)}(x,\varepsilon,\mu)| \leq M(\varepsilon^{-i/2}\exp(-bx/\varepsilon^{1/2}) + 1), \quad 0 \leq x < m < 1, \tag{24}$$

where b is an arbitrary positive constant satisfying $c - b(b + a(0)) > 0$ when $a(0) > 0$, and $c - b^2 > 0$ when $a(0) < 0$.

1.3.4. Compact formulas

Estimates (22), (23), and (24) are also formulated in the following compact form

$$|u^{(i)}(x,\varepsilon,\mu)| \leq M(\eta^{-i}\exp(-bx/\eta) + 1), \quad 0 \leq x < m < 1, \tag{25}$$

where $\eta = \mu$, b is an arbitrary positive number satisfying $c - b(b + a(0)) > 0$ when $a(0) > 0$, $\mu \geq \varepsilon^{1/2}$; $\eta = \varepsilon/\mu$, $0 < b < -a(0)$ when $a(0) < 0$, $\mu \geq \varepsilon^{1/2}$; $\eta = \varepsilon^{1/2}$ when $a(0) \neq 0$, $\mu \leq \varepsilon^{1/2}$, and b is an arbitrary positive constant satisfying $c - b(b + a(0)) > 0$ when $a(0) > 0$ and $c - b^2 > 0$ when $a(0) < 0$.

In accordance with (16), we also have the estimates

$$|u^{(i)}(x,\varepsilon,\mu)| \leq M\left(\frac{\eta^\alpha}{(\eta + x)^{\alpha+i}} + 1\right), \quad 0 \leq x < m < 1, \tag{26}$$

where α is an arbitrary positive number; $\eta = \mu$ when $a(0) > 0$, $\mu \geq \varepsilon^{1/2}$; $\eta = \varepsilon/\mu$ when $a(0) < 0$, $\mu \geq \varepsilon^{1/2}$; and $\eta = \varepsilon^{1/2}$ when $a(0) \neq 0$, $\mu \leq \varepsilon^{1/2}$.

1.3.5. Estimates of solution derivatives near boundary non-turning point $x = 1$

It is obvious that the solution derivatives in the vicinity of the non-turning point $x = 1$ ($a(1) \neq 0$) are estimated by the formulae obtained from (25) and (26) by changing the sign of $a(1)$ and substituting $1 - x$ for x, namely

$$|u^{(i)}(x, \varepsilon, \mu)| \leq M(\eta^{-i} \exp(-b(1-x)/\eta) + 1), \quad 0 \leq (1-x) < m < 1,$$

where $\eta = \mu$, b is an arbitrary positive number satisfying $c - b(b - a(1)) > 0$ when $a(1) < 0$, $\mu \geq \varepsilon^{1/2}$; $\eta = \varepsilon/\mu$, $0 < b < a(1)$ when $a(1) > 0$, $\mu \geq \varepsilon^{1/2}$; $\eta = \varepsilon^{1/2}$ when $a(1) \neq 0$, $\mu \leq \varepsilon^{1/2}$, and b is an arbitrary positive constant satisfying $c - b(b - a(1)) > 0$ when $a(1) < 0$ and $c - b^2 > 0$ when $a(1) > 0$.

In accordance with (16), we also have the estimates

$$|u^{(i)}(x, \varepsilon, \mu)| \leq M \left(\frac{\eta^{\alpha}}{(\eta + 1 - x)^{\alpha + i}} + 1 \right), \quad 0 \leq (1-x) < m < 1, \tag{27}$$

where α is an arbitrary positive number; $\eta = \mu$ when $a(1) < 0$, $\mu \geq \varepsilon^{1/2}$; $\eta = \varepsilon/\mu$ when $a(1) > 0$, $\mu \geq \varepsilon^{1/2}$; and $\eta = \varepsilon^{1/2}$ when $a(1) \neq 0$, $\mu \leq \varepsilon^{1/2}$.

1.4. Global estimates of solution derivatives

Using the previous local estimates (17), (18), (25)–(27), and (4) of solution derivatives for problem (1), we can obtain global formulae for derivatives on the interval $[0, 1]$. In particular, using the local estimates (26), (27), and (4), we obtain the following global estimates:

$$|u^{(i)}(x, \varepsilon, \mu)| \leq M \left(\frac{\eta_1^{\alpha_1}}{(\eta_1 + x)^{\alpha_1 + i}} + \frac{\eta_2^{\alpha_2}}{(\eta_2 + 1 - x)^{\alpha_2 + i}} + 1 \right), \quad 0 \leq i \leq n+1, \quad 0 \leq x \leq 1, \tag{28}$$

where α_1 and α_2 are arbitrary positive numbers, $\eta_1 = \varepsilon^{1/2}$ if $a(0) = 0$; $\eta_1 = \mu$ if $a(0) > 0$ and $\mu \geq \varepsilon^{1/2}$; $\eta_1 = \varepsilon/\mu$ if $a(0) < 0$ and $\mu \geq \varepsilon^{1/2}$; $\eta_1 = \varepsilon^{1/2}$ if $a(0) \neq 0$ and $\mu \leq \varepsilon^{1/2}$; $\eta_2 = \varepsilon^{1/2}$ if $a(1) = 0$; $\eta_2 = \mu$ if $a(1) < 0$ and $\mu \geq \varepsilon^{1/2}$; $\eta_2 = \varepsilon/\mu$ if $a(1) > 0$ and $\mu \geq \varepsilon^{1/2}$; and $\eta_2 = \varepsilon^{1/2}$ if $a(1) \neq 0$ and $\mu \leq \varepsilon^{1/2}$.

1.5. Transformations eliminating layers

The numerical algorithm proposed in this paper for solving problem (1) is based on piecewise smooth layer-damping coordinate transformations $x(\xi, \varepsilon) : [0, 1] \to [0, 1]$ in compliance with a basic principle: they are to eliminate singularities of high order of solutions $u(x, \varepsilon)$ at each subinterval $[a_i, b_i]$ of smoothness; i. e., the high-order derivatives of any concrete solution with respect to the new coordinate ξ are to have the bounds:

$$\left| \frac{\mathrm{d}^i}{\mathrm{d}\xi^i} u[x(\xi, \varepsilon), \varepsilon] \right| \leq M, \quad i \leq n, \quad a_i \leq \xi \leq b_i, \tag{29}$$

where the constant M does not depend on the parameter ε and the number n depends on the order of the approximation of the problem: the higher the order, the larger the number n will be. With the help of such transformations, any problem can be solved using high-order approximations in the physical interval x on layer-resolving grids defined by mapping the nodes of a uniform grid in $[0, 1]$ with suitable coordinate transformations $x(\xi, \varepsilon)$, as in [7, 21]. It is proposed that by using the layer-resolving grids obtained by transformations $x(\xi,\varepsilon)$ satisfying (29), ε-uniform high-order convergence will be demonstrated for schemes of high order in the physical interval x. Moreover, the numerical solution can be interpolated ε-uniformly with high-order accuracy on the entire interval $[0,1]$.

1.5.1. Basic transformation

To generate layer-resolving grids near a boundary point, we will use as a template the following universal coordinate transformation of the class $C^l[0,1]$ described in [4, 5]:

$$x_{left}(\xi,\eta,a) = \begin{cases} c_1\eta((1-d\xi)^{-1/a}-1), & 0\le\xi\le\xi_0, \\ c_1\left[\eta(1-\beta/a)-\eta+\left(\dfrac{\eta}{(1-d\xi)^{1/a}}\right)'(\xi_0)(\xi-\xi_0)+\right. & \\ +\dfrac{1}{2}\left(\dfrac{\eta}{(1-d\xi)^{1/a}}\right)''(\xi_0)(\xi-\xi_0)^2+\ldots+ & \\ \left.+\dfrac{1}{l!}\left(\dfrac{\eta}{(1-d\xi)^{1/a}}\right)^{(l)}(\xi_0)(\xi-\xi_0)^l+c_0(\xi-\xi_0)^{l+1}\right], & \xi_0\le\xi\le 1, \end{cases} \tag{30}$$

where η is a small parameter, $d=\dfrac{1-\eta^\beta}{\xi_0}\ge 1+m_1>1$, $n\ge l$, $\beta=\dfrac{a}{1+na}$, $1-\dfrac{\beta}{a}=\dfrac{na}{1+na}$, a is an arbitrary positive constant satisfying $0<a\le\alpha/n^2$, and $c_0>0$, while $c_1>0$ is such as satisfies the necessary boundary condition $x_{left}(1,\varepsilon,a)=1$,

$$\left(\frac{\eta}{(1-d\xi)^{1/a}}\right)^{(i)}(\xi_0)=d^i\frac{1}{a}\left(\frac{1}{a}+1\right)\ldots\left(\frac{1}{a}+i-1\right)\eta^{a(n-i)/(1+na)},\quad i\ge 1. \tag{31}$$

Note that formula (30) and all further formulas are valid for $\eta\in(0,1)$. It was shown in [5, 7] that this transformation eliminates up to n both power-of-type-1 singularities $\eta^\alpha/(\eta+x)^{\alpha+i}$ and exponential singularities $(1/\eta^i)\exp(-bx/\eta)$.

A simpler form of transformation (30) for an arbitrary $a>0$ was originally published in [22], while for $a=1$ in [18]. Paper [23] shows that the grid obtained using the transformation (30) is the most effective for numerical modelling of viscous flows over a plate, compared to results obtained with the grids often used by many.

Assuming in (30) $l=2$ as we typically do, taking into account (31), transformation (30) is as follows:

$$x_{left}(\xi,\eta,a) = \begin{cases} c_1\eta\left((1-d\xi)^{-1/a}-1\right), & 0\le\xi\le\xi_0, \\ c_1\left[\eta^{an/(1+na)}-\eta+d\dfrac{1}{a}\eta^{a(n-1)/(1+na)}(\xi-\xi_0)+\right. & \\ \left.+\dfrac{1}{2}d^2\dfrac{1}{a}\left(\dfrac{1}{a}+1\right)\eta^{a(n-2)/(1+na)}(\xi-\xi_0)^2c_0(\xi-\xi_0)^3\right], & \xi_0\le\xi\le 1, \end{cases} \tag{32}$$

where $d=\dfrac{1-\eta^{a/(1+na)}}{\xi_0}$, a is an arbitrary positive constant satisfying $0<a\le\alpha/n^2$,

$$\frac{1}{c_1}=\eta^{n/(1+na)}-\eta+d\frac{1}{a}\eta^{a(n-1)/(1+na)}(1-\xi_0)+\frac{1}{2}d^2\frac{1}{a}\left(\frac{1}{a}+1\right)\eta^{a(n-2)/(1+na)}(1-\xi_0)^2+c_0(1-\xi_0)^3.$$

An explicit transformation to generate a grid with node clustering near $x = 1$, denoted as $x_{right}(\xi, \eta, a)$, which eliminates up to n both power-of-type-1 singularities $\eta^\alpha/(\eta + 1 - x)^{\alpha+i}$ and exponential singularities $(1/\eta^i)\exp(m(x-1)/\eta)$, can be defined by the formula

$$x_{right}(\xi, \eta, a) = 1 - x_{left}(1 - \xi, \eta, a), \quad 0 \le \xi \le 1, \tag{33}$$

where $x_{left}(\eta, \xi, a)$ is given by general (30) or special (32) formulae. For example, by using formula (32), where $l = 2$, $a = 2$, $\xi_0 = 1/2$, we get:

$$x_{right}(\xi,\eta,2) = \begin{cases} 1 - c_1\Big(\eta^{2n/(1+2n)} - \eta + (1-\eta^{2/(1+2n)})\eta^{2(n-1)/(1+2n)}(1/2-\xi) + \\ + \dfrac{3}{2}(1-\eta^{2/(1+2n)})^2\eta^{2(n-2)/(1+2n)}(1/2-\xi)^2 + c_0(1/2-\xi)^3\Big), & 0 \le \xi \le \dfrac{1}{2}, \\ 1 - c_1\eta\{[1-2(1-\eta^{2/(1+2n)})(1-\xi)]^{-1/2} - 1\}, & \dfrac{1}{2} \le \xi \le 1, \end{cases}$$

where the constant c_1 is such that $\varepsilon^{1/2}(0, \eta, 2) = 0$.

An explicit transformation for generating a grid with node clustering near $x = 0$ and $x = 1$, denoted $x_{global}(\xi, \eta_1, a_1, \eta_2, a_2)$, for eliminating up to n power-of-type-1 singularities near both boundaries:

$$\frac{\eta_1^\alpha}{(\eta_1 + x)^{\alpha+i}} + \frac{\eta_2^\alpha}{(\eta_2 + 1 - x)^{\alpha+i}},$$

where η_1 and η_2 are small parameters, as in (28), and corresponding exponential singularities, can be defined by the combination of two transformations — either (30) and (33), or (32) and (33), namely, by formula

$$x_{global}(\xi, \eta_1, a_1, \eta_2, a_2) = x_{left}(x_{right}(\xi, \eta_2, a_2), \eta_1, a_1),$$

or

$$x_{global}(\xi, \eta_1, a_1, \eta_2, a_2) = x_{right}(x_{left}(\xi, \eta_1, a_1), \eta_2, a_2),$$

where a_i is an arbitrary positive constant independent of the small parameters η_1 and η_2 and satisfying $0 < a_i \le \alpha_i/n^2$, $i = 1, 2$.

Another way to define a similar transformation is to use $x_{left}(\xi, \eta, a)$:

$$x_{global}(\xi, \eta_1, a_1, \eta_2, a_2) = \begin{cases} 0.5x_{left}(2\xi, \eta_1, a_1), & 0 \le \xi \le 0.5, \\ 1 - 0.5x_{left}(2(1-\xi), \eta_2, a_2), & 0.5 \le \xi \le 1, \end{cases} \tag{34}$$

or to use $x_{right}(\xi, \eta, a)$:

$$x_{global}(\xi, \eta_1, a_1, \eta_2, a_2) = \frac{1}{2}\left(1 + \overline{y}(\theta, \eta_1, a_1, \eta_2, a_2)\right), \quad \theta = -1 + 2\xi, \quad 0 \le \xi \le 1,$$

where

$$\overline{y}(\theta, \eta_1, a_1, \eta_2, a_2) = \begin{cases} -x_{right}(-\theta, \eta_1, a_1), & -1 \le \eta \le 0, \\ x_{right}(\theta, \eta_2, a_2), & 0 \le \eta \le 1. \end{cases}$$

2. Numerical algorithm and experiments

2.1. Upwind numerical algorithm

As an approximation to the singularly perturbed boundary-value problem (1), we use the standard upwind finite-difference scheme on a non-uniform grid x_i, $i = 0, 1, \ldots, N$, $x_0 = 0 < x_1 < \ldots < x_N = 1$:

$$L^N\left[u_i^N\right] \equiv -\frac{2\varepsilon}{h_i+h_{i-1}}\left(\frac{u_{i+1}^N-u_i^N}{h_i}-\frac{u_i^N-u_{i-1}^N}{h_{i-1}}\right)+a_-(x_i)\frac{u_{i+1}^N-u_i^N}{h_i}+$$
$$+a_+(x_i)\frac{u_i^N-u_{i-1}^N}{h_{i-1}}+f(x_i,u_i)=0,\quad i=1,2,\ldots,N-1,\ u_0^N=A_0,\ u_N^N=A_1,\quad (35)$$

where $h_i = x_{i+1} - x_i$ and $a_\pm(x) = (a(x) \pm |a(x)|)/2$. The nodes x_i, $i = 0, \ldots, N$, of the layer-resolving grid are obtained explicitly using layer-damping transformation (34), namely,

$$x_i = x_{global}(ih, \eta_1, a_1, \eta_2, a_2),\quad i = 0, 1, \ldots, N,\quad h = 1/N.$$

For estimating the accuracy of the numerical algorithm, the following characteristic is introduced based on the double-mesh principle:

$$r_{t,\varepsilon} = \max_{0\le i\le N_t}\left|u_i^{N_t} - u_{2i}^{N_{t+1}}\right|,\quad t = 0, 1, \ldots,$$

where $u_i^{N_t} = u^N(x_i)$ when upwind scheme (35) is used, with N_t number of mesh points. In addition, one more characteristic,

$$du_{t,\varepsilon} = \max_{0\le i\le N_t}\left|u_{i+1}^{N_t} - u_i^{N_t}\right|,\quad i = 0, 1, \ldots, N_t - 1,$$

is introduced, related to the jump of the numerical solution at the neighboring nodes.

The characteristic $r_{t,\varepsilon}$ is applied to estimate the order of accuracy of the numerical solution:

$$\beta_1 = \log_2(r_{t,\varepsilon}/r_{t+1,\varepsilon}),\quad t = 0, 1, \ldots,$$

and, consequently, $du_{t,\varepsilon}$ to estimate the order of numerical-solution jump in the neighboring nodes

$$\beta_3 = \log_2(du_{t,\varepsilon}/du_{t+1,\varepsilon}),\quad t = 0, 1, \ldots$$

Note that if a solution to (1) has neither boundary nor interior layers, then for the numerical solution of this problem the value β_1 is close to l_0, while β_3 is close to 1 through the use of a stable scheme of order l_0 on the uniform grid $x_i = ih$.

Theorem 2.1. *If* $x_i = x_{global}(i/N, \eta_1, a_1, \eta_2, a_2)$, $i = 0, 1, \ldots, N$, *where* $x_{global}(\xi, \eta_1, a_1, \eta_2, a_2)$: $[0, 1] \to [0, 1]$ *is a coordinate transformation defined through (30) and (34) with* $l \ge 2$, $n \ge l$, *then*

$$|u_i^N - u(x_i, \varepsilon, \mu)| \le M/N,\quad i = 0, \ldots, N,$$

where M *is independent of* N.

This theorem is proved similarly to [4, sect. 7.4.1 and 7.4.2] for a semi-linear problem with one small parameter ε on the grid obtained using the same basic transformation (30).

2.2. Numerical examples

2.2.1. Example 1

For our first numerical experiment we consider the following problem:

$$-\varepsilon u'' + \mu(x - 2x^2)u' + u = 0.1\exp(x),\quad 0 \le x \le 1,$$
$$u(0, \varepsilon) = 0.05,\quad u(1, \varepsilon) = 0.15,$$

with $\mu > \varepsilon^{1/2}$. For this problem, $a(0) = 0$, $a'(0) > 0$, $a(1) = -1$, $c(x) = 1$, and thus estimates (28) are as follows:

$$|u^{(i)}(x,\varepsilon,\mu)| \le M\left(1 + \frac{\varepsilon^{\alpha_1/2}}{(\varepsilon^{1/2}+x)^{\alpha_1+i}} + \frac{\mu^{\alpha_2}}{(\mu+1-x)^{\alpha_2+i}}\right), \quad 0 \le i \le n+1, \quad 0 \le x \le 1,$$

for arbitrary $\alpha_1 > 0$ and $\alpha_2 > 0$. This singularity is eliminated up to n by coordinate transformation (34) with $\eta_1 = \varepsilon^{1/2}$, $\eta_2 = \mu$.

Figure 1 and Table 1 show the numerical solution and values of the characteristics β_1 and β_3 for $\varepsilon = 10^{-6}$, $\mu = 10^{-2}$ calculated using difference scheme (35) on the grid $x_i = x_{global}(i/N, \eta_1, a_1, \eta_2, a_2)$, $i = 0, 1, \dots, N$, where $x_{global}(\xi, \eta_1, a_1, \eta_2, a_2) : [0,1] \to [0,1]$ is a coordinate transformation (34) with $\eta_1 = \varepsilon^{1/2} = 10^{-3}$, $a_1 = 1/20$, $\eta_2 = \mu = 10^{-2}$, $a_2 = 1/20$. Figure 2 shows the coordinate transformation $x_{global}(\xi, \eta_1, a_1, \eta_2, a_2)$.

Table 2 shows the values of characteristics β_1 and β_3 of the numerical solution for $\varepsilon = 10^{-8}$, $\mu = 10^{-3}$, calculated with difference scheme (35) on the grid $x_i = x_{global}(i/N, \eta_1, a_1, \eta_2, a_2)$, $i = 0, 1, \dots, N$, where $x_{global}(\xi, \eta_1, a_1, \eta_2, a_2) : [0,1] \to [0,1]$ is a coordinate transformation (34) with $\eta_1 = \varepsilon^{1/2} = 10^{-4}$, $a_1 = 1/20$, $\eta_2 = \mu = 10^{-3}$, $a_2 = 1/20$.

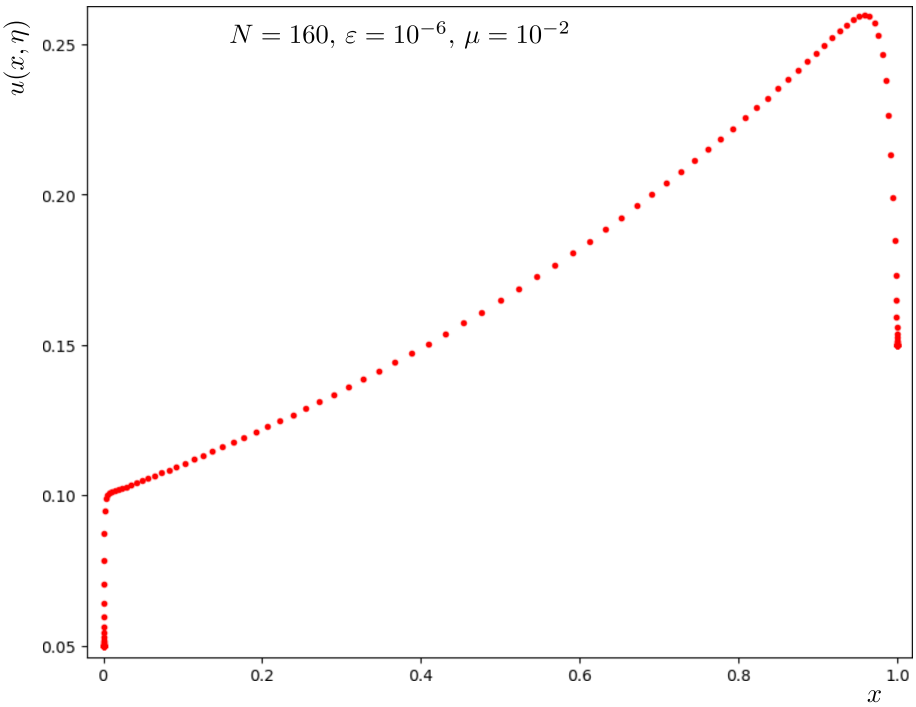

Fig. 1. Example 1

T a b l e 1. Order of solution convergence and solution jump for $\varepsilon = 10^{-6}$, $\mu = 10^{-2}$

t	N	r	β_1	du	β_3
1	160	0.0038	0.000000	0.008802	0.837976
2	320	0.00216	0.819031	0.004384	1.00558
3	640	0.00117	0.881370	0.002206	0.990747
4	1280	0.000607	0.945925	0.001105	0.997741
5	2560	0.00031	0.972436	0.000553	0.999506
6	5120	0.000156	0.985953	0.000276	0.999899

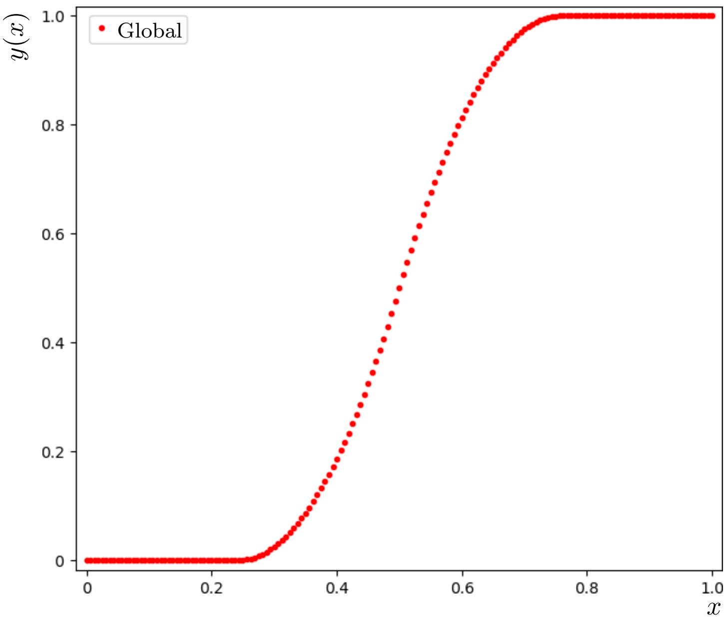

Fig. 2. Example 1

T a b l e 2. Order of solution convergence and solution jump for $\varepsilon = 10^{-8}$, $\mu = 10^{-3}$

t	N	r	β_1	du	β_3
4	160	0.001611	0.676430	0.026067	0.991273
5	320	0.000916	0.814067	0.012987	1.005161
6	640	0.000491	0.898392	0.006485	1.001811
7	1280	0.000256	0.941229	0.003240	1.001279
8	2560	0.000130	0.972124	0.001619	1.000632
9	5120	0.000066	0.985845	0.000809	1.000304

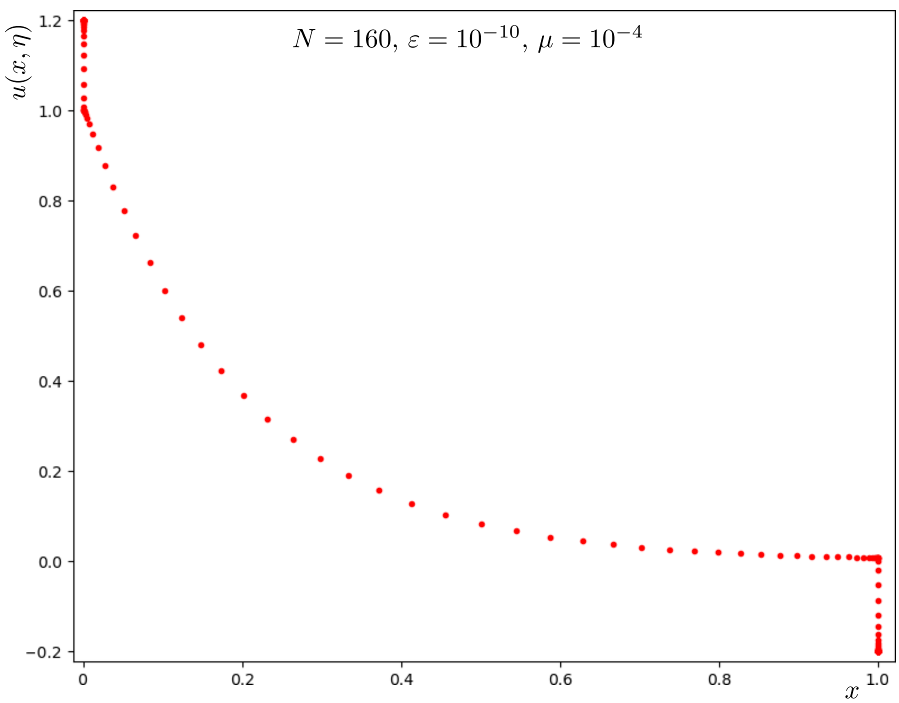

Fig. 3. Example 2

T a b l e 3. Order of solution convergence and solution jump for $\varepsilon = 10^{-10}$, $\mu = 10^{-4}$

t	N	r	β_1	du	β_3
1	160	0.0126	0.000000	0.008802	1.47946
2	320	0.00398	1.662363	0.004384	1.19617
3	640	0.000989	2.008477	0.002206	1.08667
4	1280	0.000247	2.001993	0.001105	1.04041
5	2560	6.16e-05	2.002023	0.000553	1.01946
6	5120	1.54e-05	2.003634	0.000276	1.08059

2.2.2. Example 2

For the second numerical experiment we consider the problem:

$$-\varepsilon u'' + \mu x(1-x)u' + u = \exp(-5x), \quad 0 \le x \le 1,$$
$$u(0,\varepsilon) = 1.2, \quad u(1,\varepsilon) = -0.2.$$

For this problem, $a(0) = 0$, $a(1) = 0$, and so estimates (28) are as follows:

$$|u^{(i)}(x,\varepsilon,\mu)| \le M\left(1 + \frac{\varepsilon^{\alpha_1/2}}{(\varepsilon^{1/2}+x)^{\alpha_1+i}} + \frac{\varepsilon^{\alpha_2/2}}{(\varepsilon^{1/2}+1-x)^{\alpha_2+i}}\right), \quad 0 \le i \le n+1, \quad 0 \le x \le 1,$$

for arbitrary $\alpha_1 > 0$ and $\alpha_2 > 0$.

This singularity is eliminated up to n by coordinate transformation (34) with $\eta_1 = \varepsilon^{1/2}$, $\eta_2 = \varepsilon^{1/2}$.

Figure 3 and Table 3 show the numerical solution and values of characteristics β_1 and β_3 for $\varepsilon = 10^{-10}$, $\mu = 10^{-4}$ calculated using difference scheme (35) on the grid $x_i = x_{global}(i/N, \eta_1, a_1, \eta_2, a_2)$, $i = 0, 1, \ldots, N$, where $x_{global}(\xi, \eta_1, a_1, \eta_2, a_2) : [0,1] \to [0,1]$ is a coordinate transformation (34) with $\eta_1 = \varepsilon^{1/2} = 10^{-5}$, $a_1 = 1/20$, $\eta_2 = 10^{-5}$, $a_2 = 1/20$.

Conclusion

The paper addresses a semi-linear singularly perturbed problem with two small parameters and turning points. Estimates of solution derivatives and construction of layer-resolving grids based on layer-eliminating coordinate transformations are described. The convergence of numerical solutions obtained using an upwind scheme on the layer-resolving grids is analysed. Theoretical results are confirmed by numerical experiments.

Acknowledgements. This study was funded by the Federal Research Center for Information and Computational Technologies (project “Development and investigations of computational technologies for solving fundamental and applied problems of air-, hydro-, and wave dynamics”).

The author wishes to express appreciation for the valuable comments provided by reviewers. Thanks go as well to native speaker (American) G. Lukas for correcting the author’s English.

Declaration of competing interest. The author declares that he has no financial or other conflicts of interest.

References

[1] **Liseikin V.D.** Survey of layer structures and coordinate transformations eliminating layers. World Journal of Physics. 2024; 2(1):143–171.

[2] **Kadalbajoo M.K., Patidar K.C.** A survey of numerical techniques for solving singularly perturbed ordinary differential equations. Applied Mathematics and Computation. 2002; 130(2):457–510.

[3] **Kadalbajoo M.K., Gupta V.** A brief survey on numerical methods for solving singularly perturbed problems. Applied Mathematics and Computation. 2010; 217(8):3641–3716.

[4] **Liseikin V.D.** Layer resolving grids and transformations for singular perturbation problems. VSP, Utrecht; 2001: 284.

[5] **Liseikin V.D.** Grid generation for problems with boundary and interior layers. Novosibirsk: Novosibirskiy Gosudarstvennyy Universitet; 2018: 296.

[6] **Liseikin V.D., Petrenko V.E.** Adaptive invariant method for the numerical solution of problems with boundary and interior layers. Novosibirsk: Computer Center SD AS USSR; 1989: 258. (In Russ.)

[7] **Liseikin V.D., Karasuljić S., Paasonen V.I.** Numerical grids and high-order schemes for problems with boundary and interior layers. Novosibirsk: Novosibirskiy Gosudarstvennyy Universitet; 2021: 180.

[8] **Berger A.E., Han H., Kellog R.B.** A priori estimates and analysis of a numerical method for a turning point problem. Mathematics of Computation. 1984; 42(166):465–492.

[9] **Liseikin V.D.** Theoretical and numerical analysis of problems with an interior turning point and a variable diffusion coefficient. Computational Technologies. 2023; 28(5):101–113. DOI:10.25743/ICT.2023.28.5.009.

[10] **Vulanović R., Farrell P.A.** Continuous and numerical analysis of a multiple boundary turning point problem. SIAM Journal of Numerical Analysis. 1993; (30):1400–1418.

[11] **Vulanović R.** A higher-order scheme for quasilinear boundary value problems with two small parameters. Computing. 2001; 67(4):287–303.

[12] **O'Malley Jr.** Two-parameter singular perturbation problems for second order equations. Journal of Mathematics and Mechanics. 1967; (16):1143–1164.

[13] **Vulanović R.** On numerical solution of quasilinear boundary value problems with two small parameters. Journal of Computational Mathematics. 1991; 9(4):321–329.

[14] **Gracia J.L., O'Riordan E., Pickett M.L.** A parameter robust second order numerical method for a singularly perturbed two-parameter problem. Applied Numerical Mathematics. 2006; 56(7):962–980.

[15] **Linss T., Roos H.-G.** Analysis of a finite-difference scheme for a singularly perturbed problem with two small parameters. Mathematical Analysis and Applications. 2004; (289):355–366. Available at: `www.elsevier.com/locate/jmaa`.

[16] **Shishkin G.I., Titov V.A.** A difference scheme for a differential equation with two small parameters at the derivatives. Chislennye Metody Mekhaniki Sploshnoy Sredy. 1976; 7(2):145–155. (In Russ.)

[17] **Liseikin V.D.** Estimates for derivatives of solutions to differential equations with boundary and interior layers. Sibirskii Matematicheskii Zhurnal. 1992; 33(6):106–117. (In Russ.)

[18] **Vulanović R.** Mesh construction for numerical solution of a type of singular perturbation problems. Numerical Methods and Approximation Theory. 1984: 137–142.

[19] **Linss T.** A posteriori error estimation for a singularly perturbed problem with two small parameters. International Journal Analysis and Modeling. 2010; 7(3):491–506.

[20] **Brdar M., Zarin H.** A singularly perturbed problem with two parameters on a Bakhvalov-type mesh. Journal of Computational and Applied Mathematics. 2015; 292(3):1–26.

[21] **Liseikin V.D.** On the numerical solution of singularly perturbed equations with a turning point. USSR Computational Mathematics and Mathematical Physics. 1984; 24(12):1812–1818.

[22] **Liseikin V.D.** On the numerical solution of an equation with a power boundary layer on a non-uniform grid. Chislennye Metody Mekhaniki Sploshnoy Sredy. 1984; 15(2):90–97. (In Russ.)

[23] **Kudryavtsev A.N., Liseikin V.D., Mukhortov A.V.** Analysis of boundary-layer mesh refinement rules as applied to the computation of viscous gas flow over a plate. Computational Mathematics and Mathematical Physics. 2022; 62(8):1354–1369.

Вычислительные технологии, 2024, том 29, № 6, с. 108–124. © ФИЦ ИВТ, 2024 ISSN 1560-7534
Computational Technologies, 2024, vol. 29, no. 6, pp. 108–124. © FRC ICT, 2024 eISSN 2313-691X

ВЫЧИСЛИТЕЛЬНЫЕ ТЕХНОЛОГИИ

DOI:10.25743/ICT.2024.29.6.007

Теоретический и численный анализ полулинейной задачи с двумя малыми параметрами и точками поворота

В. Д. Лисейкин

Федеральный исследовательский центр информационных и вычислительных технологий, 630090, Новосибирск, Россия
Контактный автор: Лисейкин Владимир Дмитриевич, e-mail: liseikin.v@gmail.com

Поступила 29 марта 2024 г., доработана 04 июня 2024 г., принята в печать 01 июля 2024 г.

Аннотация

Рассматривается двухточечная полулинейная краевая задача с двумя малыми параметрами и точками поворота. Доказываются оценки производных решения задачи, на основе которых строятся координатные преобразования, устраняющие слои. Анализируется сходимость численного решения задачи с помощью схемы с направленными разностями на полученных сетках. Численными расчетами подтверждена равномерная сходимость решения для разных значений малых параметров к точному решению.

Ключевые слова: адаптивная сетка, слои степенного типа, гибридные пограничные слои, схема с направленными разностями.

Цитирование: Лисейкин В.Д. Теоретический и численный анализ полулинейной задачи с двумя малыми параметрами и точками поворота. Вычислительные технологии. 2024; 29(6):108–124. DOI:10.25743/ICT.2024.29.6.007. (на английском)