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ON THE PRODUCT OF THE ULTRA-HYPERBOLIC OPERATOR RELATED TO THE ELASTIC WAVES

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В статье исследуется элементарное решение оператора произведения $\Box^k_{c_1}\Box^k_{c_2}$, где $\Box^k_{c_1}$ и $\Box^k_{c_2}$ — k-е степени ультрагиперболических операторов, определяемые равенствами

$$\Box^k_{c_1} = \left(\frac{1}{c_1^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k$$

и

$$\Box^k_{c_2} = \left(\frac{1}{c_2^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k,$$

где $(x_1, x_2, \ldots, x_n) \in R^n$, $p + q = n$, c_1 и c_2 — положительные константы, а k — неотрицательное целое число.

Показано, что элементарное решение операторов $\Box^k_{c_1}\Box^k_{c_2}$ связано с задачей об упругих волнах, зависящих от p, q, k, c_1 и c_2.

1. Introduction

We know from Trione [2, p. 11], that the generalized function $R^H_{2k}(x)$ defined by (2.1) is an elementary solution of the operator $\Box^k$, that is $\Box^k R^H_{2k} = \delta$ where $\Box^k$ is the ultra-hyperbolic operator iterated k-times, defined by

$$\Box^k = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \tag{1.1}$$

the point $x = (x_1, x_2, ..., x_n) \in R^n$ and δ is the Dirac-delta distribution.

In this paper, we developed the operator of (1.1) to be

$$\Box^k_{c_1} = \left(\frac{1}{c_1^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \tag{1.2}$$

and

$$\Box^k_{c_2} = \left(\frac{1}{c_2^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k. \tag{1.3}$$

We study the elementary solution of the equation

$$\Box_{c_1}^k \Box_{c_2}^k u(x) = \delta. \tag{1.4}$$

We can obtain

$$u(x) = S_{2k}^H(x) * T_{2k}^H(x) \tag{1.5}$$

as an elementary solution of (1.2) where the symbol * denote the convolution $S_{2k}^H(x)$ and $R_{2k}^H(x)$ are defined by (2.3) and (2.4) respectively with $\alpha = 2k$ and $x = (x_1, x_2, ..., x_n) \in R^n$. In particular if $k = 1$, $p = 1$ with $x_1 = t$(time), c_1 and c_2 are velocity then (1.3) becomes the elementary solution of the Elastic Waves of fourth order. Moreover,in the case of Elastic equilibium $\left(\frac{\partial u}{\partial t} = 0\right)$ we obtain the elementary solution of the equation $\triangle^{2k} u(x) = \delta$ where $\triangle^{2k}$ is the Laplace operator iterated $2k$ defined by

$$\triangle^{2k} = \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \cdots + \frac{\partial^2}{\partial x_q^2}\right)^{2k}, \tag{1.6}$$

where $x = (x_2, x_3, ..., x_q) \in R^{q-1}$.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean Space R^n. Denote by $v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$, $p + q = n$, the nondegenerated quadratic form. By Γ we designate the interior of the forward cone, $\Gamma_+ = \{x \in R^n : x_1 > 0 \text{ and } v > 0\}$ and by $\overline{\Gamma}_+$ designate its closure. For any complex number α define

$$R_\alpha^H(x) = \begin{cases} \dfrac{v^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma, \\ 0 \text{ for } x \notin \Gamma, \end{cases} \tag{2.1}$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2+\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}. \tag{2.2}$$

The function $R_\alpha^H(x)$ was introduced by Nozaki [3, p. 72]. It is well known that $R_\alpha^H(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geq n$ and is distribution of α if $\mathrm{Re}(\alpha) < n$.

Let $\mathrm{supp} R_\alpha^H(x) \subset \overline{\Gamma}_+$, where $\mathrm{supp} R_\alpha^H(x)$ denote the support of $R_\alpha^H(x)$.

From (2.1) we redefine

$$S_\alpha^H(x) = \begin{cases} \dfrac{V^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 \text{ for } x \notin \Gamma_+, \end{cases} \tag{2.3}$$

$$T_\alpha^H(x) = \begin{cases} \dfrac{W^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 \text{ for } x \notin \Gamma_+, \end{cases} \tag{2.4}$$

where $V = c_1^2(x_1^2 + x_2^2 + \cdots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$ and $W = c_2^2(x_1^2 + x_2^2 + \cdots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$, c_1 and c_2 are positive constants.

By putting $p = 1$ in (2.2), (2.3) and (2.4) and using the Legendre's duplication of $\Gamma(z)$.

$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2})$ then the formulae (2.3) and (2.4) reduced to

$$M_\alpha^H(x) = \begin{cases} \dfrac{V^{(\alpha-n)/2}}{H_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.5}$$

$$N_\alpha^H(x) = \begin{cases} \dfrac{W^{(\alpha-n)/2}}{H_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.6}$$

here $V = c_1^2x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, $W = c_2^2x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$ and $H_n(\alpha) = \pi^{(n-2)/2}2^{\alpha-1} \times \Gamma\left(\dfrac{\alpha-n+2}{2}\right)$, $M_\alpha^H(x)$ and $N_\alpha^H(x)$ are, precisely, the hyperbolic Kernel of Marcel Riesz.

Lemma 2.1. *Given the equations*

$$\Box_{c_1}^k u(x) = \delta, \tag{2.7}$$

and

$$\Box_{c_2}^k u(x) = \delta, \tag{2.8}$$

where $\Box_{c_1}^k$ *and* $\Box_{c_2}^k$ *are defined by (1.2) and (1.3) respectively,* $x = (x_1, x_2, ..., x_n) \in R^n$ *and* δ *is the Dirac-delta distribution. Then* $u(x) = S_{2k}^H(x)$ *and* $u(x) = T_{2k}^H(x)$ *are the elementary solution of (2.7) and (2.8) respectively, where* $S_{2k}^H(x)$ *and* $T_{2k}^H(x)$ *are defined by (2.3) and (2.4) respectively, with* $\alpha = 2k$.

Proof. See [2, p. 11].

Lemma 2.2. (*The convolution* $S_\alpha^H(x) * T_\alpha^H(x)$).

The function $S_\alpha^H(x)$ *and* $T_\alpha^H(x)$ *are tempered distributions. The convolution* $S_\alpha^H(x) * T_\alpha^H(x)$ *exists and also a tempered distribution.*

Proof. See [4].

3. Main Results

Theorem. *Given the equation*

$$\Box_{c_1}^k \Box_{c_2}^k u(x) = \delta, \tag{3.1}$$

where $\Box_{c_1}^k$ *and* $\Box_{c_2}^k$ *are defined by (1.2) and (1.3) respectively,* δ *is the Dirac-delta distribution,* $x = (x_1, x_2, ...x_n) \in R^n$. *Then*

$$u(x) = S_{2k}^H(x) * R_{2k}^H(x) \tag{3.2}$$

is an elementary solution of (3.1), where $S_{2k}^H(x)$ *and* $R_{2k}^H(x)$ *are defined by (2.1) and (2.3) respectively, with* $\alpha = 2k$. *Moreover, in particular if* $p = 1$ *with* $x_1 = t$, *and* $c_1 \neq c_2$ *then (3.2) becomes* $u(x) = M_{2k}^H(x) * N_{2k}^H(x)$ *is an elementary solution of Elastic Wave equation*

$$\left(\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2} - \sum_{i=2}^{n}\frac{\partial^2}{\partial x_i^2}\right)^k \left(\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2} - \sum_{i=2}^{n}\frac{\partial^2}{\partial x_i^2}\right)^k u(x) = \delta,$$

where $M_{2k}^H(x)$ *and* $N_{2k}^H(x)$ *are defined by (2.5) and (2.6) respectively. If elastic equilibrium* $\left(\frac{\partial u}{\partial t}=0\right)$ *then (3.1) becomes* $\triangle^{2k}u(x)=\delta$*, where* $\triangle^{2k}$ *is defined by (1.6) and we obtain* $u(x)=R_{4k}^e(x)$*, where* $x=(x_2,x_3,...,x_q)\in R^{q-1}$ *is an elementary solution of such equation and* $R_\alpha^e(x)$ *defined by*

$$R_\alpha^e(x)=\frac{|x|^{\alpha-n}}{W_n(\alpha)}, \tag{3.3}$$

where $|x|=(x_1^2+x_2^2+\cdots+x_n^2)^{1/2}$, $W_n(\alpha)=\dfrac{\pi^{n/2}2^\alpha\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$, *$\alpha$ is a complex parameter.*

Proof. Convolving both sides of (3.1) by $S_{2k}^H(x)$ we obtain

$$S_{2k}^H(x)*\Box_{c_1}^k\Box_{c_2}^k u(x)=S_{2k}^H(x)*\delta=S_{2k}^H(x),$$

$$\Box_{c_1}^k S_{2k}^H(x)*\Box_{c_2}^k u(x)=\delta*\Box_{c_2}^k u(x)=S_{2k}^H(x),$$

or $\Box_{c_2}^k u(x)=S_{2k}^H(x)$ by Lemma 2.1. Convolving both sides of the equation again by $T_{2k}^H(x)$ and Lemma 2.1 we obtain $u(x)=T_{2k}^H(x)*S_{2k}^H(x)$. Since $T_{2k}^H(x)*S_{2k}^H(x)=S_{2k}^H(x)*T_{2k}^H(x)$ exists by Lemma 2.2.

Thus $u(x)=S_{2k}^H(x)*T_{2k}^H(x)$ is an elementary solution of (3.1). In particular, if $p=1$ with $x_1=t$ and $c_1\neq c_2$ the function $S_\alpha^H(x)$ reduces to $M_\alpha^H(x)$ defined by (2.5) and $T_\alpha^H(x)$ reduces to $N_\alpha^H(x)$ defined by (2.6). Thus the equation (3.2) becomes $u(x)=M_{2k}^H(x)*N_{2k}^H(x)$ as the elementary solution of the Elastic Wave. Moreover if elastic equilibrium $\left(\frac{\partial u}{\partial t}=0\right)$ we obtain $u(x)=R_{4k}^e(x)$ is an elementary solution of the Laplace equation $\triangle^{2k}u(x)=\delta$, see [1, p. 31, Lemma 2.4], where $R_{4k}^e(x)$ is defined by (3.3) and $\triangle^{2k}$ is defined by (1.6).

References

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