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Analysis and numerical results for modified Signorini problem with nonlocal friction in electro-elasticity

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The focus of this paper is on a mathematical model that depicts the state equilibrium of a piezoelectric structure in contact with a conductive foundation, taking into account the presence of friction. The constitutive law governing the electro-elastic behavior of the system is considered to be non-linear, while the contact is modelled using Signorini's modified contact conditions. These conditions are supplemented by a non-local Coulomb friction law and an electrical conductivity condition that has been regularized. A weak formulation of the model is presented as a coupled system that relates the displacement and electric potential fields. The weak solution is shown to be both unique and existent through the invoke of Banach fixed-point theorem and arguments of abstract elliptic quasi-variational inequalities. Additionally, we explore the problem's finite element approximation and derive estimate of its associated error. In conclusion, an iterative method is introduced to solve the finite element system resulting from the analysis, and the convergence analysis of the method is considered under appropriate conditions.

Keywords: piezoelectric body, conductive foundation, Signorini modified contact conditions, Coulomb friction law, quasi-variational inequality, Banach fixed-point, iterative method.

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Introduction

Piezoelectrics are a prevalent and important type of material used in a variety of applications in engineering and real life. Although they were discovered over 100 years ago by brothers Pierre and Jacques Curie, scientists still find new ways to use them. The conversion of electrical energy into mechanical energy and vice versa is a defining characteristic of piezoelectric materials, which results in observed properties that exhibit electromechanical coupling.

In recent years, there has been considerable interest in mathematical investigations of contact problems that involve piezoelectric materials. However, the choice of appropriate contact boundary conditions continues to be a significant challenge in modelling these problems. One of the most widely used boundary conditions in both engineering and mathematical literature are the so-called Signorini conditions. They were introduced by Signorini in [1], in which the problem of unilateral contact between a linearly elastic body and a rigid foundation is formulated. It follows the work of Fichera [2] where the Signorini problem was solved using arguments of elliptic variational inequalities. Several authors have been interested in question of the existence and uniqueness of weak solutions to contact problems. More precisely, the early attempt to study frictional contact problems within the framework of variational inequalities was started with the monograph of Duvaut and Lions [3].

The documentation of piezoelectric modelling is very extensive; see, for instance [4–7]. Relevant models for elastic materials with piezoelectric effects can be found in [8–10] and the references therein. Some theoretical results for contact models with static friction taking into account the interaction between electrical and mechanical fields have been obtained in [11, 12], under the assumption that the foundation is electrically insulated, and in [13] assuming that the foundation is electrically conductive. Moreover, analysis and numerical simulation of contact with or without friction for piezoelectric materials can be found in [8, 11, 14–26] and the references therein. Several research papers have examined the topic of static and quasi-static contact problems in thermo-viscoelasticity and thermo-piezoelectricity involving friction. These papers can be found in [16, 23, 24, 26–29].

In this paper, we investigate a mathematical model that describes the static frictional contact between a piezoelectric body and a conductive foundation. The body is assumed to be electro-elastic, with a non-linear elasticity operator. In contrast to the models discussed in [9, 10, 13, 30, 31], we assume herein that the contact is modelled using the Signorini modified contact conditions (see [3, ch. 3, p. 147]), nonlocal Coulomb friction law with slip dependent friction coefficient and a regularized electrical conductivity condition, taking into account the conductivity of the foundation as in [22, 26, 30], which involve a coupling between the mechanical and the electrical unknowns. This work is divided into two parts. The first one is devoted to the existence and uniqueness of the solution. The second one is reserved for the numerical approximation of the variational formulation by the finite element method combined with an iteration method. This research stands apart from the studies referenced in [9, 13, 22, 30] by examining different boundary conditions and employing a distinct approximation approach. The modified Signorini contact conditions lead to a variational formulation which differs from the one presented in [30] by the presence of the nondifferentiable terms and represents a new mathematical model for piezoelectric materials. To our knowledge, this model with Signorini modified contact conditions for piezoelectric materials has not been studied yet and no result has been obtained for this type of problem. An important extention of this paper is the numerical analysis of the model. Numerical simulations will be presented in a forthcoming work.

The outline of the paper is as follows. Basic notations and preliminary material for the rest of the paper are recalled in Sect. 1. The mechanical problem is stated in Sect. 2. In Sect. 3, we present the variational problem and state assumptions about the given data. The unknowns for the variational problems are the displacement field and the electric potential. In Sect. 4, we state and prove our main results. The proofs are based on arguments from elliptic variational inequalities and Banach fixed-point properties of certain maps. Moreover, in Sect. 5, we study the finite element approximation of the variational problem, and we derive error estimates. Finally, in Sect. 6, we propose an iterative method to solve the resulting finite element system, which converges under certain assumptions.

1. Notations and preliminaries

In this section, we introduce the notations and various functional spaces that will play a crucial role in formulating and analyzing the mechanical problem. For more in-depth information, interested readers can refer to the following references: [27, 32–34].

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is:

$$\mathbf{u}\,.\,\mathbf{v} = u_i\, v_i\ ; \quad \|\mathbf{v}\| = (\mathbf{v}\,.\,\mathbf{v})^{1/2} \quad \forall \mathbf{u},\, \mathbf{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\, \tau_{ij}\ ; \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2} \quad \forall \boldsymbol{\sigma},\, \boldsymbol{\tau} \in \mathbb{S}^d.$$

The summation convention over repeated indices is used, all indices take values in $1, \ldots, d$.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2,\ 3)$ be an open and bounded domain with a Lipschitz boundary Γ that is divided into three open disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and on two measurable parts Γ_a and Γ_b, on the other hand, such that $\text{meas}(\Gamma_1) > 0$ and $\text{meas}(\Gamma_a) > 0$. Since the boundary is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ is defined a. e. on Γ.

We use the notation u_ν and $\mathbf{u}_\tau$ for the normal and tangential displacement that is $u_\nu = \mathbf{u} \cdot \boldsymbol{\nu}$ and $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$. We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and tangential tress given by $\sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$.

We define, respectively, the positive and the negative part of v_ν by:

$$v_\nu^+ = \max(0,\, v_\nu), \quad v_\nu^- = \max(-v_\nu,\, 0). \tag{1}$$

We introduce the following functional spaces:

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d, \quad \mathcal{H} = \{\, \boldsymbol{\sigma} = (\sigma_{ij})\,;\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$\mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H}\,;\ \text{Div}\, \sigma \in H\}, \quad \mathcal{W} = \{\mathbf{D} = (\text{D}_i) \in L^2(\Omega)^d\,;\ \text{div}\, \mathbf{D} \in L^2(\Omega)\}.$$

These spaces are real Hilbert spaces equipped with the following inner products:

$$(\mathbf{u},\, \mathbf{v})_H = \int_\Omega u_i\, v_i\, d\mathbf{x},\ \ (\boldsymbol{\sigma},\, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\, \tau_{ij}\, d\mathbf{x},\ \ (\mathbf{u},\, \mathbf{v})_{H_1} = (\mathbf{u},\, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}),\, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}},$$
$$(\boldsymbol{\sigma},\, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\, \boldsymbol{\tau})_{\mathcal{H}} + (\text{Div}\, \boldsymbol{\sigma}, \text{Div}\, \boldsymbol{\tau})_H, \quad (\mathbf{D}, \mathbf{E})_{\mathcal{W}} = (\mathbf{D}, \mathbf{E})_H + (\text{div}\, \mathbf{D}, \text{div}\, \mathbf{E})_{L^2(\Omega)},$$

with the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\mathcal{H}_1}$ and $\|\cdot\|_{\mathcal{W}}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H_1$, we also use the notation $\mathbf{v}$ to note the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ. Let H'_Γ be the dual of H_Γ and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between H'_Γ and H_Γ. For every $\boldsymbol{\sigma} \in \mathcal{H}_1$, $\boldsymbol{\sigma}\boldsymbol{\nu}$ can be defined as the element in H'_Γ which satisfying Green's formula as follows:

$$\langle \boldsymbol{\sigma}\boldsymbol{\nu},\, \gamma\mathbf{v} \rangle = (\boldsymbol{\sigma},\, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\text{Div}\boldsymbol{\sigma},\, \mathbf{v})_H \quad \forall \mathbf{v} \in H_1. \tag{2}$$

Moreover, if $\boldsymbol{\sigma}$ is continuously differentiable on $\overline{\Omega}$, then:

$$\langle \boldsymbol{\sigma}\boldsymbol{\nu}, \gamma\mathbf{v} \rangle = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \mathbf{v}\, da,$$

for all $\mathbf{v} \in H_1$, where da is the surface measure element. We also introduce $H^{1/2}(\Gamma_3) \subset L^2(\Gamma_3)$ the space of normal traces on Γ_3:

$$H^{1/2}(\Gamma_3) = \left\{ v_\nu \in L^2(\Gamma_3); \quad \exists \mathbf{v} \in H_1, \quad v_\nu = \gamma \mathbf{v} \cdot \boldsymbol{\nu} \right\},$$

and its dual $H^{-1/2}$, with norms correspondingly:

$$\|v_\nu\|_{H^{1/2}(\Gamma_3)} = \inf_{\mathbf{v} \in H_1} \{ \|\mathbf{v}\|_{H_1} ; \; v_\nu = \gamma \mathbf{v} \cdot \boldsymbol{\nu} \} \quad \forall v_\nu \in H^{1/2}(\Gamma_3),$$

$$\|\sigma_\nu\|_{H^{-1/2}(\Gamma_3)} = \sup_{\substack{v_\nu \in H^{1/2}_{\Gamma_3}, \\ v_\nu \neq 0_{H^{1/2}_{\Gamma_3}}}} \frac{\langle \sigma_\nu, \, v_\nu \rangle}{\|v_\nu\|_{H^{1/2}(\Gamma_3)}} \quad \forall \sigma_\nu \in H^{-1/2}(\Gamma_3), \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between $H^{-1/2}(\Gamma_3)$ and $H^{1/2}(\Gamma_3)$.

Bearing in mind the boundary conditions, we introduce the displacement and the electric potential spaces:

$$V = \{ \mathbf{v} \in H_1 \, ; \; \mathbf{v} = 0 \text{ on } \Gamma_1 \}, \quad W = \{ \xi \in H^1(\Omega) \, ; \; \xi = 0 \text{ on } \Gamma_a \}.$$

Since meas$(\Gamma_1) > 0$ and meas$(\Gamma_a) > 0$, Korn's and Friedrichs−Poincaré inequalities hold: There exists $c_K > 0$ and $c_F > 0$ which depends only on Ω, Γ_1 and Γ_a such that:

$$\begin{aligned} \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} &\geq c_K \|\mathbf{v}\|_{H_1} \quad \forall \mathbf{v} \in V, \\ \|\nabla \xi\|_H &\geq c_F \|\xi\|_{H^1(\Omega)} \quad \forall \xi \in W. \end{aligned} \tag{4}$$

Therefore, the space V equipped with the inner product $(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}$ is a real Hilbert space, and its associated norm $\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}}$ is equivalent on V to the usual norm $\| \cdot \|_{H_1}$. On W, we consider the inner product: $(\varphi, \xi)_W = (\nabla \varphi, \nabla \xi)_H$. It is straightforward from (4) that $\| \cdot \|_{H^1(\Omega)}$ and $\| \cdot \|_W$ are equivalent norms on W and thus $(W, \| \cdot \|_W)$ is a real Hilbert space. By Sobolev's trace theorem, there exist two positive constants c_0 and c_1 which depends only on Ω, Γ_3, Γ_1 and Γ_a such that:

$$\|\mathbf{v}\|_{L^2(\Gamma)^d} \leq c_0 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V, \tag{5}$$

$$\|\xi\|_{L^2(\Gamma_3)} \leq c_1 \|\xi\|_W \quad \forall \xi \in W. \tag{6}$$

2. Physical model and its mathematical formulation

The physical setting of the contact problem is as follows. We consider a piezoelectric body occupying, in its reference configuration, an open and bounded domain Ω with a sufficiently smooth boundary $\partial\Omega = \Gamma$. As outlined in the preceding section, this boundary is divided accordingly. The body is subjected to an action of body forces of density $\mathbf{f}_0$ and a volume electric charges of density q_0. It is clamped on Γ_1 and a surface traction of density $\mathbf{f}_2$ act on Γ_2. Moreover, the electric potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. On Γ_3 the body is in contact with friction with a conductive obstacle, the so-called foundation. We model the frictional contact with the Signorini modified contact conditions and nonlocal Coulomb's friction law. We assume that the foundation is electrically conductive and its potential is maintained at φ_F. The mechanical problem associated with the static frictional contact between a piezoelectric body and a deformable conductive foundation, considering that the contact is described using modified Signorini contact conditions, can be expressed as follows:

Problem (P). Find a displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \to \mathbb{R}$ and an electric displacement field $\mathbf{D} : \Omega \to \mathbb{R}^d$ such that:

$$\boldsymbol{\sigma} = \mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\mathcal{P}}^*\mathbf{E}(\varphi) \quad \text{in } \Omega, \tag{7}$$

$$\mathbf{D} = \boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\beta}\mathbf{E}(\varphi) \quad \text{in } \Omega, \tag{8}$$

$$\mathrm{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = 0 \quad \text{in } \Omega, \tag{9}$$

$$\mathrm{div}\mathbf{D} = q_0 \quad \text{in } \Omega, \tag{10}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1, \tag{11}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2, \tag{12}$$

$$\left.\begin{array}{l} g_1(\|\mathbf{u}\|) \le \sigma_\nu(\mathbf{u},\, \varphi) \le g_2(\|\mathbf{u}\|), \\ g_1(\|\mathbf{u}\|) < \sigma_\nu(\mathbf{u},\, \varphi) < g_2(\|\mathbf{u}\|) \Rightarrow u_\nu = 0, \\ \sigma_\nu = g_1(\|\mathbf{u}\|) \Rightarrow u_\nu \ge 0, \\ \sigma_\nu = g_2(\|\mathbf{u}\|) \Rightarrow u_\nu \le 0, \end{array}\right\} \quad \text{on } \Gamma_3, \tag{13}$$

$$\left.\begin{array}{l} \|\boldsymbol{\sigma}_\tau\| \le \mu(\|\mathbf{u}_\tau\|)|\mathrm{R}\sigma_\nu(\mathbf{u},\, \varphi)|, \\ \|\boldsymbol{\sigma}_\tau\| < \mu(\|\mathbf{u}_\tau\|)|\mathrm{R}\sigma_\nu(\mathbf{u}, \varphi)| \Rightarrow \mathbf{u}_\tau = 0, \\ \boldsymbol{\sigma_\tau} = -\mu(\|\mathbf{u}_\tau\|)|\mathrm{R}\sigma_\nu(\mathbf{u},\, \varphi)|\dfrac{\mathbf{u}_\tau}{\|\mathbf{u}_\tau\|} \Rightarrow \mathbf{u}_\tau \ne 0, \end{array}\right\} \quad \text{on } \Gamma_3, \tag{14}$$

$$\varphi = 0 \quad \text{on } \Gamma_a, \tag{15}$$

$$\mathbf{D} \cdot \boldsymbol{\nu} = q_b \quad \text{on } \Gamma_b, \tag{16}$$

$$\mathbf{D} \cdot \boldsymbol{\nu} = \psi(u_\nu)\phi_L(\varphi - \varphi_F) \quad \text{on } \Gamma_3. \tag{17}$$

Equations (7) and (8) represent the electro-elastic constitutive law of the material in which $\mathfrak{F}$ denotes the elasticity operator, assumed to be non-linear, in which $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, $\boldsymbol{\mathcal{P}}$ is the piezoelectric tensor verifying:

$$\boldsymbol{\mathcal{P}}\boldsymbol{\sigma} \cdot \mathbf{v} = \boldsymbol{\sigma}\boldsymbol{\mathcal{P}}^* \cdot \mathbf{v} \quad \forall \boldsymbol{\sigma} \in \mathbb{S}^d,\, \mathbf{v} \in \mathbb{R}^d.$$

$\boldsymbol{\mathcal{P}}^*$ is its transpose, and $\boldsymbol{\beta}$ denotes the electric permittivity tensor. Equations (8), (9) and (10) represents the equilibrium equations for the stress and electric displacement fields, respectively. Relations (11) and (12) are the displacement and traction boundary conditions, respectively, and (15), (16) represents the electric boundary conditions. Relations (13) embody Signorini's modified contact law (see [3, ch. 3, p. 147]), wherein the thresholds g_1 and g_2 delineate critical limits for surface contact pressure, which must not be exceeded to prevent localized crushing of the material due to excessive pressure in the contact zone. The inequality $g_1(\|\mathbf{u}\|) < \sigma_\nu < g_2(\|\mathbf{u}\|)$ indicates that when the normal stress (contact pressure) remains within a specified range, it avoids exceeding critical thresholds. As a result, the normal displacement is zero, i. e., $u_\nu = 0$. If the normal stress reaches the upper threshold $\sigma_\nu = g_2(\|\mathbf{u}\|)$ (which is positive), a displacement normal to the contact surface occurs, resulting in a negative value for u_ν ($u_\nu \le 0$). Conversely, if the contact pressure reaches the lower threshold $\sigma_\nu = g_1(\|\mathbf{u}\|)$ (which is negative), a displacement normal to the contact surface occurs, resulting in a positive value for u_ν ($u_\nu \ge 0$). Relations (14) represents the Coulomb's friction law in which μ is the coefficient of friction and R is a regularization operator. Finally, (17) represents the regularized electrical contact condition on Γ_3, which was considered in [26], where ψ and ϕ_L are a regularization function and the truncation function, respectively, such that:

$$\phi_L(s) = \begin{cases} -L, & \text{if } s < -L, \\ s, & \text{if } -L \le s \le L, \\ L, & \text{if } s > L. \end{cases} \qquad \psi(r) = \begin{cases} 0, & \text{if } r < 0, \\ k\delta r, & \text{if } 0 \le r \le 1/\delta, \\ k, & \text{if } r > 1/\delta, \end{cases}$$

in which L is a large positive constant, $\delta > 0$ denotes a small parameter and $k \ge 0$ is the electrical conductivity coefficient.

3. Variational formulation of the problem

In order to state the unique solvability of Problem (P), we need the following hypotheses:
(H_1) The elasticity operator $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfy the following conditions:
(a) There exists $\mathcal{M}_{\mathfrak{F}} > 0$, such that:

$$\|\mathfrak{F}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathfrak{F}(\mathbf{x}, \boldsymbol{\xi}_2)\| \le \mathcal{M}_{\mathfrak{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \quad \forall \boldsymbol{\xi}_1,\ \boldsymbol{\xi}_2 \in \mathbb{S}^d,\ \text{a. e. } \mathbf{x} \in \Omega.$$

(b) There exists $m_{\mathfrak{F}} > 0$, such that:

$$(\mathfrak{F}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathfrak{F}(\mathbf{x}, \boldsymbol{\xi}_2))(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathfrak{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2 \quad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d,\ \text{a. e. } \mathbf{x} \in \Omega.$$

(c) The mapping $\mathbf{x} \mapsto \mathfrak{F}(\mathbf{x}, \boldsymbol{\xi})$ is Lebesgue measurable on Ω, for all $\boldsymbol{\xi} \in \mathbb{S}^d$.
(d) The mapping $\mathbf{x} \mapsto \mathfrak{F}(\mathbf{x}, 0)$ belongs to $\mathcal{H}$.
(H_2) The piezoelectric tensor $\boldsymbol{\mathcal{P}} = (p_{ijk})$ satisfies: $p_{ijk} = p_{ikj} \in L^\infty(\Omega)$.
(H_3) The electric permittivity tensor $\boldsymbol{\beta} = (\beta_{ij})$ satisfies:
(a) $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$.
(b) $\exists m_{\boldsymbol{\beta}} > 0$, such that:

$$\beta_{ij}\xi_i\xi_j \ge m_{\boldsymbol{\beta}} \|\boldsymbol{\xi}\|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d,\ \text{a. e. } \mathbf{x} \in \Omega.$$

Notice that the above conditions allows us to define:

$$\mathcal{M}_{\boldsymbol{\mathcal{P}}} = \sup_{ij} \|p_{ijk}\|_{L^\infty(\Omega)}, \quad \mathcal{M}_{\boldsymbol{\beta}} = \sup_{ij} \|\boldsymbol{\beta}_{ij}\|_{L^\infty(\Omega)}.$$

(H_4) The surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfy:
(a) $\exists L_\psi > 0$, such that:

$$|\psi(\cdot, w_1) - \psi(\cdot, w_2)| \le L_\psi |w_1 - w_2| \quad \forall w_1,\ w_2 \in \mathbb{R}.$$

(b) $\exists M_\psi > 0$, such that:

$$|\psi(x, w)| \le M_\psi \quad \forall w \in \mathbb{R},\ \text{a. e. } x \in \Gamma_3.$$

(c) $\mathbf{x} \mapsto \psi(\mathbf{x}, w)$ is measurable on Γ_3, for all $w \in \mathbb{R}$.
(d) $\mathbf{x} \mapsto \psi(\mathbf{x}, w) = 0$, for all $w \le 0$.
(H_5) The coefficient of friction $\mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy:
(a) $\exists L_\mu > 0$, such that:

$$|\mu(\cdot, w_1) - \mu(\cdot, w_2)| \le L_\mu |w_1 - w_2| \quad \forall w_1, w_2 \in \mathbb{R}_+.$$

(b) $\exists \mu^* > 0$, such that:

$$\mu(\mathbf{x}, w) \le \mu^* \quad \forall w \in \mathbb{R}_+,\ \text{a. e. } \mathbf{x} \in \Gamma_3.$$

(c) The mapping $\mathbf{x} \mapsto \mu(\mathbf{x}, w)$ is measurable on Γ_3, for all $w \in \mathbb{R}_+$.

(H_6) The function $g_1 : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_-$ satisfies the following conditions:
(a) $\exists L_{g_1} > 0$, such that:

$$|g_1(\mathbf{x}, w_1) - g_1(\mathbf{x}, w_2)| \le L_{g_1}|w_1 - w_2| \quad \forall w_1, w_2 \in \mathbb{R}_+, \text{ a.e. } \mathbf{x} \in \Gamma_3.$$

(b) $\exists M_{g_1} > 0$, such that:

$$|g_1(\mathbf{x}, w)| \le M_{g_1} \quad \forall w \in \mathbb{R}_+, \text{ a.e. } \mathbf{x} \in \Gamma_3.$$

(c) The mapping $\mathbf{x} \mapsto g_1(\mathbf{x}, w)$ is measurable on Γ_3, for all $w \in \mathbb{R}_+$.

(H_7) The function $g_2 : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions:
(a) $\exists L_{g_2} > 0$, such that:

$$|g_2(\mathbf{x}, w_1) - g_2(\mathbf{x}, w_2)| \le L_{g_2}|w_1 - w_2| \quad \forall w_1, w_2 \in \mathbb{R}_+, \text{ a.e. } \mathbf{x} \in \Gamma_3.$$

(b) $\exists M_{g_2} > 0$, such that:

$$|g_2(\mathbf{x}, w)| \le M_{g_2} \quad \forall w \in \mathbb{R}_+, \text{ a.e. } \mathbf{x} \in \Gamma_3.$$

(c) The mapping $\mathbf{x} \mapsto g_2(\mathbf{x}, w)$ is measurable on Γ_3, for al $w \in \mathbb{R}_+$.

(H_8) The body forces, the traction, the volume and surface charge densities, also the given potential satisfy:

$$\mathbf{f}_0 \in L^2(\Omega)^d, \quad \mathbf{f}_2 \in L^2(\Gamma_3)^d, \quad q_0 \in L^2(\Omega), \quad q_b \in L^2(\Gamma_b), \quad \varphi_F \in L^2(\Gamma_3).$$

(H_9) The mapping $\mathrm{R} : H_{\Gamma_3}^{-1/2} \to L^\infty(\Gamma_3)$ is linear and continuous with $\|\mathrm{R}\| = c_\mathrm{R}$.

Next, we define the elements $\mathbf{f} \in V$ and $\mathbf{q} \in W$, respectively, by:

$$(\mathbf{f}, \mathbf{v})_V = \int_\Omega \mathbf{f}_0 \cdot \mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v}\, da \quad \forall \mathbf{v} \in V, \tag{18}$$

$$(q_e, \xi)_W = \int_\Omega q_0 \xi\, dx - \int_{\Gamma_b} q_b\, \xi\, da \quad \forall \xi \in W. \tag{19}$$

We define the mappings $J : V \times W \times V \to \mathbb{R}$ and $\chi : V \times W \times W \to \mathbb{R}$, respectively, by:

$$J(\mathbf{u}, \varphi, \mathbf{v}) = \int_{\Gamma_3} \mu(\|\mathbf{u}_\tau\|)|\mathrm{R}\sigma_\nu(\mathbf{u}, \varphi)|\ \|\mathbf{v}_\tau\|\, da + \int_{\Gamma_3} g_2(\|\mathbf{u}\|)\ v_\nu^-\, da - \int_{\Gamma_3} g_1(\|\mathbf{u}\|)\ v_\nu^+\, da, \tag{20}$$

$$\chi(\mathbf{u}, \varphi, \xi) = \int_{\Gamma_3} \psi(u_\nu)\ \phi_L(\varphi - \varphi_F)\ \xi\, da. \tag{21}$$

Keeping in mind assumptions (H_4)–(H_8) it follows that the integrals in (18)–(21) are well-defined. Thus, according to these notations and by using a standard procedure based on Green's formula, we can state the variational formulation of Problem (P), in the terms of displacement field and electric potential.

Problem (PV). Find a displacement field $\mathbf{u} \in V$ and an electric potential $\varphi \in W$ such that:

$$(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}),\, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_\mathcal{H} + (\boldsymbol{\mathcal{P}}^*\nabla\varphi,\, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\mathbf{u}))_H + J(\mathbf{u},\, \varphi,\, \mathbf{v}) - J(\mathbf{u},\, \varphi,\, \mathbf{u})$$
$$\ge (\mathbf{f},\, \mathbf{v} - \mathbf{u})_V \quad \forall \mathbf{v} \in V, \tag{22}$$

$$(\boldsymbol{\beta}\nabla\varphi, \nabla\xi)_H - (\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u}), \nabla\xi)_H + \chi(\mathbf{u}, \varphi, \xi) = (q_e, \xi)_W \quad \forall \xi \in W. \tag{23}$$

4. Existence and uniqueness results

The following theorem establishes the existence and uniqueness of the solution to the Problem (PV).

Theorem 1. *Assume that the hypotheses* (H_1)-(H_9) *hold true, there exists* $L^* > 0$ *such that if*

$$(L_\mu + \mu^* + L_{g_1} + L_{g_2} + L_\psi L + M_\psi) < L^*,$$

then, Problem (PV) has a unique solution.

The proof of Theorem 1 will be divided into several steps. We suppose in the sequel that the hypotheses of Theorem 1 are fulfilled. Before stating and proving our main results, we consider the product spaces $X = V \times W$, and $Y = L^2(\Gamma_3)^4$, together with the inner products:

$$(\mathbf{x}, \mathbf{y})_X = (\mathbf{u}, \mathbf{v})_V + (\varphi, \xi)_W, \quad (\boldsymbol{\eta}, \boldsymbol{\theta})_Y = (\eta_i, \theta_i)_{L^2(\Gamma_3)}, \tag{24}$$

for all $\mathbf{x} = (\mathbf{u}, \varphi)$, $\mathbf{y} = (\mathbf{v}, \xi) \in X$, $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3, \eta_4)$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4) \in Y$ and the associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We define the operator $A : X \times X \to X$, the functions $\widetilde{J}$, $\widetilde{\chi}$ on $X \times X$ and the element $\mathbf{f}_3 \in X$ by equalities:

$$(A\mathbf{x}, \mathbf{y})_X = (\boldsymbol{\mathfrak{F}}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\boldsymbol{\beta}\nabla\varphi, \nabla\xi)_H + (\boldsymbol{\mathcal{P}}^*\nabla\varphi, \boldsymbol{\varepsilon}(\mathbf{v}))_H - (\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u}), \nabla\xi)_H, \tag{25}$$

$$\widetilde{J}(\mathbf{x}, \mathbf{y}) = J(\mathbf{u}, \varphi, \mathbf{v}), \tag{26}$$

$$\widetilde{\chi}(\mathbf{x}, \mathbf{y}) = \chi(\mathbf{u}, \varphi, \xi), \tag{27}$$

$$\mathbf{f}_3 = (\mathbf{f}, q_e) \in X, \tag{28}$$

for all $\mathbf{x} = (\mathbf{u}, \varphi)$ and $\mathbf{y} = (\mathbf{v}, \xi) \in X$. With the above notations, we get the following equivalent problem:

Problem ($\widetilde{\mathbf{PV}}$). Find $\mathbf{x} = (\mathbf{u}, \varphi) \in X$ such that:

$$(A\mathbf{x}, \mathbf{y} - \mathbf{x})_X + \widetilde{J}(\mathbf{x}, \mathbf{y}) - \widetilde{J}(\mathbf{x}, \mathbf{x}) + \widetilde{\chi}(\mathbf{x}, \mathbf{y} - \mathbf{x}) \geq (\mathbf{f}_3, \mathbf{y} - \mathbf{x})_X \quad \forall \mathbf{y} = (\mathbf{v}, \xi) \in X. \tag{29}$$

We start with the following technical lemmas which is frequently used in what follows.

Lemma 1. *The couple* $\mathbf{x} = (\mathbf{u}, \varphi) \in U$ *is a solution to Problem (PV) if and only if it is a solution to Problem* ($\widetilde{PV}$).

Proof. Let $\mathbf{x} = (\mathbf{u}, \varphi) \in X$ be a solution to Problem (PV) and let $\mathbf{y} = (\mathbf{v}, \xi) \in X$. We choose $(\xi - \varphi)$ as test function in (23), add the corresponding inequality to (22) and use (24)–(28) to obtain (29). Conversely, let $\mathbf{x} = (\mathbf{u}, \varphi) \in X$ be a solution to Problem ($\widetilde{\text{PV}}$). We take $\mathbf{y} = (\mathbf{v}, \varphi)$ in (29), where $\mathbf{v}$ is an arbitrary element of X and obtain (22). Then for any $\xi \in W$, we take successively $\mathbf{y} = (\mathbf{v}, \varphi + \xi)$, and $\mathbf{y} = (\mathbf{v}, \varphi - \xi)$ in (29) to obtain (23), which concludes the proof of Lemma 1. ■

Lemma 2. *The operator* $A : X \to X$ *is strongly monotone and Lipschitz continuous.*

Proof. We consider two elements $\mathbf{x}_1 = (\mathbf{u}_1, \varphi_1)$, $\mathbf{x}_2 = (\mathbf{u}_2, \varphi_2) \in X$, from the assumptions ($H_1$)–($H_3$), and (24) alongside with an algebraic manipulation similar to those used in [9, 30], we can easily prove that there exist $m_A > 0$ depend only on $\mathfrak{F}$, $\boldsymbol{\beta}$, Ω, Γ_a and there exist $M_A > 0$ depend only on $\mathfrak{F}$, $\boldsymbol{\beta}$ and $\boldsymbol{\mathcal{P}}$ such that:

$$(A\mathbf{x}_1 - A\mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_2)_X \geq m_A(\|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 + \|\varphi_1 - \varphi_2\|_W^2) = m_A\|\mathbf{x}_1 - \mathbf{x}_2\|_X^2, \tag{30}$$

$$\|A\mathbf{x}_1 - A\mathbf{x}_2\|_X \leq M_A\|\mathbf{x}_1 - \mathbf{x}_2\|_X, \tag{31}$$

for all $\mathbf{x}_1,\ \mathbf{x}_2 \in X$, where $m_A = \min(m_{\mathfrak{F}}, m_{\boldsymbol{\beta}})$, and $M_A = 4 \times \max(\mathcal{M}_{\boldsymbol{\mathcal{F}}}, \mathcal{M}_{\boldsymbol{\beta}}, \mathcal{M}_{\boldsymbol{\mathcal{P}}})$. ■

Now, let $\mathbf{z} = (z_1, z_2, z_3, z_4) \in Y$ with $z_1 \geq 0$, $z_2 \geq 0$, $z_3 \geq 0$, and we define the functions:

$$J_{\mathbf{z}}(\mathbf{v}) = \int_{\Gamma_3} z_1\|\mathbf{v}_\tau\|\, da + \int_{\Gamma_3} z_2 v_\nu^-\, da + \int_{\Gamma_3} z_3 v_\nu^+\, da \quad \forall \mathbf{v} \in V, \tag{32}$$

$$\chi_{\mathbf{z}}(\xi) = \int_{\Gamma_3} z_4 \xi\, da \quad \forall \xi \in W. \tag{33}$$

We consider the element $\mathbf{f}_{\mathbf{z}} \in X$ given by:

$$\mathbf{f}_{\mathbf{z}} = (\mathbf{f}, q_{\mathbf{z}}) \in X,$$

where

$$(q_{\mathbf{z}}, \xi)_W = (q_e, \xi)_W - \chi_{\mathbf{z}}(\xi) \quad \forall \xi \in W.$$

It follows from (19), and (33), that $q_{\mathbf{z}} \in W$. We extend the functional $J_{\mathbf{z}}$ defined by (32) to a functional $\widetilde{J}_{\mathbf{z}}$ defined on V, that is:

$$\widetilde{J}_{\mathbf{z}}(\mathbf{x}) = J_{\mathbf{z}}(\mathbf{u}) \quad \forall \mathbf{x} = (\mathbf{u}, \varphi) \in X. \tag{34}$$

Using the above notations and Lemma 1, we have the following intermediate problem.

Problem ($\mathbf{PV^z}$). Find $\mathbf{x}_{\mathbf{z}} = (\mathbf{u}_{\mathbf{z}}, \varphi_{\mathbf{z}}) \in X$ such that:

$$(A\mathbf{x}_{\mathbf{z}}, \mathbf{y} - \mathbf{x}_{\mathbf{z}})_X + \widetilde{J}_{\mathbf{z}}(\mathbf{y}) - \widetilde{J}_{\mathbf{z}}(\mathbf{x}_{\mathbf{z}}) \geq (\mathbf{f}_{\mathbf{z}}, \mathbf{y} - \mathbf{x}_{\mathbf{z}})_X \quad \forall \mathbf{y} = (\mathbf{v}, \xi) \in X. \tag{35}$$

We have the following existence and uniqueness result.

Lemma 3. *For any $\mathbf{z} = (z_1, z_2, z_3, z_4) \in Y$, suppose that the hypotheses (H_1)–(H_3) hold, then:*

(i) Problem ($PV^{\mathbf{z}}$) has a unique solution $\mathbf{x}_{\mathbf{z}} = (\mathbf{u}_{\mathbf{z}}, \varphi_{\mathbf{z}}) \in X$ which depends Lipschitz continuously on $\mathbf{z} \in Y$.

(ii) There exists a constant $c_2 > 0$ such that the solution of Problem ($PV^{\mathbf{z}}$) satisfies:

$$\|\mathbf{x}_{\mathbf{z}}\|_X \leq c_2\|\mathbf{f}_{\mathbf{z}}\|_X.$$

Proof.

(i) From Lemma 2, we have the operator $A : X \to X$ is strongly monotone and Lipschitz continuous. The functional $\widetilde{J}_{\mathbf{z}}$ given by (34) is proper, convex and Lipschitz continuous, and therefore, $\widetilde{J}_{\mathbf{z}}$ is a fortiori lower semi continuous. Indeed, firstly, it's quite easy to see that $\widetilde{J}_{\mathbf{z}}$ is proper since $\widetilde{J}_{\mathbf{z}}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$. The convexity of $\widetilde{J}_{\mathbf{z}}$ follows from of the that of the functionals $\mathbf{v} \mapsto \|\mathbf{v}_\tau\|$, $\mathbf{v} \mapsto v_\nu^+$, and $\mathbf{v} \mapsto v_\nu^-$ defined in (1). Let $\mathbf{x}_1 = (\mathbf{u}_1, \varphi_1)$, $\mathbf{x}_2 = (\mathbf{u}_2, \varphi_2) \in X$, we have:

$$\begin{aligned}\left|\widetilde{J}_{\mathbf{z}}(\mathbf{x}_1) - \widetilde{J}_{\mathbf{z}}(\mathbf{x}_2)\right| = &\left|\int\limits_{\Gamma_3} z_1\|\mathbf{u}_{1,\tau}\|\, da + \int\limits_{\Gamma_3} z_2 u_{1,\nu}^-\, da + \int\limits_{\Gamma_3} z_3 u_{1,\nu}^+\, da\right.\\ &\left. - \int\limits_{\Gamma_3} z_1\|\mathbf{u}_{2,\tau}\|\, da - \int\limits_{\Gamma_3} z_2 u_{2,\nu}^+ da - \int\limits_{\Gamma_3} z_3 u_{2,\nu}^-\, da\right|\\ &\le \|z_1\|_{L^2(\Gamma_3)}\|\mathbf{u}_{1,\tau} - \mathbf{u}_{2,\tau}\|_{L^2(\Gamma_3)^d} + \|z_2\|_{L^2(\Gamma_3)}\|u_{1,\nu}^- - u_{2,\nu}^-\|_{L^2(\Gamma_3)}\\ &\quad + \|z_3\|_{L^2(\Gamma_3)}\|u_{1,\nu}^+ - u_{2,\nu}^+\|_{L^2(\Gamma_3)}.\end{aligned}$$

Now, by (5) and (24), we find:

$$|\widetilde{J}_{\mathbf{z}}(\mathbf{x}_1) - \widetilde{J}_{\mathbf{z}}(\mathbf{x}_2)| \le c_0\left(\|z_1\|_{L^2(\Gamma_3)} + \|z_2\|_{L^2(\Gamma_3)} + \|z_3\|_{L^2(\Gamma_3)}\right)\|\mathbf{x}_1 - \mathbf{x}_2\|_X.$$

Thus, the functional $\widetilde{J}_{\mathbf{z}}$ is Lipschitz continuous and therefore it is i lower semi continuous function. Hence, it follows from standard arguments based on variational inequalities that there exists a unique solution: $\mathbf{x}_{\mathbf{z}} = (\mathbf{u}_{\mathbf{z}}, \varphi_z)$ of Problem ($PV^{\mathbf{z}}$). Following this, we will prove that this solution depends Lipschitz continuously on $\mathbf{z} \in Y$. Let $\mathbf{z}$, $\mathbf{z}' \in Y$ be given, and denote the corresponding solution of the problem (35) by $\mathbf{x}_{\mathbf{z}} = (\mathbf{u}_{\mathbf{z}}, \varphi_{\mathbf{z}})$, and $\mathbf{x}_{\mathbf{z}'} = (\mathbf{u}_{\mathbf{z}'}, \varphi_{\mathbf{z}'})$. Then, we have:

$$\begin{aligned}(A\mathbf{x}_{\mathbf{z}}, \mathbf{y} - \mathbf{x}_{\mathbf{z}})_X + \widetilde{J}_{\mathbf{z}}(\mathbf{y}) - \widetilde{J}_{\mathbf{z}}(\mathbf{x}_{\mathbf{z}}) \ge (\mathbf{f}_{\mathbf{z}}, \mathbf{y} - \mathbf{x}_{\mathbf{z}})_X \quad \forall \mathbf{y} = (\mathbf{v}, \xi) \in X,\\ (A\mathbf{x}_{\mathbf{z}'}, \mathbf{y} - \mathbf{x}_{\mathbf{z}'})_X + \widetilde{J}_{\mathbf{z}'}(\mathbf{y}) - \widetilde{J}_{\mathbf{z}'}(\mathbf{x}_{\mathbf{z}'}) \ge (\mathbf{f}_{\mathbf{z}'}, \mathbf{y} - \mathbf{x}_{\mathbf{z}'})_X \quad \forall \mathbf{y} = (\mathbf{v}, \xi) \in X.\end{aligned}$$

We take $\mathbf{y} = \mathbf{x}_{\mathbf{z}'}$ in the first inequality, and $\mathbf{y} = \mathbf{x}_{\mathbf{z}}$ in the second inequality, then we add the obtained inequalities to find:

$$\begin{aligned}&(A\mathbf{x}_{\mathbf{z}} - A\mathbf{x}_{\mathbf{z}'}, \mathbf{x}_{\mathbf{z}} - \mathbf{x}_{\mathbf{z}'})_X\\ &\le (\mathbf{f}_{\mathbf{z}} - \mathbf{f}_{\mathbf{z}'}, \mathbf{x}_{\mathbf{z}} - \mathbf{x}_{\mathbf{z}'})_X + \widetilde{J}_{\mathbf{z}}(\mathbf{x}_{\mathbf{z}'}) - \widetilde{J}_{\mathbf{z}}(\mathbf{x}_{\mathbf{z}}) + \widetilde{J}_{\mathbf{z}'}(\mathbf{x}_{\mathbf{z}}) - \widetilde{J}_{\mathbf{z}'}(\mathbf{x}_{\mathbf{z}'})\\ &\le c_0\left(\|z_1 - z_1'\|_{L^2(\Gamma_3)} + \|z_2 - z_2'\|_{L^2(\Gamma_3)} + \|z_3 - z_3'\|_{L^2(\Gamma_3)}\right)\|\mathbf{u}_{\mathbf{z}} - \mathbf{u}_{\mathbf{z}'}\|_V\\ &\quad c_1\|z_4 - z_4'\|_{L^2(\Gamma_3)}\|\varphi_1 - \varphi_2\|_W.\end{aligned}$$

Therefore, it follows from (5), (6), (24), and (30), that:

$$\|\mathbf{x}_{\mathbf{z}} - \mathbf{x}_{\mathbf{z}'}\|_X \le c_3\|\mathbf{z} - \mathbf{z}'\|_Y, \tag{36}$$

where $c_3 = 2\dfrac{\max(c_0, c_1)}{m_A}$, hence (i) follows.

(ii) Let $\mathbf{z} = (z_1, z_2, z_3, z_4) \in Y$, we take $\mathbf{y} = 0$ in the inequality (35), one has:

$$(A\mathbf{x}_{\mathbf{z}}, \mathbf{x}_{\mathbf{z}})_X + \widetilde{J}_{\mathbf{z}}(\mathbf{x}_{\mathbf{z}}) \le (\mathbf{f}_{\mathbf{z}}, \mathbf{x}_{\mathbf{z}})_X \quad \forall \mathbf{x}_{\mathbf{z}} \in X.$$

Or $z_1 > 0$, $z_2 > 0$, and $z_3 > 0$, one has:

$$(A\mathbf{x}_{\mathbf{z}}, \mathbf{x}_{\mathbf{z}})_X \le (\mathbf{f}_{\mathbf{z}}, \mathbf{x}_{\mathbf{z}})_X \quad \forall \mathbf{x}_{\mathbf{z}} \in X.$$

According to (30), we deduce:

$$\|\mathbf{x_z}\|_X \leq c_2 \|\mathbf{f_z}\|_X,$$

where $c_2 = 1/m_A$. ■

Next, we consider the operator $\Lambda : Y \to Y$ defined by:

$$\Lambda \mathbf{z} = (\mu(\|\mathbf{u}_{\mathbf{z},\boldsymbol{\tau}}\|)|\mathrm{R}\sigma_\nu(\mathbf{u_z}, \varphi_\mathbf{z})|, g_2(\|\mathbf{u_z}\|), -g_1(\|\mathbf{u_z}\|), \psi(u_{\mathbf{z},\nu})\phi_L(\varphi_\mathbf{z} - \varphi_F)) \,. \tag{37}$$

Using assumptions (H_4)–(H_7), we can easily see that operator Λ is well defined. Next, we will prove that the operator Λ has fixed-point and to this end, we need the following result:

Lemma 4. *There exist $L^* > 0$ such that if $(L_\mu + \mu_* + L_{g_1} + L_{g_2} + L_\psi L + M_\psi) \leq L^*$, then Λ has a unique fixed-point.*

Proof. Let $\mathbf{z} = (z_1, z_2, z_3, z_4)$, $\mathbf{z}' = (z'_1, z'_2, z'_3, z'_4) \in Y$. One has:

$$\begin{aligned}\|\Lambda\mathbf{z} - \Lambda\mathbf{z}'\|_Y \leq \|\mu(\|\mathbf{u}_{\mathbf{z},\boldsymbol{\tau}}\|)|\mathrm{R}\sigma_\nu(\mathbf{u_z}, \varphi_\mathbf{z})| - \mu(\|\mathbf{u}_{\mathbf{z}',\boldsymbol{\tau}}\|)|\mathrm{R}\sigma_\nu(\mathbf{u}_{\mathbf{z}'}, \varphi_{\mathbf{z}'})|\|_{L^2(\Gamma_3)} + \\ + \|g_1(\|\mathbf{u_z}\|) - g_1(\|\mathbf{u}_{\mathbf{z}'}\|)\|_{L^2(\Gamma_3)} + \|g_2(\|\mathbf{u_z}\|) - g_2(\|\mathbf{u}_{\mathbf{z}'}\|)\|_{L^2(\Gamma_3)} + \\ + \|\psi(u_{\mathbf{z},\nu})\phi_L(\varphi_\mathbf{z} - \varphi_F) - \psi(u_{\mathbf{z}',\nu})\phi_L(\varphi_{\mathbf{z}'} - \varphi_F)\|_{L^2(\Gamma_3)}.\end{aligned}$$

Therefore,

$$\|\Lambda\mathbf{z} - \Lambda\mathbf{z}'\|_Y \leq G_1 + G_2,$$

where

$$\begin{aligned}G_1 = \|\mu(\|\mathbf{u}_{\mathbf{z},\boldsymbol{\tau}}\|)|\mathrm{R}\sigma_\nu(\mathbf{u_z}, \varphi_\mathbf{z})| - \mu(\|\mathbf{u}_{\mathbf{z}',\boldsymbol{\tau}}\|)|\mathrm{R}\sigma_\nu(\mathbf{u}_{\mathbf{z}'}, \varphi_{\mathbf{z}'})|\|_{L^2(\Gamma_3)} + \\ + \|g_1(\|\mathbf{u_z}\|) - g_1(\|\mathbf{u}_{\mathbf{z}'}\|)\|_{L^2(\Gamma_3)} + \|g_2(\|\mathbf{u_z}\|) - g_2(\|\mathbf{u}_{\mathbf{z}'}\|)\|_{L^2(\Gamma_3)}.\end{aligned}$$

Using (H_5), (H_6), (H_7), (6), the properties of R, and after some algebra, we obtain:

$$\begin{aligned}G_1 &\leq L_\mu \|\mathrm{R}\sigma_\nu(\mathbf{u_z}, \varphi_\mathbf{z})\|_{L^\infty(\Gamma_3)} \|\mathbf{u_z} - \mathbf{u}_{\mathbf{z}'}\|_{L^2(\Gamma_3)^d} + \\ &\quad + \mu_* \mathrm{meas}(\Gamma_3)^{1/2} \|\mathrm{R}\sigma_\nu(\mathbf{u_z}, \varphi_\mathbf{z}) - \mathrm{R}\sigma_\nu(\mathbf{u}_{\mathbf{z}'}, \varphi_{\mathbf{z}'})\|_{L^\infty(\Gamma_3)} + \\ &\quad + L_{g_1}\|\mathbf{u_z} - \mathbf{u}_{\mathbf{z}'}\|_{L^2(\Gamma_3)^d} + L_{g_2}\|\mathbf{u_z} - \mathbf{u}_{\mathbf{z}'}\|_{L^2(\Gamma_3)^d} \leq \\ &\leq L_\mu c_0 \|\mathrm{R}\sigma_\nu(\mathbf{u_z}, \varphi_\mathbf{z})\|_{L^\infty(\Gamma_3)} \|\mathbf{u_z} - \mathbf{u}_{\mathbf{z}'}\|_V + \\ &\quad + \mu_* \mathrm{meas}(\Gamma_3)^{1/2} c_\mathrm{R} \|\sigma_\nu(\mathbf{u_z}, \varphi_\mathbf{z}) - \sigma_\nu(\mathbf{u}_{\mathbf{z}'}, \varphi_{\mathbf{z}'})\|_{H^{-1/2}(\Gamma_3)} + \\ &\quad + c_0(L_{g_1} + L_{g_2})\|\mathbf{u_z} - \mathbf{u}_{\mathbf{z}'}\|_V,\end{aligned}$$

the $H^{-1/2}$ norm, defined in (3), leads us to:

$$\|\sigma_\nu(\mathbf{u}_z, \varphi_z) - \sigma_\nu(\mathbf{u}_{z'}, \varphi_{z'})\|_{H^{-1/2}(\Gamma_3)} = \sup_{\substack{v_\nu \in H^{1/2}_{\Gamma_3}, \\ v_\nu \neq 0_{H^{1/2}_{\Gamma_3}}}} \frac{\langle \sigma_\nu(\mathbf{u}_z, \varphi_z) - \sigma_\nu(\mathbf{u}_{z'}, \varphi_{z'}), v_\nu \rangle_{\Gamma_3}}{\|v_\nu\|_{H^{1/2}(\Gamma_3)}},$$

applying Green's formula, for every $\mathbf{v} \in V$ with $\mathbf{v}_\tau = \mathbf{0}$, we have:

$$\begin{aligned}&\|\sigma_\nu(\mathbf{u}_z, \varphi_z) - \sigma_\nu(\mathbf{u}_{z'}, \varphi_{z'})\|_{H^{-1/2}(\Gamma_3)} = \\ &= \sup_{\substack{v_\nu \in H^{1/2}_{\Gamma_3}, \\ v_\nu \neq 0_{H^{1/2}_{\Gamma_3}}}} \frac{\langle \mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}_z) - \mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}_{z'}), \boldsymbol{\varepsilon}(\mathbf{v}) \rangle_\mathcal{H} + \langle \boldsymbol{\mathcal{P}}^* \nabla(\varphi_z - \varphi_{z'}), \boldsymbol{\varepsilon}(\mathbf{v}) \rangle_\mathcal{H}}{\|v_\nu\|_{H^{1/2}(\Gamma_3)}},\end{aligned}$$

since fo any v_ν in $H^{1/2}$, there exists $\mathbf{v} \in H_1$ and a constant $c_\nu > 0$ (see [14]) such that:

$$v_\nu = \gamma \mathbf{v} \quad \text{and} \quad \|v_\nu\|_{H^{1/2}} \geq c_\nu \|v\|_V,$$

thus,

$$\begin{aligned}
&\|\sigma_\nu(\mathbf{u}_z, \varphi_z) - \sigma_\nu(\mathbf{u}_{z'}, \varphi_{z'})\|_{H^{-1/2}(\Gamma_3)} \leq \\
&\leq \frac{1}{c_\nu} \sup_{\substack{v \in V, \\ v \neq 0_V}} \frac{\|\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}_z) - \mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}_{z'})\|_{\mathcal{H}} \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} + \|\boldsymbol{\mathcal{P}}^* \nabla(\varphi_z - \varphi_{z'})\|_{\mathcal{H}} \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}}}{\|\mathbf{v}\|_V} \leq \\
&\leq \frac{1}{c_\nu} \sup_{\substack{v \in V, \\ v \neq 0_V}} \left(\mathcal{M}_{\mathfrak{F}} \|\mathbf{u}_z - \mathbf{u}_{z'}\|_V + \mathcal{M}_{\boldsymbol{\mathcal{P}}} \|\nabla\left(\varphi_z - \varphi_{z'}\right)\|_H\right) \frac{\|\mathbf{v}\|_V}{\|\mathbf{v}\|_V} \leq \\
&\leq \frac{1}{c_\nu} \max\left(\mathcal{M}_{\mathfrak{F}}, \mathcal{M}_{\boldsymbol{\mathcal{P}}}\right) \left(\|\mathbf{u}_z - \mathbf{u}_{z'}\|_V + \|\varphi_z - \varphi_{z'}\|_W\right).
\end{aligned}$$

Then, it is straightforward that:

$$\begin{aligned}
G_1 \leq\ & L_\mu c_0 \|\mathrm{R}\sigma_\nu(\mathbf{u}_\mathbf{z}, \varphi_\mathbf{z})\|_{L^\infty(\Gamma_3)} \|\mathbf{u}_\mathbf{z} - \mathbf{u}_{\mathbf{z}'}\|_V + \\
&+ \frac{1}{c_\nu} \max\left(\mathcal{M}_{\mathfrak{F}}, \mathcal{M}_{\boldsymbol{\mathcal{P}}}\right) \mu_* \mathrm{meas}(\Gamma_3)^{1/2} c_\mathrm{R} \left(\|\mathbf{u}_\mathbf{z} - \mathbf{u}_{\mathbf{z}'}\|_V + \|\varphi_\mathbf{z} - \varphi_{\mathbf{z}'}\|_W\right) + \\
&+ c_0(L_{g_1} + L_{g_2}) \|\mathbf{u}_\mathbf{z} - \mathbf{u}_{\mathbf{z}'}\|_V \leq \\
\leq\ & C(L_\mu + \mu_* + L_{g_1} + L_{g_2}) \|\mathbf{x}_\mathbf{z} - \mathbf{x}_{\mathbf{z}'}\|_X,
\end{aligned} \tag{38}$$

where $C = \max\left(\|\mathrm{R}\sigma_\nu(\mathbf{u}_\mathbf{z}, \varphi_\mathbf{z})\|_{L^\infty(\Gamma_3)} c_0, \frac{\sqrt{2}}{c_\nu} \max(\mathcal{M}_{\mathfrak{F}}, \mathcal{M}_{\boldsymbol{\mathcal{P}}}) c_\mathrm{R} \mathrm{meas}(\Gamma_3)^{1/2}, c_0\right)$. Moreover, it follows from assumption (H_4), the bounds $|\phi_L(\varphi - \varphi_F)| \leq L$, (6), and (24) that:

$$\begin{aligned}
G_2 &= \|\psi(u_{\mathbf{z},\nu}) \phi_L(\varphi_\mathbf{z} - \varphi_F) - \psi(u_{\mathbf{z}',\nu}) \phi_L(\varphi_{\mathbf{z}'} - \varphi_F)\|_{L^2(\Gamma_3)} \leq \\
&\leq \|\left(\psi(u_{\mathbf{z},\nu}) - \psi(u_{\mathbf{z}',\nu})\right) \phi_L(\varphi_\mathbf{z} - \varphi_F)\|_{L^2(\Gamma_3)} + \\
&\quad + \|\psi(u_{\mathbf{z}',\nu})(\phi_L(\varphi_\mathbf{z} - \varphi_F) - \phi_L(\varphi_{\mathbf{z}'} - \varphi_F))\|_{L^2(\Gamma_3)} \leq \\
&\leq C'(L L_\psi + M_\psi) \|\mathbf{x}_\mathbf{z} - \mathbf{x}_{\mathbf{z}'}\|_X,
\end{aligned} \tag{39}$$

where $C' = \max(c_0, c_1)$.

Now, we use the two previous inequalities (38) and (39), to find that, there exist a constant $c_4 > 0$, such that:

$$\|\Lambda\mathbf{z} - \Lambda\mathbf{z}'\|_Y \leq c_4(L_\mu + \mu_* + L_{g_1} + L_{g_2} + L_\psi L + M_\psi) \|\mathbf{x}_\mathbf{z} - \mathbf{x}_{\mathbf{z}'}\|_X,$$

where $c_4 = \max(C, C')$. Finally, keeping in mind (36), we obtain

$$\|\Lambda\mathbf{z} - \Lambda\mathbf{z}'\|_Y \leq c_4 c_3(L_\mu + \mu_* + L_{g_1} + L_{g_2} + L_\psi L + M_\psi) \|\mathbf{z} - \mathbf{z}'\|_Y.$$

Let $L^* = \dfrac{1}{c_4 c_3}$, then if $(L_\mu + \mu_* + L_{g_1} + L_{g_2} + L_\psi L + M_\psi) \leq L^*$ the mapping Λ is contraction of Y. By Banach fixed-point theorem, the mapping Λ has a unique fixed-point $\mathbf{z}^*$ on Y. Let $(L_\mu + \mu_* + L_{g_1} + L_{g_2} + L_\psi L + M_\psi) \leq L^*$ and let $\mathbf{z}^*$ the fixed-point of operator Λ. We denote by $(\mathbf{u}^*, \varphi^*)$ the solution of Problem ($\mathrm{PV}^\mathbf{z}$) for $\mathbf{z} = \mathbf{z}^*$. Using (35) and (37), it is easy to see that $(\mathbf{u}^*, \varphi^*)$ is a solution of Problem (PV). This proves the existence part of Theorem 1. The uniqueness of the solution results from the uniqueness of the fixed-point of the operator Λ. ■

5. Finite element setting and discrete variational problem

This section is devoted to studying the finite element approximation of Problem (PV) and deriving an error estimate of the approximate solution. First, we consider the following finite-dimensional spaces $V^h \subset V$ and $W^h \subset W$ defined by:

$$V^h = \{\mathbf{v}^h \in \mathcal{C}(\overline{\Omega})^d, \mathbf{v}^h_{|\Omega^e} \in \mathbb{P}_1(\Omega^e); \Omega^e \in \mathcal{T}^h, \mathbf{v}^h = 0 \text{ on } \overline{\Gamma}_1\},$$
$$W^h = \{\psi^h \in \mathcal{C}(\overline{\Omega})^d, \psi^h_{|\Omega^e} \in \mathbb{P}_1(\Omega^e); \Omega^e \in \mathcal{T}^h, \psi^h = 0 \text{ on } \overline{\Gamma}_a\}.$$

Approximating the spaces V and W, where $h > 0$ is the parameter of discretization. Here Ω is assumed to be a polygonal domain, $\mathcal{T}^h$ denotes a finite element triangulation of $\overline{\Omega}$ that are compatible with the partition of the boundary, and we denote by $\mathbb{P}_1(\Omega^e)$ the space of polynomials of global degree less or equal to one in an element Ω^e of the triangulation. Thus, the discrete approximation of Problem (PV) is the following:

Problem (PV^h). Find a discrete displacement field $\mathbf{u}^h \in V^h$ and a discrete electric potential $\varphi^h \in W^h$, such that:

$$(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}^h), \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{u}^h))_{\mathcal{H}} + (\boldsymbol{\mathcal{P}}^*\nabla\varphi^h, \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{u}^h))_H + J(\mathbf{u}^h, \varphi^h, \mathbf{v}^h) - J(\mathbf{u}^h, \varphi^h, \mathbf{u}^h) \geq$$
$$\geq (\mathbf{f}, \mathbf{v}^h - \mathbf{u}^h)_V \quad \forall \mathbf{v}^h \in V^h, \tag{40}$$
$$(\boldsymbol{\beta}\nabla\varphi^h, \nabla\xi^h)_H - (\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u}^h), \nabla\xi^h)_H + \chi(\mathbf{u}^h, \varphi^h, \xi^h) = (q, \xi^h)_W \quad \forall \xi^h \in W^h. \tag{41}$$

Applying Theorem 1, for the case when V and W are replaced by V^h and W^h, respectively, we find that Problem (PV^h) have a unique solution $(\mathbf{u}^h, \varphi^h) \in V^h \times W^h$. We have the following convergence result:

Theorem 2. *Let us denote by $(\mathbf{u}, \varphi)$ and $(\mathbf{u}^h, \varphi^h)$, the solutions to Problems (PV) and (PV^h), respectively. Under the hypotheses of Theorem 1, with the same value of L^*, the following error estimates are obtained:*

$$\|\mathbf{u} - \mathbf{u}^h\|_V + \|\varphi - \varphi^h\|_W \leq$$
$$\leq C \inf_{(\mathbf{v}^h, \xi^h) \in V^h \times W^h} \Big\{ \|\mathbf{u} - \mathbf{v}^h\|_V + \|\varphi - \xi^h\|_W + \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d} + \|\varphi - \xi^h\|_{L^2(\Gamma_3)} +$$
$$+ \left(\|\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u})\|^{1/2}_{\mathcal{H}} + \|\boldsymbol{\mathcal{P}}^*\nabla\varphi\|^{1/2}_H + \|\mathbf{f}\|^{1/2}_V\right) \|\mathbf{u} - \mathbf{v}^h\|^{1/2}_V +$$
$$+ \left(\|\mu(\mathbf{u}_{\boldsymbol{\tau}})\|_{L^2(\Gamma_3)} \|R\sigma_\nu(\mathbf{u}, \varphi)\|_{L^\infty(\Gamma_3)} + (M_{g_1} + M_{g_2}) \text{meas}(\Gamma_3)^{1/2}\right)^{1/2} \|\mathbf{u} - \mathbf{v}^h\|^{1/2}_{L^2(\Gamma_3)^d} \Big\}, \tag{42}$$

where $C > 0$ independent of h.

Proof. We replace ξ by ξ^h in (23) then we subtract (41) from the resulting equation, to get:

$$\big(\boldsymbol{\beta}\nabla(\varphi - \varphi^h), \nabla\xi^h\big)_H - (\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}^h), \nabla\xi^h)_H + \chi(\mathbf{u}, \varphi, \xi^h) - \chi(\mathbf{u}^h, \varphi^h, \xi^h) = 0 \quad \forall \xi^h \in W. \tag{43}$$

and it follows that for all $\xi^h \in W^h$:

$$\big(\boldsymbol{\beta}\nabla(\varphi - \varphi^h), \nabla(\xi^h - \varphi)\big)_H + \big(\boldsymbol{\beta}\nabla(\varphi - \varphi^h), \nabla(\varphi - \varphi^h)\big)_H -$$
$$- \big(\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}^h), \nabla(\xi^h - \varphi)\big)_H - \big(\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}^h), \nabla(\varphi - \varphi^h)\big)_H +$$
$$+ \chi(\mathbf{u}, \varphi, \xi^h - \varphi) + \chi(\mathbf{u}, \varphi, \varphi - \varphi^h) - \chi(\mathbf{u}^h, \varphi^h, \xi^h - \varphi) - \chi(\mathbf{u}^h, \varphi^h, \varphi - \varphi^h) = 0.$$

Hence, for all $\xi^h \in W^h$, we have:

$$\begin{aligned}\big(\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h),\nabla(\varphi-\varphi^h)\big)_H &= (\boldsymbol{\beta}\nabla(\varphi-\varphi^h),\nabla(\varphi-\varphi^h))_H-\\ &-(\boldsymbol{\beta}\nabla(\varphi-\varphi^h),\nabla(\varphi-\xi^h))_H+(\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h),\nabla(\varphi-\xi^h))_H+\\ &+\chi(\mathbf{u},\varphi,\varphi-\varphi^h)-\chi(\mathbf{u}^h,\varphi^h,\varphi-\varphi^h)+\chi(\mathbf{u},\varphi,\xi^h-\varphi)-\chi(\mathbf{u}^h,\varphi^h,\xi^h-\varphi). \end{aligned}\tag{44}$$

Next, choosing $\mathbf{v}=\mathbf{u}^h \in V^h$ in (22), we obtain:

$$\big(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h)\big)_{\mathcal{H}}+(\boldsymbol{\mathcal{P}}^*\nabla\varphi,\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h))_H \le J(\mathbf{u},\varphi,\mathbf{u}^h)-J(\mathbf{u},\varphi,\mathbf{u})+(\mathbf{f},\mathbf{u}-\mathbf{u}^h)_V. \tag{45}$$

In addition, the formula (40) can be rewritten as:

$$\begin{aligned}&-(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}^h),\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h))_{\mathcal{H}}-(\boldsymbol{\mathcal{P}}^*\nabla\varphi^h,\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h))_H\le\\ &\le(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}^h),\boldsymbol{\varepsilon}(\mathbf{v}^h-\mathbf{u}))_{\mathcal{H}}+(\boldsymbol{\mathcal{P}}^*\nabla\varphi^h,\boldsymbol{\varepsilon}(\mathbf{v}^h-\mathbf{u}))_H+\\ &\quad+J(\mathbf{u}^h,\varphi^h,\mathbf{v}^h)-J(\mathbf{u}^h,\varphi^h,\mathbf{u}^h)+(\mathbf{f},\mathbf{u}^h-\mathbf{v}^h)_V\quad\forall\mathbf{v}^h\in V^h.\end{aligned}\tag{46}$$

Now, we use the two inequalities (45) and (46) to get:

$$\begin{aligned}&(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u})-\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}^h),\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h))_{\mathcal{H}}+(\boldsymbol{\mathcal{P}}^*\nabla(\varphi-\varphi^h),\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h))_H\le\\ &\le(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u})-\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}^h),\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}^h))_{\mathcal{H}}+(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}^h-\mathbf{u}))_{\mathcal{H}}+(\boldsymbol{\mathcal{P}}^*\nabla\varphi^h,\boldsymbol{\varepsilon}(\mathbf{v}^h-\mathbf{u}))_H+\\ &\quad+J(\mathbf{u},\varphi,\mathbf{u}^h)-J(\mathbf{u},\varphi,\mathbf{u})+J(\mathbf{u}^h,\varphi^h,\mathbf{v}^h)-J(\mathbf{u}^h,\varphi^h,\mathbf{u}^h)+(\mathbf{f},\mathbf{u}-\mathbf{v}^h)_V.\end{aligned}$$

Replacing now (44) in (46), we obtain:

$$\begin{aligned}&(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u})-\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}^h),\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h))_{\mathcal{H}}+(\boldsymbol{\beta}\nabla(\varphi-\varphi^h),\nabla(\varphi-\varphi^h))_H\le\\ &\le(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u})-\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}^h),\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}^h))_{\mathcal{H}}+(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}^h-\mathbf{u}))_{\mathcal{H}}+(\boldsymbol{\mathcal{P}}^*\nabla\varphi^h,\boldsymbol{\varepsilon}(\mathbf{v}^h-\mathbf{u}))_H+\\ &\quad+(\boldsymbol{\beta}\nabla(\varphi-\varphi^h),\nabla(\varphi-\xi^h))_H-(\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h),\nabla(\varphi-\xi^h))_H-(\mathbf{f},\mathbf{v}^h-\mathbf{u})_V+\\ &\quad\quad+J(\mathbf{u},\varphi,\mathbf{u}^h)-J(\mathbf{u},\varphi,\mathbf{u})+J(\mathbf{u}^h,\varphi^h,\mathbf{v}^h)-J(\mathbf{u}^h,\varphi^h,\mathbf{u}^h)-\\ &\quad\quad\quad-\chi(\mathbf{u},\varphi,\varphi-\varphi^h)+\chi(\mathbf{u}^h,\varphi^h,\varphi-\varphi^h)-\chi(\mathbf{u},\varphi,\xi^h-\varphi)+\chi(\mathbf{u}^h,\varphi^h,\xi^h-\varphi).\end{aligned}$$

Then, keeping in mind assumptions $(\mathrm{H}_1)(c)$, $(\mathrm{H}_3)(c)$ and the previous inequality, it's follows that:

$$m_{\mathfrak{F}}\|\mathbf{u}-\mathbf{u}^h\|_V^2+m_{\boldsymbol{\beta}}\|\varphi-\varphi^h\|_W^2\le\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5, \tag{47}$$

where

$$\begin{aligned}\mathcal{I}_1&=(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u})-\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}^h),\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}^h))_{\mathcal{H}}+(\boldsymbol{\beta}\nabla(\varphi-\varphi^h),\nabla(\varphi-\xi^h))_H-(\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h),\nabla(\varphi-\xi^h))_H,\\ \mathcal{I}_2&=(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}^h-\mathbf{u}))_{\mathcal{H}}+(\boldsymbol{\mathcal{P}}^*\nabla\varphi^h,\varepsilon(\mathbf{v}^h-\mathbf{u}))_H+J(\mathbf{u},\varphi,\mathbf{v}^h)-J(\mathbf{u},\varphi,\mathbf{u})-(\mathbf{f},\mathbf{v}^h-\mathbf{u})_V,\\ \mathcal{I}_3&=J(\mathbf{u},\varphi,\mathbf{u}^h)-J(\mathbf{u}^h,\varphi^h,\mathbf{u}^h)+J(\mathbf{u}^h,\varphi^h,\mathbf{u})-J(\mathbf{u},\varphi,\mathbf{u}),\\ \mathcal{I}_4&=J(\mathbf{u}^h,\varphi^h,\mathbf{v}^h)-J(\mathbf{u},\varphi,\mathbf{v}^h)+J(\mathbf{u},\varphi,\mathbf{u})-J(\mathbf{u}^h,\varphi^h,\mathbf{u}),\\ \mathcal{I}_5&=\chi(\mathbf{u}^h,\varphi^h,\varphi-\varphi^h)-\chi(\mathbf{u},\varphi,\varphi-\varphi^h)+\chi(\mathbf{u}^h,\varphi^h,\xi^h-\varphi)-\chi(\mathbf{u},\varphi,\xi^h-\varphi).\end{aligned}$$

Let's us now evaluate the five terms of the right-hand side of (47). For the first term, and by using the properties of the operators $\mathfrak{F}$, $\boldsymbol{\beta}$ and $\boldsymbol{\mathcal{E}}$, we have:

$$\begin{aligned}|\mathcal{I}_1|&\le\|\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u})-\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}^h)\|_{\mathcal{H}}\|\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{v}^h)\|_{\mathcal{H}}+\big(\|\boldsymbol{\beta}\nabla(\varphi-\varphi^h)\|_H+\|\boldsymbol{\mathcal{P}}\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}^h)\|_H\big)\,\|\nabla(\varphi-\xi^h)\|_H\le\\ &\le\max\left(\mathcal{M}_{\mathfrak{F}},\mathcal{M}_{\boldsymbol{\beta}},\mathcal{M}_{\boldsymbol{\mathcal{P}}}\right)\big(\|\mathbf{u}-\mathbf{u}^h\|_V\|\mathbf{u}-\mathbf{v}^h\|_V+\big(\|\varphi-\varphi^h\|_W+\|\mathbf{u}-\mathbf{u}^h\|_V\big)\,\|\varphi-\xi^h\|_W\big).\end{aligned}\tag{48}$$

Concerning the second term of the right-hand side of (47), we use $(\mathrm{H}_1)(b)$, $(\mathrm{H}_2)(b)$, $(\mathrm{H}_3)(b)$, $(\mathrm{H}_6)(b)$ and $(\mathrm{H}_7)(b)$, we find that:

$$\begin{aligned}|\mathcal{I}_2| = &\left|\big(\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}^h-\mathbf{u})\big)_{\mathcal{H}}+\big(\boldsymbol{\mathcal{P}}^*\nabla\varphi^h,\boldsymbol{\varepsilon}(\mathbf{v}^h-\mathbf{u})\big)_H+J(\mathbf{u},\varphi,\mathbf{v}^h)-J(\mathbf{u},\varphi,\mathbf{u})-(\mathbf{f},\mathbf{v}^h-\mathbf{u})_V\right|\le\\ &\le \|\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u})\|_{\mathcal{H}}\|\mathbf{u}-\mathbf{v}^h\|_V+\|\boldsymbol{\mathcal{P}}^*\nabla\varphi\|_H\|\mathbf{u}-\mathbf{v}^h\|_V+\|\mathbf{f}\|_V\|\mathbf{u}-\mathbf{v}^h\|_V+\\ &\quad+\|\mu(\mathbf{u}_{\boldsymbol{\tau}})\|_{L^2(\Gamma_3)}\|\mathrm{R}\sigma_\nu(\mathbf{u},\varphi)\|_{L^\infty(\Gamma_3)}\|\mathbf{u}-\mathbf{v}^h\|_{L^2(\Gamma_3)^d}+\\ &\quad+(M_{g_1}+M_{g_2})\mathrm{meas}(\Gamma_3)^{1/2}\|\mathbf{u}-\mathbf{v}^h\|_{L^2(\Gamma_3)^d}. \end{aligned}\tag{49}$$

About the third term of (47), using $(\mathrm{H}_5)(b)$, (c), $(\mathrm{H}_6)(c)$, and $(\mathrm{H}_7)(c)$ to obtain:

$$\begin{aligned}|\mathcal{I}_3| = &\left|J(\mathbf{u},\varphi,\mathbf{u}^h)-J(\mathbf{u}^h,\varphi^h,\mathbf{u}^h)+J(\mathbf{u}^h,\varphi^h,\mathbf{u})-J(\mathbf{u},\varphi,\mathbf{u})\right|\le\\ &\le\left|\int_{\Gamma_3}(\mu(\|\mathbf{u}_{\boldsymbol{\tau}}\|)|\mathrm{R}\sigma_\nu(\mathbf{u},\varphi)-\mu(\|\mathbf{u}^h_{\boldsymbol{\tau}}\|)|\mathrm{R}\sigma_\nu(\mathbf{u}^h,\varphi^h))(\|\mathbf{u}^h_{\boldsymbol{\tau}}\|-\|\mathbf{u}_{\boldsymbol{\tau}}\|)\,da\right|+\\ &\quad+\left|\int_{\Gamma_3}\big(g_2(\|\mathbf{u}\|)-g_2(\|\mathbf{u}^h\|\big)\,(u_\nu^{h-}-u_\nu^-)\,da+\int_{\Gamma_3}\big(g_1(\|\mathbf{u}\|)-g_1(\|\mathbf{u}^h\|)\big)\,(u_\nu^+-u_\nu^{h+})\,da\right|\le\\ &\le\big(c_0^2\|\mathrm{R}\sigma_\nu(\mathbf{u},\varphi)\|_{L^\infty(\Gamma_3)}L_\mu+\max\left(\mathcal{M}_{\mathfrak{F}},\mathcal{M}_{\boldsymbol{\mathcal{P}}}\right)\mu_*c_{\mathrm{R}}+c_0^2\left(L_{g_1}+L_{g_2}\right)\big)\,\|\mathbf{u}-\mathbf{u}^h\|^2_{V+}\\ &\quad+\frac{1}{c_\nu}\max\left(\mathcal{M}_{\mathfrak{F}},\mathcal{M}_{\boldsymbol{\mathcal{P}}}\right)\mu_*c_{\mathrm{R}}c_0\|\varphi-\varphi^h\|_W\|\mathbf{u}-\mathbf{u}^h\|_V.\end{aligned}\tag{50}$$

Analogously, for the fourth term, we have:

$$\begin{aligned}|\mathcal{I}_4| = &\left|J(\mathbf{u}^h,\varphi^h,\mathbf{v}^h)-J(\mathbf{u},\varphi,\mathbf{v}^h)+J(\mathbf{u},\varphi,\mathbf{u})-J(\mathbf{u}^h,\varphi^h,\mathbf{u})\right|\le\\ &\le\int_{\Gamma_3}|\mathrm{R}\sigma_\nu(\mathbf{u},\varphi)|\left|\mu(\|\mathbf{u}^h_{\boldsymbol{\tau}}\|)-\mu(\|\mathbf{u}_{\boldsymbol{\tau}}\|)\right|\left|\|\mathbf{v}^h_{\boldsymbol{\tau}}\|-\|\mathbf{u}_{\boldsymbol{\tau}}\|\right|\,da+\\ &\quad+\int_{\Gamma_3}\mu(\|\mathbf{u}^h_{\boldsymbol{\tau}}\|)\left||\mathrm{R}\sigma_\nu(\mathbf{u}^h,\varphi^h)|-|\mathrm{R}\sigma_\nu(\mathbf{u},\varphi)|\right|\left|\|\mathbf{v}^h_\tau\|-\|\mathbf{u}_\tau\|\right|\,da+\\ &\quad+\int_{\Gamma_3}\left|g_2(\|\mathbf{u}^h\|)-g_2(\|\mathbf{u}\|)\right|\left|v_\nu^{h-}-u_\nu^-\right|\,da+\\ &\quad+\int_{\Gamma_3}\left|g_1(\|\mathbf{u}^h\|)-g_1(\|\mathbf{u}\|)\right|\left|v_\nu^{h+}-u_\nu^+\right|\,da,\end{aligned}$$

$$\begin{aligned}|\mathcal{I}_4|\le&\big(c_0\|\mathrm{R}\sigma_\nu(\mathbf{u},\varphi)\|_{L^\infty(\Gamma_3)}L_\mu+\max(\mathcal{M}_{\mathfrak{F}},\mathcal{M}_{\boldsymbol{\mathcal{P}}})\,\mu_*c_{\mathrm{R}}+c_0(L_{g_1}+L_{g_2})\big)\|\mathbf{u}-\mathbf{u}^h\|_V\|\mathbf{u}-\mathbf{v}^h\|_{L^2(\Gamma_3)^d}+\\ &+\frac{1}{c_\nu}\max\left(\mathcal{M}_{\mathfrak{F}},\mathcal{M}_{\boldsymbol{\mathcal{P}}}\right)\mu_*c_{\mathrm{R}}\|\varphi-\varphi^h\|_W\|\mathbf{u}-\mathbf{v}^h\|_{L^2(\Gamma_3)^d}.\end{aligned}\tag{51}$$

Concerning the the last term of the right-hand side of (47), using $(\mathrm{H}_4)(b)$, (c), the bounds $|\phi_L(\varphi-\varphi_F)|\le L$, and the Lipschitz continuity of the function ϕ_L, we have:

$$\begin{aligned}|\mathcal{I}_5| = &\left|\chi(\mathbf{u}^h,\varphi^h,\varphi-\varphi^h)-\chi(\mathbf{u},\varphi,\varphi-\varphi^h)+\chi(\mathbf{u}^h,\varphi^h,\xi^h-\varphi)-\chi(\mathbf{u},\varphi,\xi^h-\varphi)\right|\le\\ &\le\left|\int_{\Gamma_3}\big(\psi(u_\nu^h)\phi_L(\varphi^h-\varphi_F)-\psi(u_\nu)\phi_L(\varphi-\varphi_F)\big)\,(\varphi-\varphi^h)\,da\right|+\\ &\quad+\left|\int_{\Gamma_3}\big(\psi(u_\nu^h)\phi_L(\varphi^h-\varphi_F)-\psi(u_\nu)\phi_L(\varphi-\varphi_F)\big)\,(\xi^h-\varphi)\,da\right|\le\\ &\le M_\psi c_1^2\|\varphi-\varphi^h\|^2_W+LL_\psi c_0c_1\|\mathbf{u}-\mathbf{u}^h\|_V\|\varphi-\varphi^h\|_W+\\ &\quad+M_\psi c_1\|\varphi-\varphi^h\|_W\|\xi^h-\varphi\|_{L^2(\Gamma_3)}+LL_\psi c_0\|\mathbf{u}-\mathbf{u}^h\|_V\|\xi^h-\varphi\|_{L^2(\Gamma_3)}.\end{aligned}\tag{52}$$

Applying Young's inequality:

$$ab \leq \eta a^2 + \frac{1}{4\eta} b^2,$$

and using (47)–(52), we find that:

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^h\|_V^2 + \|\varphi - \varphi^h\|_W^2 \leq \\
&\leq C \Big\{ \|\mathbf{u} - \mathbf{v}^h\|_V^2 + \|\varphi - \xi^h\|_W^2 + \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d}^2 + \|\varphi - \xi^h\|_{L^2(\Gamma_3)}^2 + \\
&\quad + \left(\|\mathfrak{F}\boldsymbol{\varepsilon}(\mathbf{u})\|_{\mathcal{H}} + \|\boldsymbol{\mathcal{P}}^* \nabla \varphi\|_H + \|\mathbf{f}\|_V \right) \|\mathbf{u} - \mathbf{v}^h\|_V + \\
&\qquad + \left(\|\mu(\mathbf{u}_{\boldsymbol{\tau}})\|_{L^2(\Gamma_3)} \|\mathrm{R}\sigma_\nu(\mathbf{u}, \varphi)\|_{L^\infty(\Gamma_3)} + (M_{g_1} + M_{g_2}) \mathrm{meas}(\Gamma_3)^{1/2} \right) \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d} \Big\}, \quad (53)
\end{aligned}$$

where C is a positive constant independent of h, and consequently the inequality (42) holds. ■

Theorem 3. *Under the hypothesis of Theorem 1, and in addition, assume that:*

$$\boldsymbol{\sigma}_\tau \in L^2(\Gamma_3)^d, \quad \textit{and} \quad \sigma_\nu \in L^2(\Gamma_3).$$

Then, there exists a constant C independent of h such that:

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^h\|_V + \|\varphi - \varphi^h\|_W \leq \\
&\leq C \inf_{(\mathbf{v}^h, \xi^h) \in V^h \times W^h} \Big\{ \|\mathbf{u} - \mathbf{v}^h\|_V + \|\varphi - \xi^h\|_W + \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d} + \|\varphi - \xi^h\|_{L^2(\Gamma_3)} + \\
&\quad + \Big(\|\boldsymbol{\sigma}_\tau\|_{L^2(\Gamma_3)^d} + \|\sigma_\nu\|_{L^2(\Gamma_3)} + \|\mu(\mathbf{u}_{\boldsymbol{\tau}})\|_{L^2(\Gamma_3)} \|\mathrm{R}\sigma_\nu(\mathbf{u}, \varphi)\|_{L^\infty(\Gamma_3)} + \\
&\qquad + (M_{g_1} + M_{g_2}) \mathrm{meas}(\Gamma_3)^{1/2} \Big)^{1/2} \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d}^{1/2} \Big\}, \quad (54)
\end{aligned}$$

where $C > 0$ independent of h.

Proof. We start by making an approximation of the term $\mathcal{I}_2$, under the added regularity of $\boldsymbol{\sigma}_\tau \in L^2(\Gamma_3)^d$, and $\sigma_\nu \in L^2(\Gamma_3)$. Thus, from (2), the constitutive law (7), and the boundary conditions (11), (12), we get:

$$\begin{aligned}
|\mathcal{I}_2| &= \left| \left(\boldsymbol{\mathcal{P}}^* \nabla (\varphi - \varphi^h), \boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{v}^h) \right)_H + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\mathbf{v}_\tau^h - \mathbf{u}_\tau) da + \int_{\Gamma_3} \sigma_\nu (v_\nu^h - u_\nu) da + J(\mathbf{u}, \varphi, \mathbf{v}^h) - J(\mathbf{u}, \varphi, \mathbf{u}) \right| \leq \\
&\leq \|\boldsymbol{\mathcal{P}}^* \nabla \left(\varphi - \varphi^h \right) \|_H \|\boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{v}^h)\|_{\mathcal{H}} + \left(\|\boldsymbol{\sigma}_\tau\|_{L^2(\Gamma_3)^d} + \|\sigma_\nu\|_{L^2(\Gamma_3)} \right) \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d} + \\
&\quad + \left(\|\mathrm{R}\sigma_\nu(\mathbf{u}, \varphi)\|_{L^\infty(\Gamma_3)} \|\mu(\|\mathbf{u}_{\boldsymbol{\tau}}\|)\|_{L^2(\Gamma_3)} + (M_{g_1} + M_{g_2}) \,\mathrm{meas}(\Gamma_3)^{1/2} \right) \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d},
\end{aligned}$$

subsequently, through the utilization of Young's inequality, we obtain:

$$\begin{aligned}
|\mathcal{I}_2| &\leq \eta \|\varphi - \varphi^h\|_W^2 + \frac{1}{4\eta} \|\mathbf{u} - \mathbf{v}^h\|_V^2 + \left(\|\boldsymbol{\sigma}_\tau\|_{L^2(\Gamma_3)^d} + \|\sigma_\nu\|_{L^2(\Gamma_3)} \right) \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d} + \\
&\quad + \left(\|\mathrm{R}\sigma_\nu(\mathbf{u}, \varphi)\|_{L^\infty(\Gamma_3)} \|\mu(\|\mathbf{u}_{\boldsymbol{\tau}}\|)\|_{L^2(\Gamma_3)} + (M_{g_1} + M_{g_2}) \,\mathrm{meas}(\Gamma_3)^{1/2} \right) \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d}.
\end{aligned}$$

So, using this inequality and the same arguments used in the proof of Theorem 2, we conclude that the estimation (55) is verified, which concludes the proof.

In order to evaluate the errors arising from approximating the finite element spaces V^h and W^h, it is necessary to introduce an extra assumption regarding the smoothness of the solution:

$$\mathbf{u} \in H^2(\Omega)^d, \quad \mathbf{u}_{|\Gamma_3} \in H^2(\Gamma_3)^d, \quad \varphi \in H^2(\Omega), \quad \varphi_{|\Gamma_3} \in H^2(\Gamma_3).$$

Denoting $\Pi^h\mathbf{u}$ and $\Pi^h\varphi$ the standard finite element interpolation operators of $\mathbf{u}$ and φ, respectively, then we have the interpolation error estimate (cf. [35]):

$$\left\|\mathbf{u} - \Pi^h\mathbf{u}\right\|_V \leq Ch|\mathbf{u}|_{H^2(\Omega)^d}, \tag{55}$$

$$\left\|\varphi - \Pi^h\varphi\right\|_W \leq Ch|\varphi|_{H^2(\Omega)}. \tag{56}$$

where $|.|_{H^2(\Omega)^d}$ is the semi-norm over $H^2(\Omega)^d$. The restriction of the partitions τ^h on $\bar{\Gamma}_3$ induces a regular family of finite-element partitions of $\bar{\Gamma}_3$. So, we also have the interpolation error estimate:

$$\left\|\mathbf{u} - \Pi^h\mathbf{u}\right\|_{L^2(\Gamma_3)^d} \leq Ch^2|\mathbf{u}|_{H^2(\Gamma_3)^d}, \tag{57}$$

$$\left\|\varphi - \Pi^h\varphi\right\|_{L^2(\Gamma_3)} \leq Ch^2|\varphi|_{H^2(\Gamma_3)}. \tag{58}$$

Hence, by (54) and (55)–(58), we have the following error estimate:

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^h\|_V + \|\varphi - \varphi^h\|_W \leq \\
&\leq Ch\Bigg\{|\mathbf{u}|_{H^2(\Omega)^d} + |\varphi|_{H^2(\Omega)} + h|\mathbf{u}|_{H^2(\Gamma_3)^d} + h|\varphi|_{H^2(\Gamma_3)} + \\
&\quad + \Bigg(\|\boldsymbol{\sigma}_\tau\|_{L^2(\Gamma_3)^d} + \|\sigma_\nu\|_{L^2(\Gamma_3)} + \|\mathrm{R}\sigma_\nu(\mathbf{u},\varphi)\|_{L^\infty(\Gamma_3)}\|\mu(\|\mathbf{u}_{\boldsymbol{\tau}}\|)\|_{L^2(\Gamma_3)} + \\
&\quad\quad + (M_{g_1} + M_{g_2})\,\mathrm{meas}(\Gamma_3)^{1/2}\Bigg)^{1/2} |\mathbf{u}|^{1/2}_{H^2(\Gamma_3)^d}\Bigg\}.
\end{aligned}$$

6. Iteration method

In this section, we propose an iterative method which is useful for solving Problem (PV^h) and it is based on the method of successive approximations by a fixed-point iteration method [36]. This iteration method consists of the following procedure:
Let $\mathbf{x}_n^h = (\mathbf{u}_n^h, \varphi_n^h) \in X^h = V^h \times W^h$ be the n-th approximation of the solution to Problem (PV^h). We seek for the weak solution $\mathbf{x}_{n+1}^h = (\mathbf{u}_{n+1}^h, \varphi_{n+1}^h) \in X^h$ of the linear problem.

$$\begin{aligned}
&\left(\mathbf{x}_{n+1}^h, \mathbf{y}^h - \mathbf{x}_{n+1}^h\right)_X + \rho\widetilde{J}(\mathbf{x}_n^h, \mathbf{y}^h) - \rho\widetilde{J}(\mathbf{x}_n^h, \mathbf{x}_{n+1}^h) + \rho\widetilde{\chi}(\mathbf{x}_n^h, \mathbf{y}^h - \mathbf{x}_{n+1}^h) \geq \\
&\geq \left(\mathbf{x}_n^h, \mathbf{y}^h - \mathbf{x}_{n+1}^h\right)_X - \rho\left(A\mathbf{x}_n^h - \mathbf{f}_3, \mathbf{y}^h - \mathbf{x}_{n+1}^h\right)_X \quad \forall \mathbf{y}^h \in X^h,
\end{aligned} \tag{59}$$

where $\rho > 0$ is a constant.

We have the following result.

Lemma 5. *There exists a unique solution* $\mathbf{x}_{n+1}^h = (\mathbf{u}_n^h, \varphi_n^h) \in V^h \times W^h$, *satisfying* (59).

Proof. Let us write the variational inequality (59) in the form:

$$\left\{\begin{array}{l}
\text{Find } \mathbf{z}^h \in X^h \text{ such that:} \\
b(\mathbf{z}^h, \mathbf{y}^h - \mathbf{z}^h) + \phi(\mathbf{y}^h) - \phi(\mathbf{z}^h) \geq (G, \mathbf{y}^h - \mathbf{z}^h)_X \quad \forall \mathbf{y}^h \in X^h,
\end{array}\right. \tag{60}$$

where $\mathbf{z}^h = \mathbf{x}^h_{n+1}$ and

$$b(\mathbf{z}^h, \mathbf{y}^h - \mathbf{z}^h) = (\mathbf{x}^h_{n+1}, \mathbf{y}^h - \mathbf{x}^h_{n+1})_X, \quad \phi(\mathbf{y}^h) = \rho \widetilde{J}(\mathbf{x}^h_n, \mathbf{y}^h), \quad \phi(\mathbf{z}^h) = \rho \widetilde{J}(\mathbf{x}^h_n, \mathbf{x}^h_{n+1}),$$
$$(G, \mathbf{y}^h - \mathbf{z}^h)_X = (\mathbf{x}^h_n, \mathbf{y}^h - \mathbf{x}^h_{n+1})_X - \rho(A\mathbf{x}^h_n - \mathbf{f}_3, \mathbf{y}^h - \mathbf{x}^h_{n+1})_X - \rho\widetilde{\chi}(\mathbf{x}^h_n, \mathbf{y}^h - \mathbf{x}^h_{n+1}).$$

Since $b(\mathbf{z}^h, \mathbf{y}^h)$ is a continuous and X^h-elliptic bilinear from, $\phi(\mathbf{z}^h)$ is a proper, convex and lower semi-continuous function and G is linear and continuous functional, we deduce that the variational inequality (60) has a unique solution (see [36]). ■

Theorem 4. *Let* $\mathbf{x}^h$ *and* $\mathbf{x}^h_{n+1}$ *be the solutions of* (40), (41) *and* (59), *respectively. Under the assumptions of Theorem 1, with the same value of* L^*, $\mathbf{x}^h_{n+1}$ *converges strongly to* $\mathbf{x}^h$ *in* X^h *for:*

$$0 < \rho < \frac{2(m_A - \alpha)}{M_A^2 - \alpha^2}.$$

Proof. In the first phase of our demonstration, we will show that the solution of (59), $\mathbf{x}^h_{n+1}$, weakly converges to $\mathbf{x}^h$, the solution of (PV^h). In order to do this, we consider $\mathbf{x}^h_{n+1}$ and $\mathbf{x}^h_{n+2}$ as two successive solutions of the variational inequality (59):

$$\begin{aligned}
&\left(\mathbf{x}^h_{n+1}, \mathbf{y}^h - \mathbf{x}^h_{n+1}\right)_X + \rho\widetilde{J}(\mathbf{x}^h_n, \mathbf{y}^h) - \rho\widetilde{J}(\mathbf{x}^h_n, \mathbf{x}^h_{n+1})_X + \rho\widetilde{\chi}(\mathbf{x}^h_n, \mathbf{y}^h - \mathbf{x}^h_{n+1}) \geq \\
&\geq \left(\mathbf{x}^h_n, \mathbf{y}^h - \mathbf{x}^h_{n+1}\right)_X - \rho\left(A\mathbf{x}^h_n - \mathbf{f}_3, \mathbf{y}^h - \mathbf{x}^h_{n+1}\right)_X, && (61)\\
&\left(\mathbf{x}^h_{n+2}, \mathbf{y}^h - \mathbf{x}^h_{n+2}\right)_X + \rho\widetilde{J}(\mathbf{x}^h_{n+1}, \mathbf{y}^h) - \rho\widetilde{J}(\mathbf{x}^h_{n+1}, \mathbf{x}^h_{n+2}) + \rho\widetilde{\chi}(\mathbf{x}^h_{n+1}, \mathbf{y}^h - \mathbf{x}^h_{n+2}) \geq \\
&\geq \left(\mathbf{x}^h_{n+1}, \mathbf{y}^h - \mathbf{x}^h_{n+2}\right)_X - \rho\left(A\mathbf{x}^h_{n+1} - \mathbf{f}_3, \mathbf{y}^h - \mathbf{x}^h_{n+2}\right)_X. && (62)
\end{aligned}$$

By adding the inequalities resulting from $\mathbf{y}^h = \mathbf{x}^h_{n+2}$ and $\mathbf{y}^h = \mathbf{x}^h_{n+1}$ in (61) and (62), respectively, we obtain:

$$\begin{aligned}
&\left(\mathbf{x}^h_{n+2} - \mathbf{x}^h_{n+1}, \mathbf{x}^h_{n+2} - \mathbf{x}^h_{n+1}\right)_X \leq \\
&\leq \rho\left(\widetilde{J}(\mathbf{x}^h_n, \mathbf{x}^h_{n+2}) - \widetilde{J}(\mathbf{x}^h_{n+1}, \mathbf{x}^h_{n+2}) + \widetilde{J}(\mathbf{x}^h_{n+1}, \mathbf{x}^h_{n+1}) - \widetilde{J}(\mathbf{x}^h_n, \mathbf{x}^h_{n+1})\right) + \\
&\quad + \rho\left(\widetilde{\chi}(\mathbf{x}^h_n, \mathbf{x}^h_{n+2} - \mathbf{x}^h_{n+1}) - \widetilde{\chi}(\mathbf{x}^h_{n+1}, \mathbf{x}^h_{n+2} - \mathbf{x}^h_{n+1})\right) + \\
&\quad\quad + \left(\mathbf{x}^h_{n+1} - \mathbf{x}^h_n - \rho(A\mathbf{x}^h_{n+1} - A\mathbf{x}^h_n), \mathbf{x}^h_{n+2} - \mathbf{x}^h_{n+1}\right)_X.
\end{aligned}$$

Then, it follows that:

$$\|\mathbf{x}^h_{n+2} - \mathbf{x}^h_{n+1}\|^2_X \leq \rho\mathcal{S}_1 + \mathcal{S}_2, \tag{63}$$

where

$$\begin{aligned}
\mathcal{S}_1 =& \widetilde{J}(\mathbf{x}^h_n, \mathbf{x}^h_{n+2}) - \widetilde{J}(\mathbf{x}^h_n, \mathbf{x}^h_{n+1}) + \widetilde{J}(\mathbf{x}^h_{n+1}, \mathbf{x}^h_{n+1}) - \widetilde{J}(\mathbf{x}^h_{n+1}, \mathbf{x}^h_{n+2}) + \\
&+ \widetilde{\chi}(\mathbf{x}^h_n, \mathbf{x}^h_{n+2}) - \widetilde{\chi}(\mathbf{x}^h_n, \mathbf{x}^h_{n+1}) + \widetilde{\chi}(\mathbf{x}^h_{n+1}, \mathbf{x}^h_{n+1}) - \widetilde{\chi}(\mathbf{x}^h_{n+1}, \mathbf{x}^h_{n+2}),\\
\mathcal{S}_2 =& \left(\mathbf{x}^h_{n+1} - \mathbf{x}^h_n - \rho\left(A\mathbf{x}^h_{n+1} - A\mathbf{x}^h_n\right), \mathbf{x}^h_{n+2} - \mathbf{x}^h_{n+1}\right)_X.
\end{aligned}$$

From algebraic manipulations similar to those in the proof of Theorem 2, we obtain:

$$\mathcal{S}_1 \leq \rho\alpha\|\mathbf{x}^h_{n+2} - \mathbf{x}^h_{n+1}\|_X \|\mathbf{x}^h_{n+1} - \mathbf{x}^h_n\|_X. \tag{64}$$

On the other hand, using Cauchy – Schwarz inequality, we get:

$$\mathcal{S}_2 \leq \|\mathbf{x}^h_{n+1} - \mathbf{x}^h_n - \rho\left(A\mathbf{x}^h_{n+1} - A\mathbf{x}^h_n\right)\|_X \|\mathbf{x}^h_{n+2} - \mathbf{x}^h_{n+1}\|_X,$$

and since, we have:

$$\begin{aligned}&\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n-\rho\left(A\mathbf{x}^h_{n+1}-A\mathbf{x}^h_n\right)\|^2_X=\\&=\left(\mathbf{x}^h_{n+1}-\mathbf{x}^h_n-\rho\left(A\mathbf{x}^h_{n+1}-A\mathbf{x}^h_n\right),\,\mathbf{x}^h_{n+1}-\mathbf{x}^h_n-\rho\left(A\mathbf{x}^h_{n+1}-A\mathbf{x}^h_n\right)\right)_X=\\&=\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n\|^2_X-2\rho\big(\mathbf{x}^h_{n+1}-\mathbf{x}^h_n,\,A\mathbf{x}^h_{n+1}-A\mathbf{x}^h_n\big)+\rho^2\|A\mathbf{x}^h_{n+1}-A\mathbf{x}^h_n\|^2_X,\end{aligned}$$

moreover by using (30), and (31), we obtain:

$$\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n-\rho\left(A\mathbf{x}^h_{n+1}-A\mathbf{x}^h_n\right)\|^2_X\le\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n\|^2_X-2\rho m_A\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n\|^2_X+\rho^2M_A^2\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n\|^2_X,$$

then, we have:

$$\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n-\rho\left(A\mathbf{x}^h_{n+1}-A\mathbf{x}^h_n\right)\|_X\le\sqrt{1-2\rho m_A+\rho^2M_A^2}\;\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n\|_X.$$

Hence, using the above inequality, we find that:

$$\mathcal{S}_2\le\sqrt{1-2\rho m_A+\rho^2M_A^2}\;\|\mathbf{x}^h_{n+2}-\mathbf{x}^h_{n+1}\|_X\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n\|_X. \tag{65}$$

Combining (63), (64), and (65), we obtain:

$$\|\mathbf{x}^h_{n+2}-\mathbf{x}^h_{n+1}\|_X\le\lambda(\rho)\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n\|_X,$$

where $\lambda(\rho)=\rho\alpha+\sqrt{1-2\rho m_A+\rho^2M_A^2}$. i.e.,

$$\|\mathbf{x}^h_{n+1}-\mathbf{x}^h_n\|_X\le\lambda(\rho)^{n+1}\|\mathbf{x}^h_1-\mathbf{x}^h_0\|_X.$$

We can choose ρ such that:

$$0<\rho<\frac{2(m_A-\alpha)}{M_A^2-\alpha^2}\quad\text{for}\quad\frac{\alpha}{m_A}<1,$$

we obtain $\lambda(\rho)<1$. Then, we deduce that $(\mathbf{x}^h_n)$ is a Cauchy sequence. Hence $(\mathbf{x}^h_n)$ is bounded in X^h, so there exist $\mathbf{x}^*\in X$, and a subsequence still denoted by $(\mathbf{x}^h_n)$, such that:

$$\mathbf{x}^h_n\rightharpoonup\mathbf{x}^*\quad\text{weakly in } X^h,\quad\text{as}\quad n\to+\infty. \tag{66}$$

Next, we proof that $\mathbf{x}^*$ is a solution of (PV^h). Since the trace map $\gamma:V\times W\to L^2(\Gamma_3)^d\times L^2(\Gamma_3)$ is a compact operator, from the weak convergence $\mathbf{x}^h_n\rightharpoonup\mathbf{x}^*$ in X^h, we obtain the convergence $\mathbf{x}^h_n\to\mathbf{x}^*$ strongly in $L^2(\Gamma_3)^d\times L^2(\Gamma_3)$. From (59), we have:

$$\begin{aligned}&\rho\left(A\mathbf{x}^h_n,\mathbf{y}^h-\mathbf{x}^h_{n+1}\right)_X+\rho\widetilde{J}(\mathbf{x}^h_n,\mathbf{y}^h)-\rho\widetilde{J}(\mathbf{x}^h_n,\mathbf{x}^h_{n+1})+\rho\widetilde{\chi}(\mathbf{x}^h_n,\mathbf{y}^h-\mathbf{x}^h_{n+1})\ge\\&\ge\left(\mathbf{x}^h_n-\mathbf{x}^h_{n+1},\mathbf{y}-\mathbf{x}^h_{n+1}\right)_X+\rho(\mathbf{f}_3,\mathbf{y}^h-\mathbf{x}^h_{n+1})_X.\end{aligned}$$

Now, from (66), the properties of R, ψ, g_1, g_2 and ϕ_L, we have:

$$\left.\begin{aligned}&\widetilde{J}(\mathbf{x}^h_n,\mathbf{y}^h)-\widetilde{J}(\mathbf{x}^h_n,\mathbf{x}^h_{n+1})\to\widetilde{J}(\mathbf{x}^*,\mathbf{y}^h)-\widetilde{J}(\mathbf{x}^*,\mathbf{x}^*),\\&\widetilde{\chi}(\mathbf{x}^h_n,\mathbf{y}^h-\mathbf{x}^h_{n+1})\to\widetilde{\chi}(\mathbf{x}^*,\mathbf{y}^h-\mathbf{x}^*),\\&(\mathbf{x}^h_n,\mathbf{y}^h-\mathbf{x}^h_{n+1})_X-(\mathbf{x}^h_{n+1},\mathbf{y}^h-\mathbf{x}^h_{n+1})_X\to(\mathbf{x}^*,\mathbf{y}^h-\mathbf{x}^*)_X-(\mathbf{x}^*,\mathbf{y}^h-\mathbf{x}^*)_X,\\&(\mathbf{f}_3,\mathbf{y}^h-\mathbf{x}^h_{n+1})\to(\mathbf{f}_3,\mathbf{y}^h-\mathbf{x}^*)_X,\end{aligned}\right\}\text{ as } n\to+\infty.$$

Then, we find that:

$$\limsup_{n\to+\infty} \rho(A\mathbf{x}_n^h, \mathbf{x}_{n+1}^h - \mathbf{y}^h)_X \le \rho(\mathbf{f}_3, \mathbf{x}^* - \mathbf{y}^h) + \rho\widetilde{J}(\mathbf{x}^*, \mathbf{y}^h) - \rho\widetilde{J}(\mathbf{x}^*, \mathbf{x}^*) + \rho\widetilde{\chi}(\mathbf{x}^*, \mathbf{y}^h - \mathbf{x}^*),$$

or

$$\begin{aligned}\limsup_{n\to+\infty} \rho(A\mathbf{x}_n^h, \mathbf{x}_{n+1}^h - \mathbf{x}^*)_X &= \limsup_{n\to+\infty} \rho(A\mathbf{x}_n^h, \mathbf{x}_{n+1}^h - \mathbf{y}^h)_X + \limsup_{n\to+\infty} \rho(A\mathbf{x}_n^h, \mathbf{y}^h - \mathbf{x}^*)_X \le \\ &\le \limsup_{n\to+\infty} \rho(A\mathbf{x}_n^h, \mathbf{x}_{n+1}^h - \mathbf{y}^h)_X + \limsup_{n\to+\infty} \rho\|A\mathbf{x}_n^h\|_X \|\mathbf{y}^h - \mathbf{x}^*\|_X \le \\ &\le \rho\left(\mathbf{f}_3, \mathbf{x}^* - \mathbf{y}^h\right) + \rho\widetilde{J}(\mathbf{x}^*, \mathbf{y}^h) - \rho\widetilde{J}(\mathbf{x}^*, \mathbf{x}^*) + \rho\widetilde{\chi}(\mathbf{x}^*, \mathbf{y}^h - \mathbf{x}^*) + \\ &\quad + \limsup_{n\to+\infty} \rho\|A\mathbf{x}_n^h\|_X \|\mathbf{y}^h - \mathbf{x}^*\|_X,\end{aligned}$$

for all $\mathbf{y}^h = (\mathbf{v}^h, \xi^h) \in X^h$. Note that $\|A\mathbf{x}_n^h\|_X$ is bounded, and we may then substitute $\mathbf{y}^h = \mathbf{x}^*$ into the previous inequality to obtain:

$$\limsup_{n\to+\infty} \rho(A\mathbf{x}_n^h, \mathbf{x}_{n+1}^h - \mathbf{x}^*)_X \le 0.$$

Furthermore, we use the pseudo-monotonicity of the operator A to conclude:

$$\rho(A\mathbf{x}^*, \mathbf{x}^* - \mathbf{y}^h)_X \le \liminf_{n\to+\infty} \rho(A\mathbf{x}_n^h, \mathbf{x}_{n+1}^h - \mathbf{y}^h)_X.$$

Hence, we have:

$$(A\mathbf{x}^*, \mathbf{y}^h - \mathbf{x}^*)_X + \widetilde{J}(\mathbf{x}^*, \mathbf{y}) - \widetilde{J}(\mathbf{x}^*, \mathbf{x}^*) + \widetilde{\chi}(\mathbf{x}^*, \mathbf{y}^h - \mathbf{x}^*) \ge (\mathbf{f}_3, \mathbf{y}^h - \mathbf{x}^*)_X \quad \forall \mathbf{x}^* \in X^h.$$

From (72), we find that $\mathbf{x}^*$ is a solution of Problem (PV^h), and from the uniqueness of the solution to this variational inequality we obtain $\mathbf{x}^* = \mathbf{x}^h$. We conclude that $\mathbf{x}^h = (\mathbf{u}^h, \varphi^h)$ is the unique weak limit in $X^h = V^h \times W^h$ of any subsequence of the sequence $(\mathbf{x}_n^h)$ and therefore, we find that the whole sequence $(\mathbf{x}_n^h)$ converges weakly to element $\mathbf{x}^h$.

In the second phase of our demonstration, we will proof that $\mathbf{x}_{n+1}^h$, the solution of (59), converges strongly to $\mathbf{x}^h$ the solution of (PV^h) as $n \to +\infty$.
(i) The couple $\mathbf{x}^h = (\mathbf{u}^h, \varphi^h)$ is a solution of (PV^h) if only if

$$(A\mathbf{x}^h, \mathbf{y}^h - \mathbf{x}^h)_X + \widetilde{J}(\mathbf{x}^h, \mathbf{y}^h) - \widetilde{J}(\mathbf{x}^h, \mathbf{x}^h) + \widetilde{\chi}(\mathbf{x}^h, \mathbf{y}^h - \mathbf{x}^h) \ge (\mathbf{f}_3, \mathbf{y}^h - \mathbf{x}^h)_X, \tag{67}$$

for all $\mathbf{y}^h = (\mathbf{v}^h, \xi^h) \in X^h$.
(ii) The couple $\mathbf{x}_{n+1}^h = (\mathbf{u}_{n+1}^h, \varphi_{n+1}^h)$ is a solution of (59) if only if

$$\begin{aligned}&(\mathbf{x}_{n+1}^h, \mathbf{y}^h - \mathbf{x}_{n+1}^h)_X + \rho\widetilde{J}(\mathbf{x}_n^h, \mathbf{y}^h) - \rho\widetilde{J}(\mathbf{x}_n^h, \mathbf{x}_{n+1}^h) + \rho\widetilde{\chi}(\mathbf{x}_n^h, \mathbf{y}^h - \mathbf{x}_{n+1}^h) \ge \\ &\ge (\mathbf{x}_n^h, \mathbf{y}^h - \mathbf{x}_{n+1}^h)_X - \rho(A\mathbf{x}_n^h - \mathbf{f}_3, \mathbf{y}^h - \mathbf{x}_{n+1}^h)_X \quad \forall \mathbf{y}^h \in X^h.\end{aligned} \tag{68}$$

Multiplying both sides of the inequality (67) by ρ, then taking $\mathbf{y}^h = \mathbf{x}_{n+1}^h$ in (67), $\mathbf{y}^h = \mathbf{x}^h$ in (68) and adding the obtained inequalities, we get:

$$\begin{aligned}&(\mathbf{x}_{n+1}^h - \mathbf{x}_n^h, \mathbf{x}^h - \mathbf{x}_{n+1}^h)_X + \rho(A\mathbf{x}^h - A\mathbf{x}_n^h, \mathbf{x}_{n+1}^h - \mathbf{x}^h)_X + \\ &\quad + \rho\left[\widetilde{J}(\mathbf{x}_n^h, \mathbf{x}^h) - \widetilde{J}(\mathbf{x}_n^h, \mathbf{x}_{n+1}^h) + \widetilde{J}(\mathbf{x}^h, \mathbf{x}_{n+1}^h) - \widetilde{J}(\mathbf{x}^h, \mathbf{x}^h)\right] + \\ &\quad + \rho\left[\widetilde{\chi}(\mathbf{x}_n^h, \mathbf{x}^h - \mathbf{x}_{n+1}^h) + \widetilde{\chi}(\mathbf{x}^h, \mathbf{x}_{n+1}^h - \mathbf{x}^h)\right] \ge 0.\end{aligned} \tag{69}$$

The inequality (69), can be rewritten as follows:

$$(\mathbf{x}_{n+1}^h - \mathbf{x}^h, \mathbf{x}_{n+1}^h - \mathbf{x}^h)_X \leq \mathcal{G}_1 + \mathcal{G}_2, \tag{70}$$

where

$$\mathcal{G}_1 = \rho\Big[\widetilde{J}(\mathbf{x}_n, \mathbf{x}^h) - \widetilde{J}(\mathbf{x}_n^h, \mathbf{x}_{n+1}^h) + \widetilde{J}(\mathbf{x}^h, \mathbf{x}_{n+1}) - \widetilde{J}(\mathbf{x}^h, \mathbf{x}^h) + \widetilde{\chi}(\mathbf{x}_n^h, \mathbf{x}^h - \mathbf{x}_{n+1}^h) + \widetilde{\chi}(\mathbf{x}^h, \mathbf{x}_{n+1}^h - \mathbf{x}^h)\Big],$$

$$\mathcal{G}_2 = \big(\mathbf{x}_n^h - \mathbf{x}^h - \rho(A\mathbf{x}^h - A\mathbf{x}_n^h), \mathbf{x}_{n+1}^h - \mathbf{x}^h\big)_X.$$

One has:

$$\mathcal{G}_1 \leq \rho\alpha \|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X \|\mathbf{x}_n^h - \mathbf{x}^h\|_X. \tag{71}$$

Moreover, it follows from (30), (31) and Cauchy Schwarz inequality that:

$$\mathcal{G}_2 \leq \sqrt{1 - 2\rho m_A + \rho^2 M_A^2} \|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X \|\mathbf{x}_n^h - \mathbf{x}^h\|_X. \tag{72}$$

Then, in virtue of (70), (71), and (72), we get:

$$\|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X^2 \leq \lambda(\rho) \|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X \|\mathbf{x}_n^h - \mathbf{x}^h\|_X.$$

Next, we use the triangular inequality to conclude that:

$$\begin{aligned} \|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X^2 &\leq \lambda(\rho) \|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X (\|\mathbf{x}_n^h - \mathbf{x}_{n+1}^h\|_X + \|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X) \leq \\ &\leq \lambda(\rho) \|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X^2 + \lambda(\rho) \|\mathbf{x}_n^h - \mathbf{x}_{n+1}^h\|_X \|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X. \end{aligned}$$

Hence, we have:

$$\|\mathbf{x}_{n+1}^h - \mathbf{x}^h\|_X \leq \frac{\lambda(\rho)}{1 - \lambda(\rho)} \|\mathbf{x}_{n+1}^h - \mathbf{x}_n^h\|_X.$$

Finally, from the above inequality, letting $n \to +\infty$, we obtain $\mathbf{x}_n^h \to \mathbf{x}^h$. ■

Conclusion

The presented paper outlines a model that deals with the static process of frictional contact between an electrically conductive foundation and a piezoelectric body, where the electro-elastic constitutive law is considered to be nonlinear. The model used in this study incorporated Signorini modified contact conditions and Coulomb's friction law, while also taking into account the electrical conductivity condition. By applying the theory of variational inequalities and a fixed-point theorem, the existence of a unique weak solution for the problem was established. Additionally, a finite element method was utilized to approximate the solution, and an iteration method was proposed to numerically solve the problem, with its convergence being established.

Conflicts of interest statement

The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers' bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ

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Анализ и численные результаты для модифицированной задачи Синьорини с нелокальным трением в электроупругости

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Аннотация

В статье основное внимание уделяется математической модели, которая описывает состояние равновесия пьезоэлектрической структуры, находящейся в контакте с проводящим основанием, с учетом трения. Основной закон, регулирующий электроупругое поведение системы, считается нелинейным, а контакт моделируется с использованием модифицированных контактных условий Синьорини. Эти условия дополняются нелокальным законом трения Кулона и регуляризованным условием электропроводности. Слабая формулировка модели представлена как связанная система, которая связывает поля смещения и электрического потенциала. Показано, что слабое решение существует и единственно, при этом используются теоремы Банаха о неподвижной точке и аргументов абстрактных эллиптических квазивариационных неравенств. Кроме того, исследовано конечно-элементное приближение задачи и выведена оценка связанной с ним погрешности. Представлен итерационный метод для решения системы конечных элементов, полученной в результате анализа, и рассмотрен анализ сходимости метода при соответствующих условиях.

Ключевые слова: пьезоэлектрическое тело, проводящее основание, модифицированные контактные условия Синьорини, закон трения Кулона, квазивариационное неравенство, банахова неподвижная точка, итерационный метод.

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