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A constrained weighted Tchebychev program for multiple objective integer linear programming

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In this paper, an algorithm for enumerating all non-dominated vectors of multiple objective integer linear programs is presented. Starting from an initial non-dominated vector, at each iteration, the procedure determines a new solution by solving a constrained weighted Tchebychev program. Progressively more constraints are added to this program in order to reduce the admissible research set.

Keywords: multiple objective integer programming, Tchebychev norm, branch and bound.

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Introduction

Multiple objective integer linear programming (MOILP) is very useful for many areas of application as any model, that incorporates discrete phenomena requires the consideration of integer variables (such as, for modelling investment choices, production levels, fixed charges, logical conditions or disjunctive constraints).

Over the last decades, several methods have been developed to solve MOILP problems, some methods require the presence of human decision maker (DM) (interactive) and generate only a subset of non-dominated vectors, and other methods consist in enumerating all non-dominated vectors without intervention of DM. In general, the approaches can be classified as exact or heuristic and grouped according to the methodological concepts they use. Among others, the concepts employed in exact algorithms include branch and bound [1, 2], dynamic programming [3], implicit enumeration [4–6], reference directions [7], weighted norms [8–10]; weighted sums with additional constraints [11, 12], and 0–1 programmation [13] and lexicographic method [14]. Heuristic approaches, such as simulated annealing, tabu search, and evolutionary algorithms, have been proposed for multiobjective integer programs with an underlying combinatorial structure [15]. Several survey articles have already been published in this area. Teghem and Kunsch [16] presented a survey of interactive methods for multiobjective integer and mixed-integer linear programming, a brief overview of MOILP approaches can be found in [17].

The algorithm presented in this work based on a parameterized exploration of the outcome space that defines a sequence of progressively more constrained single-objective mixed-integer problems that successively eliminates undesirable points. The main idea is to use the Weighted Tchebychev Program (WTP) for identifying the non-dominated objective vectors. It is known that WTP program is a mixed-integer linear program (MILP) which can be examined using standard integer-linear programming techniques such as branch and bound. Thus, it may yield several optimal solutions which some can be non-dominated or weakly non-dominated by others. In order to avoid the delicate situation lies this norm and the weakly non-dominated vectors, we try to modify the program WTP by adding some constraints. This technique of additional constraints known in the literature as the "Corner constraints" is developed by Klein Hannan [4], also used by Sylva Crema [5].

The organization of the paper is as follows: Section 1 briefly reviews basic definitions, results and foundations of Tchebychev norms. The algorithm is presented in Section 2 and a number of propositions are provided to support finiteness and convergence properties, an illustrative example is also provided.

1. Basic results and Tchebychev metrics

The MOILP problem under consideration has the following form:

$$(P) \quad V\max\{C\mathbf{x},\ \mathbf{x} \in D\}.$$

Where $D = S \cap \mathbb{Z}$ with $S = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{b}; \mathbf{x} \geq 0\}$ is nonempty bounded set; $A \in \mathbb{Z}^{m\times n}$, $\mathbf{b} \in \mathbb{Z}^m$, $C = (c^i)_{i\in\{1,\ldots,p\}} \in \mathbb{Z}^{p\times n}$ and $p \geq 2$.

We denote by Z the image of D in outcome space defined by the objective vector function.

Unlike single-objective problems, the resolution of multiple criteria problems imposes a set of feasible solutions, using the property that no improvement on any criterion is possible without sacrificing on at least one other criterion. These solutions are called efficient solutions or non-dominated solutions, which are defined as follows.

A feasible solution $\hat{\mathbf{x}} \in D$ is said to be an *efficient solution* of MOILP if and only if, there is no feasible solution $\mathbf{x} \in D$ such that $C\mathbf{x} \geq C\hat{\mathbf{x}}$ and $C\mathbf{x} \neq C\hat{\mathbf{x}}$ ($c^i\mathbf{x} \geq c^i\hat{\mathbf{x}}$ for all $i = 1,\ldots,p$ and $c^i\mathbf{x} > c^i\hat{\mathbf{x}}$ for at least one i). The point $\hat{\mathbf{z}} = C\hat{\mathbf{x}}$ is then called *non-dominated vector*. Otherwise, $\hat{\mathbf{x}}$ is not efficient and $\hat{\mathbf{z}} = C\hat{\mathbf{x}}$ is said to be dominated by $\mathbf{z} = C\mathbf{x}$.

$\hat{\mathbf{x}} \in D$ is called weakly efficient if there is no $\mathbf{x} \in D$ such that $C\mathbf{x} > C\hat{\mathbf{x}}$, i. e. $c^i\mathbf{x} > c^i\hat{\mathbf{x}}$ for all $i = 1,\ldots,p$. The point $\hat{\mathbf{z}} = C\hat{\mathbf{x}}$ is then called weakly non-dominated objective vector.

$E(P)$ and $Z(P)$ will be used henceforth to denote, respectively, the set of all efficient solutions of problem P and their image in outcome space defined by the objective vector function. Since the feasible region of P is nonconvex, unsupported non-dominated solutions may exist. A non-dominated point $\mathbf{z} \in Z(P)$ is called unsupported if it is dominated by a convex combination (which does not belong to Z) of other non-dominated objective vector (belonging to Z).

The ranges of the non-dominated objective vectors in the outcome space provide valuable information about the problem MOILP considered if the objective functions are bounded over the feasible region. Upper bounds of the non-dominated solutions set are available in the ideal objective vector $\mathbf{z}^\star \in \mathbb{R}^p$. Its components $z_i^\star$ are obtained by maximizing each of the objective functions individually subject to the feasible region D. A vector strictly better than $\mathbf{z}^{\star}$ can be called a utopian objective vector $\mathbf{z}^{\star\star}$. In this work, we use the utopian and not the ideal objective values in order to avoid dividing by zero in all occasions. Thus, the components of the matrix C are assumed integer, then we can set $\mathbf{z}^{\star\star} = \mathbf{z}^{\star} + 1$.

The Tchebychev theory, whose foundation originated from Bowman [18], has been successfully exploited within the scope of interactive algorithms for multiple objective optimization in Steuer and Choo [9] and since, the scalarization techniques based on Tchebychev norms was intensively used to solve multiple objective programming problem involving discrete decisions. However, Bowman [18] proved that the Tchebychev scalarization norms is appropriate for generating the non-dominated objective vectors set, in particular those which are unsupported (see for example [19–21]).

We denote by Δ the weighting vectors space defined as

$$\boldsymbol{\beta} \in \Delta = \left\{ \boldsymbol{\beta} \in \mathbb{R}^p \mid 0 < \beta_i < 1, \sum_{i=1}^{p} \beta_i = 1 \right\}.$$

Given a point $\mathbf{z} \in Z$, the weighted Tchebychev norm of $\mathbf{z}$ in $\mathbb{R}^p$ according to $\mathbf{z}^{\star\star}$ is defined as

$$\|\mathbf{z}^{\star\star} - \mathbf{z}\|^{\beta} = \max_{i=1,\dots,p} \left\{ \beta_i |z_i^{\star\star} - z_i| \right\}. \tag{1}$$

Here $\boldsymbol{\beta} \in \Delta$ represents its weighted vector which can be calculated as follows

$$\beta_i = \frac{1}{z_i^{\star\star} - z_i} \left[\sum_{i=1}^{p} \frac{1}{z_i^{\star\star} - z_i} \right]^{-1} \quad \forall\, 1 \le i \le p. \tag{2}$$

The aim for introducing this norm is to measure the distance between any $\mathbf{z}$ and the utopian objective vector $\mathbf{z}^{\star\star}$. Therefore, this technique consists in selecting the feasible objective vectors with minimum weigh distance from $\mathbf{z}^{\star\star}$. In others words, for a given $\boldsymbol{\beta}$, to reach this goal one has to solve the so-called *minimization of the norm* problem defined as follows

$$\min_{\mathbf{z} \in Z} \left\{ ||\mathbf{z}^{\star\star} - \mathbf{z}||^{\boldsymbol{\beta}} \right\}. \tag{3}$$

Bowman [18] has proposed to solve an equivalent problem called *weighted Tchebychev program* defined as follows

$$P(\boldsymbol{\beta}) \begin{cases} \min\ \omega \\ \omega \ge \beta_i (z_i^{\star\star} - z_i),\ 1 \le i \le p, \\ z_i = c^i x, \\ x \in D, \\ \omega \ge 0. \end{cases} \tag{4}$$

Problem $P(\boldsymbol{\beta})$ is a mixed-integer linear program (MILP) which can be examined using standard integer-linear programming techniques such as branch and bound. However, $P(\boldsymbol{\beta})$ may yield several optimal solutions of which some can be non-dominated or weakly non-dominated by others. In order to eliminate the weakly non-dominated solutions we can use the augmented weighted Tchebychev program $P_\rho(\boldsymbol{\beta})$ for a small $\rho > 0$ as folows:

$$P_\rho(\boldsymbol{\beta}) \begin{cases} \min\ \omega + \rho \sum_{1}^{p} (z_i^{\star\star} - z_i) \\ \omega \ge \beta_i (z_i^{\star\star} - z_i),\ 1 \le i \le p, \\ z_i = c^i x, \\ x \in D, \\ \omega \ge 0. \end{cases} \tag{5}$$

We have the following results.

Theorem 1 (**see [22]**). *Let Z be finite and*

$$M = \{\mathbf{z} \in Z \mid (\mathbf{x}, \mathbf{z}, \omega) \text{ is a minimal solution of } P(\boldsymbol{\beta}) \text{ for some } \boldsymbol{\beta} \in \Delta\}.$$

Then there exists $\bar{\mathbf{z}} \in M$ such that $\bar{\mathbf{z}} \in Z(P)$.

Theorem 2 (**see [18]**). $\mathbf{z} = C\hat{\mathbf{x}}$, $\hat{\mathbf{x}} \in D$ *is non-dominated solution for MOILP only if it is a solution to $P(\boldsymbol{\beta})$ for some $\boldsymbol{\beta}$.*

Eswaran et al. [8], Ted et al. [10] developed two algorithms based on solving $P(\boldsymbol{\beta})$ for enumerating all non-dominated vectors of MOILP but solely with two objectives where the technique of comparison is used to eliminate the weakly non-dominated solutions. For a problem having more than two objective functions this technique is not appropriate.

In this work, we propose to solve the weighted Tchebychev program augmented by adding some constraints in order to avoid the trap related by the weighted norm and the weakly non-dominated vectors. The main idea of our technique consists in moving from a non-dominated solution to another nearby solution by solving the modified Tchebychev program according to the weighted vector which is obtained from some non-dominated vectors. The technique of additional constraints, firstly developed by Klein and Hannan [4] and also used by Sylva and Crema [5], consists in reducing progressively the admissible space and eliminating the non-dominated solution previously found.

Proposition 1 (**see [5]**). *Let $\hat{\mathbf{x}}^s$ be efficient solution to problem (P) and $D_s = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^n_+, C\mathbf{x} \leq C\hat{\mathbf{x}}^s\}$. Let $\hat{\mathbf{x}}^*$ be an efficient solution to the multiple objective integer problem "max"$\{C\mathbf{x}, \mathbf{x} \in D - D_s\}$. Then, $\hat{\mathbf{x}}^*$ is an efficient solution to problem (P).*

Proposition 2. *Let $\hat{\mathbf{z}} = C\hat{\mathbf{x}}$ be non-dominated solution to problem (P) and $D_s = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^n_+, C\mathbf{x} \leq C\hat{\mathbf{x}}^s\}$ and $\hat{\boldsymbol{\beta}}$ its weight vector defined as in formula* (2). *If $\bar{\mathbf{z}}$ is an optimal solution to problem $P_\rho(\boldsymbol{\beta})$ with $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$ such that*

$$P_\rho(\boldsymbol{\beta}) \begin{cases} \min\ \omega + \rho \displaystyle\sum_1^p (z_i^{\star\star} - z_i) \\ \omega \geq \beta_i(z_i^{\star\star} - z_i),\ 1 \leq i \leq p, \\ z_i = c^i x,\ 1 \leq i \leq p, \\ x \in D - D_s, \\ \omega \geq 0. \end{cases}$$

Then $\bar{\mathbf{z}}$ is non-dominated objective solution to problem P.

Proof. Let as suppose there exists an efficient solution $\mathbf{x}' \in D$ such that $C\bar{\mathbf{x}} \leq C\mathbf{x}'$ with at least one strict inequality. $\mathbf{x}'$ cannot belong to D_s, $C\mathbf{x}'$ is not dominated by $\hat{\mathbf{z}}$. However, for all $i, (z_i^{\star\star} - C^i\mathbf{x}') \leq (z_i^{\star\star} - C^i\bar{\mathbf{x}})$, since $\forall \mathbf{z} \in Z, \mathbf{z} < \mathbf{z}^{\star\star}$ then for all i $\hat{\beta}_i > 0$, thus

$$\hat{\beta}_i(z_i^{\star\star} - C^i\mathbf{x}') \leq \hat{\beta}_i(z_i^{\star\star} - C^i\bar{\mathbf{x}})\ \forall i, \tag{6}$$

we must have

$$\max_{i=1,\dots,p} \hat{\beta}_i(z_i^{\star\star} - C^i\mathbf{x}') \leq \max_{i=1,\dots,p} \hat{\beta}_i(z_i^{\star\star} - C^i\bar{\mathbf{x}}). \tag{7}$$

By the substitutions $\mathbf{z}' = C\mathbf{x}'$ and $\bar{\mathbf{z}} = C\bar{\mathbf{x}}$ in each of the p constraints of $P(\hat{\boldsymbol{\beta}})$, we get

$$\bar{\omega} \geq \hat{\beta}_i(z_i^{\star\star} - C^i\bar{\mathbf{x}})\ \forall i,$$
$$\omega' \geq \hat{\beta}_i(z_i^{\star\star} - C^i\mathbf{x}')\ \forall i.$$

According to the definition of the weighted Tchebychev norm and the inequality (7), we should have

$$\omega' \leq \bar{\omega}. \tag{8}$$

According to inequalities (6) and (8) we have

$$\omega' + \rho \sum_{1}^{p} (z_i^{\star\star} - C^i \mathbf{x}') \leq \bar{\omega} + \rho \sum_{1}^{p} (z_i^{\star\star} - C^i \bar{\mathbf{x}}). \tag{9}$$

Two cases are to be discussed: If $\omega' + \rho \sum_{1}^{p} (z_i^{\star\star} - C^i \mathbf{x}') < \bar{\omega} + \rho \sum_{1}^{p} (z_i^{\star\star} - C^i \bar{\mathbf{x}})$ then the optimality of $\bar{\mathbf{z}}$ is not preserved. Otherwise, contradiction with non efficiency of $\bar{\mathbf{x}}$. □

2. Algorithm

The algorithm we proposed here is proved to enumerate all non-dominated objective vectors for the problem MOILP. After having calculated the ideal objective vector $\mathbf{z}^\star$, we can set the utopian objective value as $\mathbf{z}^{\star\star} = \mathbf{z}^\star + 1$. The procedure starts with an initial non-dominated solution $\hat{\mathbf{z}}^0 = C\hat{\mathbf{x}}^0$ which can be calculated by solving the parametric problem defined as $P(\boldsymbol{\lambda}) \equiv \max\{\boldsymbol{\lambda} C\mathbf{x}, \mathbf{x} \in D\}$ for an arbitrary $\boldsymbol{\lambda} \in \Delta$. Consider $\boldsymbol{\beta}^0 = \boldsymbol{\lambda}$ as an initial weighted vector for the iterative procedure. At each iteration k, the weighted Tchebychev program $P(\boldsymbol{\beta})$ is solved in the reduced space $D - D_s$ such that $D_s = \{\mathbf{x}, \mathbf{x} \in \mathbb{Z}^n, C\mathbf{x} \leq C\hat{\mathbf{x}}^s, s = 1, \ldots, k-1\}$, let be $\boldsymbol{\beta} = \boldsymbol{\beta}^{k-1}$ such that $\boldsymbol{\beta}^{k-1}$ is a weighted vector of $\hat{\mathbf{z}}^{k-1} = C\hat{\mathbf{x}}^{k-1}$. If $P(\boldsymbol{\beta})$ is feasible then, according to proposition 2 a new efficient/non-dominated solution $(\hat{\mathbf{x}}^k, \hat{\mathbf{z}}^k)$ is obtained in the neighborhood of the last non-dominated solution found. The current associated weighted vector $\boldsymbol{\beta}^k$ can be calculated as in (2) and considered for the next iteration. Otherwise, the process ends with all non-dominated objective vectors.

Mathematically, the technique of additional constraints can be formulated as

$$D - \bigcup_{s=1}^{k} D_s = \left\{ \begin{array}{ll} c^i\mathbf{x} \geq (c^i\hat{\mathbf{x}}^s + 1)y_i^s + M_i(1 - y_i^s), & i = 1, 2, \ldots, p, \; s = 1, 2, \ldots, k, \\ \sum_{i=1}^{p} y_i^s \geq 1, & \\ y_i^s \in \{0, 1\}, & i = 1, 2, \ldots, p, \; s = 1, 2, \ldots, k, \\ \mathbf{x} \in D, & \end{array} \right\}$$

where M_i is a lower bound for any feasible value of the i^{th} objective function such that, if $c_j^i \geq 0$, $j = 1, \ldots, n$, $M_i = \min\{c^i\mathbf{x} \mid \mathbf{x} \in D\}$, otherwise $M_i = 0$. The variables y_i^s, $i = 1, \ldots, p$, associated to $\hat{\mathbf{x}}^s$ and additional constraints are added in order to impose an improvement on at least the objective function. Note that when $y_i^s = 0$, the associated constraint is redundant and when $y_i^s = 1$, a strict improvement is forced in the i^{th} objective function evaluated in $\hat{\mathbf{x}}^s$.

Proposition 3 (see [5]). *Let $\hat{\mathbf{x}}^1$, $\hat{\mathbf{x}}^2$, ..., $\hat{\mathbf{x}}^k$ be efficient solutions to problem MOILP and $D_s = \left\{\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}_+^n, C\mathbf{x} \leq C\hat{\mathbf{x}}^s\right\}$. Let $\hat{\mathbf{x}}^*$ be an efficient solution to the multiple objective integer problem $(P_k) \equiv \max\left\{C\mathbf{x}, \mathbf{x} \in D - \bigcup_{s=1}^{k} D_s\right\}$. Then, $\hat{\mathbf{x}}^*$ is an efficient solution to problem MOILP. Furthermore, if (P_k) is unfeasible then $\{C\hat{\mathbf{x}}^s\}_{s=1}^{k}$ is the entire set of non-dominated objective vectors for MOILP.*

Proposition 4. *Let $\hat{\mathbf{x}}^1$, $\hat{\mathbf{x}}^2$, ..., $\hat{\mathbf{x}}^k$ be efficient solutions to a problem (P) and $D_s = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^n_+, C\mathbf{x} \leq C\hat{\mathbf{x}}^s\}$. Let $\hat{\boldsymbol{\beta}}^k$ the weighted vector of $\hat{\mathbf{z}} = C\hat{\mathbf{x}}^k$ and $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}^k$ and for a small $\rho > 0$, if $P_\rho(\boldsymbol{\beta})$ such that*

$$P_\rho(\boldsymbol{\beta}) \begin{cases} \min\ \omega + \rho \sum_1^p (z_i^{\star\star} - z_i) \\ \omega \geq \beta_i(z_i^{\star\star} - c^i x),\ 1 \leq i \leq p, \\ z_i = c^i\mathbf{x},\ 1 \leq i \leq p, \\ \mathbf{x} \in D - \bigcup_{s=1}^k D_s, \\ \omega \geq 0, \end{cases}$$

is unfeasible then $\{C\hat{\mathbf{x}}\}_{s=1}^k$ is the entire set of non-dominated objective vectors for MOILP.

Proof. According to Proposition 2, for any weighted vector $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$ of non-dominated objective vector, the weighted Tchebychev program $P(\boldsymbol{\beta})$ admits one optimal solution which is non-dominated. If for some $\hat{\boldsymbol{\beta}}^k$, $P(\boldsymbol{\beta})$ is unfeasible, then $D \subseteq \bigcup_{s=1}^k D_s$, for any $\mathbf{x} \in D$ there exists $\mathbf{x}^s \in \bigcup_{s=1}^k D_s$ such that $C\mathbf{x} \leq C\mathbf{x}^s$, we must have that $C\mathbf{x} = C\mathbf{x}^s$ and $C\mathbf{x} \in \{C\mathbf{x}^s\}_{s=1}^k$ or $C\mathbf{x} \leq C\mathbf{x}^s$ with at least one strict inequality (and $C\mathbf{x}$ is a dominated vector). □

A Constrained Weighted Tchebychev Program for MOILP

Input

$\downarrow A_{(m\times n)}$: matrix of constraints;
$\downarrow \mathbf{b}_{(m\times 1)}$: RHS(Right Hand Side) vector;
$\downarrow C_{(p\times n)}$: matrix of criteria;

Output

$\uparrow Z(P)$: The entire of non-dominated vector solution.
$\uparrow E(P)$: Subset of efficient solutions.

Initialization

- Let $z_i^{\star\star} = z_i^\star + 1$ the utopian vector such that $z_i^\star = \max\{c^i\mathbf{x}, \mathbf{x} \in D, i = 1, \ldots, p\}$ and for the lower bounds where $\forall i = 1, \ldots, p$ $M_i = \min\{c^i\mathbf{x} \mid \mathbf{x} \in D\}$, if $c_j^i \geq 0$, $j = 1, \ldots, n$, else set $M_i = 0$.
- Let $(\hat{\mathbf{x}}^0, \hat{\mathbf{z}}^0)$ the efficient/non-dominated objective vector solution of $\max\left\{\sum_{i=1}^p \frac{1}{2} c^i\mathbf{x}, x \in D\right\}$.
- $k = 0$, compute $\hat{\boldsymbol{\beta}}^0$ of $\hat{\mathbf{z}}^0$ defined as in (2).
- Let $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}^0$, $E(P) = \{\hat{\mathbf{x}}^0\}$ and $Z(P) = \{\hat{\mathbf{z}}^0\}$.

Repeat

- Solve $P_\rho(\boldsymbol{\beta})$ in $D - \bigcup_{s=1}^k D_s$ where $D_s = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^n_+, C\mathbf{x} \leq C\hat{\mathbf{x}}^s\}$ and $\hat{\mathbf{x}}^s \in E(P)$.
  - If $P_\rho(\boldsymbol{\beta})$ is unfeasible, Stop.
  - Otherwise, let be $(\hat{\mathbf{x}}^k, \hat{\mathbf{z}}^k)$ its optimal solution and compute $\hat{\boldsymbol{\beta}}^k$ of $\hat{\mathbf{z}}^k$.
- Set $E(P) = E(P) \bigcup \{\hat{\mathbf{x}}^k\}$, $Z(P) = Z(P) \bigcup \{\hat{\mathbf{z}}^k\}$, $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}^k$ and $k = k + 1$.

Until $P_\rho(\boldsymbol{\beta})$ is unfeasible.

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3. Numerical example

Let us consider the MOILP problem

$$\begin{cases} {}''\max'' \ (x_1 + x_2, x_1 - x_2), \\ D = \begin{cases} 3x_1 + x_2 \leq 5, \\ x_1, x_2 \geq 0 \text{ and integer.} \end{cases} \end{cases} \tag{10}$$

This example contains five efficient solutions/non-dominated objective vectors where $(4; -4)$ is unsupported (see Fig. 1). The parameter ρ has been fixed at 0.002.

Initialization step. Firstly we compute the ideal vector $\mathbf{z}^\star$ and the utopian vector $\mathbf{z}^{\star\star}$ such that $z_1^\star = \max\{(1,1)\mathbf{x}, \mathbf{x} \in D\}$, $z_2^\star = \max\{(1,-1)\mathbf{x}, \mathbf{x} \in D\}$, then we obtained $\mathbf{z}^\star = (5; 1)$ and $\mathbf{z}^{\star\star} = (6; 2)$. Secondly we calculate the lower bounds of the corresponding objective functions $M_1 = 0$, $M_2 = 5$. To obtain an initial non-dominated solution, we can take the optimal solution of parametric problem $P\left(\frac{1}{2}; \frac{1}{2}\right) \equiv \max\left\{\frac{1}{2}c^1\mathbf{x} + \frac{1}{2}c^2\mathbf{x}, 3x_1 + x_2 \leq 5, x_1, x_2 \geq 0 \text{ and integer}\right\}$ which is $(\hat{\mathbf{x}}^0, \hat{\mathbf{z}}^0) = ((1; 2), (3; -1))$ and $\hat{\boldsymbol{\beta}}^0 = \left(\frac{1}{2}; \frac{1}{2}\right)$. Let $E(P) = \{\hat{\mathbf{x}}^0\}$, $Z(P) = \{\hat{\mathbf{z}}^0\}$ and $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}} = \left(\frac{1}{2}; \frac{1}{2}\right)$.

Iteration 1. Solve $P_\rho(\boldsymbol{\beta})$ in the reduced space by adding the additional constraints in order to eliminate the point $(\hat{\mathbf{x}}^1, \hat{\mathbf{z}}^1)$

$$P_\rho(\boldsymbol{\beta}^\star) \begin{cases} \min \ \omega - 0.004x_1 \\ \omega \geq \frac{1}{2}(6 - x_1 - x_2), \\ \omega \geq \frac{1}{2}(2 - x_1 + x_2), \\ \omega \geq 0, \\ D - D_1 \begin{cases} 3x_1 + x_2 \leq 5, \\ x_1, x_2 \geq 0, \\ x_1 + x_2 \geq (3+1)y_1^1, \\ x_1 - x_2 \geq (-1+1)y_2^1 - 5(1 - y_2^1), \\ y_1^1 + y_2^1 \geq 1, \\ y_1^1, y_2^1 \in \{0, 1\}, \end{cases} \end{cases}$$

$(\hat{\mathbf{x}}^1, \hat{\mathbf{z}}^1) = \{(1; 1), (2; 0)\}$ and $y_1^1 = 0$, $y_2^1 = 1$ is obtained as an optimal solution, the relative constraint to y_1^1 is redundant, the current space of new solution is only constrained by $x_1 - x_2 \geq 0$ (see Fig. 1). We compute $\hat{\boldsymbol{\beta}}^1$ of $\hat{\mathbf{z}}^1$ as defined as in formula (2), we must have $\hat{\boldsymbol{\beta}}^1 = \left(\frac{1}{3}, \frac{2}{3}\right)$. We let $E = E \bigcup \hat{\mathbf{x}}^1$ $Z(P) = Z(P) \bigcup \hat{\mathbf{z}}^1$ and $\boldsymbol{\beta}^\star = \left(\frac{1}{3}, \frac{2}{3}\right)$.

Iteration 2. In this iteration the problem $P_\rho(\boldsymbol{\beta}^\star)$ is solved in the space that reduced by two solutions (non-dominated vectors) previously found

$$P_\rho(\boldsymbol{\beta}^\star)\begin{cases} \min\ \omega - 0.004x_1 \\ \omega \geq \frac{1}{3}(2-x_1-x_2), \\ \omega \geq \frac{2}{3}(0-x_1+x_2), \\ \omega \geq 0, \\ D-D_2 \begin{cases} \mathbf{x}, y_j^1 \in D-D_1,\ j=1,2, \\ x_1+x_2 \geq (2+1)y_1^2, \\ x_1-x_2 \geq (0+1)y_2^2 - 5(1-y_2^2), \\ y_1^2+y_2^2 \geq 1, \\ y_j^i \in \{0,1\},\ i,j=1,2. \end{cases} \end{cases}$$

As showed in Fig. 2, $(\hat{\mathbf{x}}^2, \hat{\mathbf{z}}^2) = \{(1;0),(1;1)\}$ is a new efficient/non-dominated solution with secondary variables $y_1^1 = 0$, $y_2^1 = 1$, $y_1^2 = 0$, $y_2^2 = 1$, the constraints associated to y_1^1,

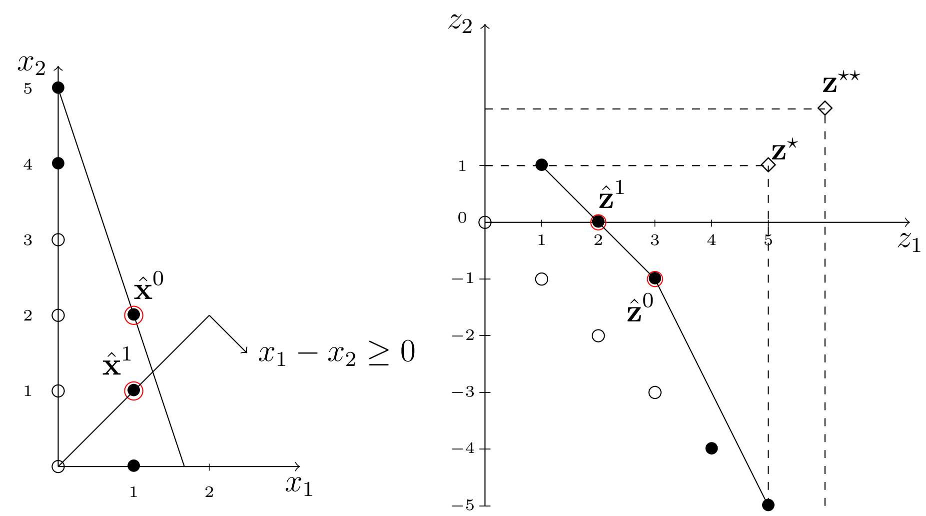

Fig. 1. Iteration 1

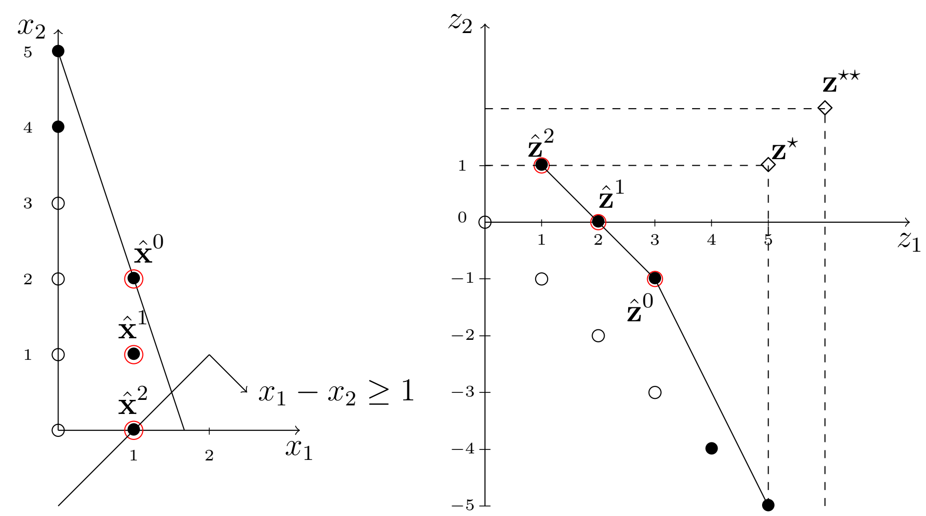

Fig. 2. Iteration 2

y_2^1, y_1^2 are redundant; according to the formula (2), the weighted vector $\hat{\boldsymbol{\beta}}^2$ is $\left(\frac{1}{6}, \frac{5}{6}\right)$. Set $E = E \bigcup \hat{\mathbf{x}}^2$ $Z(P) = Z(P) \bigcup \hat{\mathbf{z}}^2$ and $\boldsymbol{\beta}^\star = \left(\frac{1}{6}, \frac{5}{6}\right)$.

Iteration 3. The following step adds constraints that delete the efficient points previously found

$$P_\rho(\boldsymbol{\beta}^\star)\begin{cases} \min\ \omega - 0.004x_1 \\ \omega \geq \frac{1}{6}(1 - x_1 - x_2), \\ \omega \geq \frac{5}{6}(1 - x_1 + x_2), \\ \omega \geq 0, \\ D - D_3 \begin{cases} \mathbf{x}, y_j^i \in D - D_2,\ i = 1, 2,\ j = 1, 2, \\ x_1 + x_2 \geq (1+1)y_1^3, \\ x_1 - x_2 \geq (1+1)y_2^3 - 5(1 - y_2^3), \\ y_1^3 + y_2^3 \geq 1, \\ y_1^3, y_2^3 \in \{0, 1\}. \end{cases} \end{cases}$$

The optimal solution to the last problem is $(\hat{\mathbf{x}}^3, \hat{\mathbf{z}}^3) = \{(0; 4), (4; -4)\}$ and $y_1^1 = 1$, $y_2^1 = 0$, $y_1^2 = 1$, $y_2^2 = 0$ $y_1^3 = 1$, $y_2^3 = 0$. The additional constraints to y_2^1, y_1^2, y_1^3, y_2^3, y_2^2 are redundant. This solution is unsupported and gotten relatively to the constraint $x_1 + x_2 \geq 4$ according to the variable y_1^1, see Fig. 3, the current weighted vector is $\hat{\boldsymbol{\beta}}^3 = \left(\frac{3}{4}, \frac{1}{4}\right)$. Let $E = E \bigcup \hat{\mathbf{x}}^3$ $Z(P) = Z(P) \bigcup \hat{\mathbf{z}}^3$ and $\boldsymbol{\beta}^\star = \left(\frac{3}{4}, \frac{1}{4}\right)$.

Iteration 4. Now, problem $P_\rho(\boldsymbol{\beta}^\star)$ is defined as:

$$P(\boldsymbol{\beta}^\star)\begin{cases} \min\ \omega - 0.004x_1 \\ \omega \geq \frac{3}{4}(4 - x_1 - x_2), \\ \omega \geq \frac{2}{7}(-4 - x_1 + x_2), \\ \omega \geq 0, \\ D - D_4 \begin{cases} \mathbf{x}, y_j^i \in D - D_3,\ i = 1, 2, 3,\ j = 1, 2, \\ x_1 + x_2 \geq (4+1)y^4_, \\ x_1 - x_2 \geq (-4+1)y_2^4 - 5(1 - y_2^4), \\ y_1^4 + y_2^4 \geq 1, \\ y_1^4, y_2^4 \in \{0, 1\}, \end{cases} \end{cases}$$

$(\hat{\mathbf{x}}^4, \hat{\mathbf{z}}^4) = \{(0; 5), (5; -5)\}$, $y_1^1 = 1$, $y_2^1 = 0$, $y_1^2 = 1$, $y_2^2 = 0$, $y_1^3 = 1$, $y_2^3 = 0$ and $y_1^4 = 1$, $y_2^4 = 0$ is the optimal solution to the current Tchebychev program. This optimal solution is obtained from the only constraint $x_1 + x_2 \geq 5$ imposed by the first objective function relatively to the additional variable y_1^4, see Fig. 4; for this iteration, we obtain $\hat{\boldsymbol{\beta}}^4 = \left(\frac{7}{8}, \frac{1}{8}\right)$. Set $E = E \bigcup \hat{\mathbf{x}}^4$ $Z(P) = Z(P) \bigcup \hat{\mathbf{z}}^4$ and $\boldsymbol{\beta}^\star = \left(\frac{7}{8}, \frac{1}{8}\right)$.

Iteration 5. The next Tchebychev problem $P_\rho(\boldsymbol{\beta}^\star)$ to be solved is

$$P(\boldsymbol{\beta}^{\star})\begin{cases} \min\ \omega - 0.004x_1 \\ \omega \geq \dfrac{7}{8}(5 - x_1 - x_2), \\ \omega \geq \dfrac{1}{8}(-5 - x_1 + x_2), \\ \omega \geq 0, \\ D - D_5 \begin{cases} \mathbf{x}, y_j^i \in D - D_4,\ i = 1,2,3,4,\ j = 1,2, \\ x_1 - 2x_2 \geq (0+1)y_1^5, \\ -x_1 + 3x_2 \geq (1+1)y_2^5 - 5(1 - y_2^5), \\ y_1^5 + y_2^5 \geq 1, \\ y_1^5, y_2^5 \in \{0,1\}. \end{cases} \end{cases}$$

The current Tchebychev problem $P_\rho(\boldsymbol{\beta}^{\star})$ is unfeasible, then the process is complete and we have the complete set of non-dominated objective vectors, and the subset of the corresponding efficient solutions

$$E(P) = \{(1;2),(1;1),(1;0),(0;4),(0;5)\}, \quad Z(P) = \{(3;-1),(2;0),(1;1),(4;-4),(5;-5)\}.$$

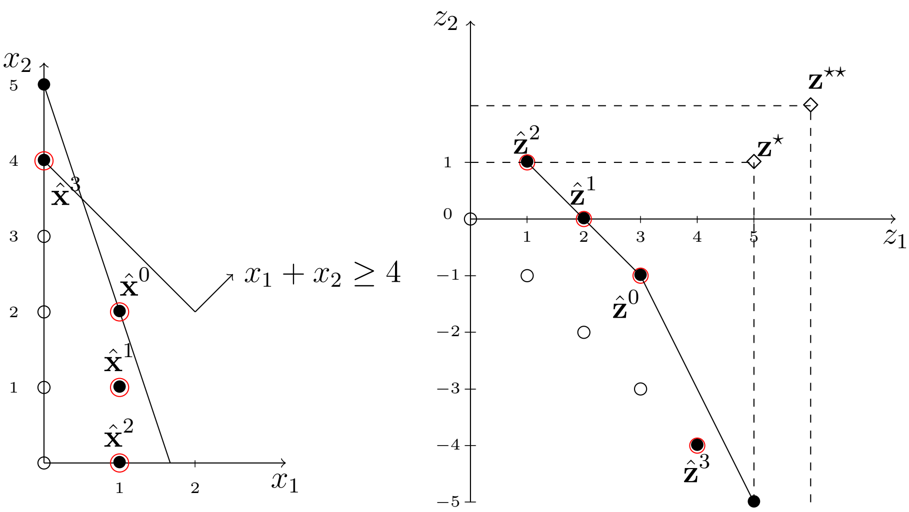

Fig. 3. Iteration 3

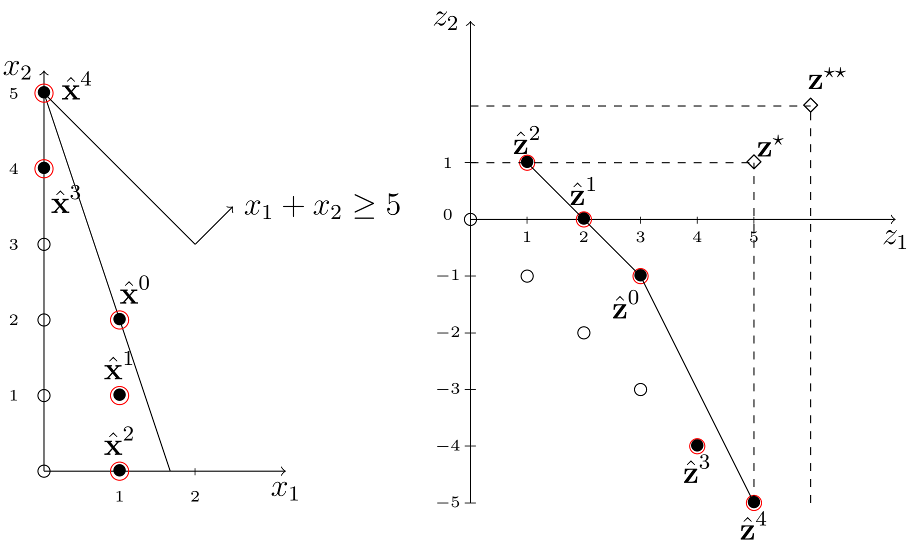

Fig. 4. Iteration 4

4. Computational results

The algorithm described above was implemented in the MATLAB environment and run on a PC (Intel Pentium dual-core 2.66 GHz processor), with 2 Gb of RAM, under Windows Vista. The CPLEX 12.9 library was the choice for solving scalar programs. The main feature of the algorithm lies in the resolution of the weighted Tchebychev program. In order to test the performance of the algorithm, only an overall look is necessary to see how both computing time and number of efficient solutions gone over which is equal to a number of iterations performed.

The algorithm was tested with MOILP problems randomly generated from discrete uniform distribution. The components of the matrices A, C and the vector b were drawn in the ranges $[1, 30]$, $[-20, 20]$ and $[100, 300]$, respectively.To avoid infeasibilities, all the constraints of each problem are of the $\leq$ kind. Furthermore, since all the coefficients of A and b are positives, the boundedness of the feasible region is assured. The number of objective functions p was taken as 2, 3 and 5. A total of 540 problems was grouped according to the number of variables, constraints and objective functions into 54 categories. For each category of problems, 10 instances were solved. The average CPU time (in seconds for $p = 1$, $p = 2$ and in minutes for $p = 5$) and the average number of iterations required, also, the minimum and maximum values of each measure are reported in brackets as shown in Table.

The results obtained show that the proposed algorithm is efficient in terms of the CPU time (in the average). Nevertheless, the mean time required by the algorithm is sometimes meaningful in some problems, perhaps its due to degeneracy especially in the presence of binary variables. Obviously, the difficulties encountered in the solution of the considered problem are closely related to their dimensions, moreover to their aspect multiple objective and discrete nature.

Computational results

m	n	$p = 2$		$p = 3$		$p = 5$	
		CPU, s	Iter	CPU, s	Iter	CPU, min.	Iter
5	10	1.35[0.62;3.44]	8.4[7;10]	8.08[4.19;20.61]	13.4[10;22]	1.82[0.58;3.53]	31.5[18;69]
	15	1.95[0.9;3.71]	16.8[6;27]	12.05[4.28;29.74]	13.7[8;29]	2.01[1.40;5.45]	49.6[33;78]
	20	2.52[1.06;4.04]	15.4[10;27]	9.81[5.01;19.54]	15.7[10;29]	2.54[1.33;5.58]	45.2[29;61]
10	10	1.36[0.44;2.4]	8.2[4;22]	10.02[6.71;31.91]	19.5[12;34]	3.72[2.18;5.05]	38.1[28;57]
	15	2.23[0.65;3.52]	9.5[6;17]	21.71[15.74;39.12]	20.1[11;27]	5.65[4.88;8.02]	40.1[31;71]
	20	1.67[0.99;1.84]	13.2[7;22]	8.52[5.09;24.01]	19.8[12;31]	7.34[3.41;12.56]	45[26;76]
20	20	1.75[1.00;3.90]	9.4[7;18]	4.98[2.02;12.5]	29.4[16;36]	6.83[3.78;11.51]	48.2[25;60]
	30	2.02[0.95;3.50]	10.5[8;17]	23.49[10.1;31.53]	31.5[16;49]	8.05[4.52;15.20]	47.3[30;79]
	50	2.22[1.23;3.02]	13.0[10;18]	65.62[6.9;217.7]	34.2[24;69]	9.35[3.97;20.4]	36.4[27;97]
30	30	2.02[0.8;2.22]	15.4[11;20]	37.22[24.8;58.52]	39.5[19;56]	8.95[5.04;18.27]	28.8[20;83]
	50	2.82[2.05;4.01]	14.5[11;19]	26.5[10.69;35.13]	37.7[23;46]	10.62[5.38;21.45]	31.5[16;79]
	100	5.20[2.58;6.83]	19.5[14;25]	239.8[24.4;1160.4]	50.2[22;61]	8.55[2.42;19.65]	26.6[12;76]
50	50	2.7[2.08;4.58]	14.2[11;16]	13.9[3.25;38.92]	29.5[20;53]	11.84[4.06;25.36]	40.7[31;82]
	100	6.79[4.50;8.13]	15.8[8;20]	371.53[8.1;2251.1]	38.1[24;63]	13.52[5.08;22.71]	52.5[23;77]
	200	14.37[5.46;19.3]	13.6[7;23]	788.4[67.2;1033.1]	41.1[10;69]	21.38[8.22;31.52]	61.2[26;91]
100	100	55.52[24.4;191.6]	14.6[11;25]	165.3[19.1;195.75]	31.8[15;52]	30.66[20.45;38.64]	69.5[19;88]
	150	64.3[19.3;197.17]	12.1[10;19]	314.8[101.9;1054.6]	37.5[27;56]	37.82[26.78;41.35]	63.9[25;79]
	200	51.2[12.7;443.6]	14.1[11;20]	326.3[90.3;1112.7]	35.6[31;58]	52.44[28.33;66.82]	57.7[30;71]

Conclusion

We have described an algorithm for multiple objective integer linear programs based on the weighted Tchebychev norm. The general idea of the technique used in the algorithm is the same as that of [5], the difference is based in the using of the weighted Tchebychev norm which defined the derivative problems.

The main idea is to combine the weighted Tchebychev program with the cuts idea of [5]. This technique not just the dominated solutions which are eliminated but even so weakly non-dominated solutions, knowing that this last can be generated by weighted Tchebychev program(see for example [19, 22]).

The procedure possesses the advantage of the displacement of a solution to another neighbor solution, it can be very useful in the interactive procedures, specially at early stages even though we don't match information on the DMs preferences.

The algorithm was coded using the MATLAB environment and The CPLEX 12.9 library to solve the mixed integer programs involving in the method. The algorithm is tested with several problems randomly generated from discrete uniform distribution and the results obtained are very encouraging. For future research, we suggest an updated survey for a comparison between several methods.

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ВЫЧИСЛИТЕЛЬНЫЕ ТЕХНОЛОГИИ

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Взвешенная программа Чебышева с ограничениями для многоцелевого целочисленного линейного программирования

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Аннотация

Представлен алгоритм перебора всех недоминируемых векторов в задаче многоцелевого целочисленного линейного программирования (MOLP). Начиная с начального недоминируемого вектора, на каждой итерации процедура определяет новое решение с использованием взвешенной чебышевской нормы. Постепенно добавляются дополнительные ограничения, чтобы уменьшить допустимое исследуемое множество.

Ключевые слова: многоцелевое целочисленное программирование, норма Чебышева, метод ветвей и границ.

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