ON THE MULTIPLICATIVE PRODUCT OF THE DIRAC-DELTA DISTRIBUTION ON THE HYPER-SURFACE

A. Kananthai
Department of Mathematics, Chiangmai University
Chiangmai 50200, Thailand

Определяется мультипликативное произведение распределений $\frac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}\frac{\delta(cr+s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}$, где δ — дельта-функция Дирака, $r=\sqrt{x_1^2+x_2^2+...+x_p^2}$, $s=\sqrt{x_{p+1}^2+x_{p+2}^2+...+x_{p+q}^2}$, а $p>1$ и $q>1$ $(p+q=n)$ — размерности в евклидовом пространстве R^n, $(x_1,x_2,...,x_p)\in R^p$, $(x_{p+1},x_{p+2},...,x_{p+q})\in R^q$, c — вещественное число. При некоторых ограничениях на p и n в таком мультипликативном произведении получена формула для функции Грина в квантовых теориях поля.

1. Introduction

Let $x=(x_1,x_2,...,x_n)$ be a point in the Euclidean space R^n and $P(x_1,x_2,...,x_n)$ be any sufficiently smooth function such that on $P=P(x_1,x_2,...,x_n)=0$ we have grad $P\neq 0$, that means there are no singular points on $P=0$. Then the generalized function $\delta^{(k-1)}(P)$ is defined in [1, p. 211] by

$$\langle\delta^{(k-1)}(P),\varphi\rangle=(-1)^{k-1}\int\Psi_{u_1}^{(k-1)}(0,u_2,u_3,...,u_n)du_2du_3...du_n. \tag{1.1}$$

We write $u_1=P$ and choose the remaining u_i coordinate (with $i=2,3,...,n$) arbitary except that the Jacobian of the x_i with respect to the u_i, which we shall denote by $D\binom{x}{u}$, fail to vanish (which is always possible so long as grad $P\neq 0$ on $P=0$). In (1.1), write

$$\Psi(u)=\Psi(u_1,u_2,...,u_n)=\varphi_1 D\left(\begin{array}{c}x\\u\end{array}\right),$$

$$\varphi_1(u_1,u_2,...,u_n)=\varphi(x_1,x_2,...,x_n).$$

The integral of (1.1) is taken over the $P=0$ surface, where $\varphi(x_1,x_2,...,x_n)$ is an infinitely differentiable function with bounded support.

Now, consider the nondegenerated quadratic form

$$u=c^2(x_1^2+x_2^2+...+x_p^2)-(x_{p+1}^2+x_{p+2}^2+...+x_{p+q}^2) \tag{1.2}$$

where $p > 1$ and $q > 1$ with $p+q=n$ is the dimension of the space R^n and c is a real number. Write $r=\sqrt{x_1^2+x_2^2+...+x_p^2}$ and $s=\sqrt{x_{p+1}^2+x_{p+2}^2+...+x_{p+q}^2}$ then (1.2) can be written

$$u=c^2r^2-s^2=(cr-s)(cr+s). \tag{1.3}$$

Taking $P=cr-s$ in (1.1), then (1.1) can be written in the form

$$\langle\delta^{(k-1)}(cr-s),\varphi\rangle=(-1)^{k-1}\int\left[\frac{\partial^{k-1}}{\partial P^{k-1}}\varphi(u_1,u_2,...u_n)D\left(\begin{array}{c}x\\u\end{array}\right)\right]_{p=0}du_2du_3...du_n. \tag{1.4}$$

In this paper we study the multiplicative product $\dfrac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}\ \dfrac{\delta(cr+s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}$ which is the generalization of the work of M. Aguirre Tellez [2].

2. The generalized functions $\frac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}$ and $\frac{\delta(cr+s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}$

From (1.2), let $y_1=cx_1$, $y_2=cx_2$, ..., $y_p=cx_p$ and the coordinate $x=(y_1,y_2,...,y_p,x_{p+1},x_{p+2},...,x_{p+q})\in R^n$ and $dx=dy_1dy_2...dy_pdx_{p+1}dx_{p+2}...dx_{p+q}=c^pdx_1dx_2...dx_pdx_{p+1}...dx_{p+q}, p+q=n$. We define

$$\langle\delta^{(k-1)}(cr-s),\varphi(x)\rangle=\int\limits_{P=0}\delta^{(k-1)}(cr-s)\varphi(x)dx=$$

$$=\int\limits_{P=0}\delta^{(k-1)}(cr-s)\varphi(x)dy_1...dy_pdx_{p+1}...dx_{p+q}=c^p\int\limits_{P=0}\delta^{(k-1)}(cr-s)\varphi(x)dx_1dx_2...dx_{p+q}. \tag{2.1}$$

Let us transform to bipolar coordinate defined by $x_1=r\omega_1$, $x_2=r\omega_2$, ..., $x_p=r\omega_p$ and $x_{p+1}=s\omega_{p+1}$, $x_{p+2}=s\omega_{p+2}$, ..., $x_{p+q}=r\omega_{p+q}$ where $r=\sqrt{x_1^2+x_2^2+...+x_p^2}$, $s=\sqrt{x_{p+1}^2+x_{p+2}^2+...+x_{p+q}^2}$. Thus

$$dx_1dx_2...dx_{p+q}=r^{p-1}s^{q-1}drdsd\Omega^{(p)}d\Omega^{(q)} \tag{2.2}$$

where $d\Omega^{(p)}$ and $d\Omega^{(q)}$ are the elements of surface area on the unit sphere in the space R^p and R^q respectively.

Choose the coordinates to be P,r and ω_i, then (2.2) becomes

$$dx_1dx_2...dx_{p+q}=(cr-p)^{q-1}r^{p-1}drdPd\Omega^{(p)}d\Omega^{(q)} \tag{2.3}$$

where $P=cr-s$. By (1.4) and (2.3), the equation (2.1) can be written in the form

$$\langle\delta^{(k-1)}(cr-s),\varphi(x)\rangle=c^p(-1)^{k-1}\int\left[\frac{\partial^{k-1}}{\partial P^{k-1}}\left\{(cr-P)^{q-1}\varphi(r,s)\right\}\right]_{cr=s}r^{p-1}drd\Omega^{(p)}d\Omega^{(q)}. \tag{2.4}$$

Since $P=cr-s$, then $\dfrac{\partial^{k-1}}{\partial P^{k-1}}=(-1)^{k-1}\dfrac{\partial^{k-1}}{\partial S^{k-1}}$.

Thus (1.8) can be written as

$$\langle\delta^{(k-1)}(cr-s),\varphi(x)\rangle=c^P\int\left[\frac{\partial^{k-1}}{\partial s^{k-1}}\left\{s^{q-1}\varphi(r,s)\right\}\right]_{cr=s}r^{(p-1)}drd\Omega^{(p)}d\Omega^{(q)}.$$

Let

$$\Psi(r,s)=\int_{\Omega}\varphi(r,s)d\Omega^{(p)}d\Omega^{(q)} \tag{2.5}$$

then

$$\langle\delta^{(k-1)}(cr-s),\varphi(x)\rangle=c^p\int_{r=0}^{\infty}\left[\frac{\partial^{k-1}}{\partial s^{k-1}}\left\{s^{q-1}\Psi(r,s)\right\}\right]_{cr=s}r^{p-1}dr.$$

For $k=1$, we have

$$\langle\delta(cr-s),\varphi(x)\rangle=c^p\int_{r=0}^{\infty}\left[\,s^{q-1}\Psi(r,s)\right]_{cr=s}r^{p-1}dr$$

or

$$\left\langle\frac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}},\varphi(x)\right\rangle=c^p\int_{r=0}^{\infty}\left[\,s^{\frac{q-1}{2}}\Psi(r,s)\right]_{cr=s}r^{\frac{p-1}{2}}dr. \tag{2.6}$$

Similarly, for $P=cr+s$ we obtain

$$\left\langle\frac{\delta(cr+s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}},\varphi(x)\right\rangle=c^p\int_{-\infty}^{0}\left[\,s^{\frac{q-1}{2}}\Psi(r,s)\right]_{cr=-s}r^{\frac{p-1}{2}}dr. \tag{2.7}$$

3. The generalized functions $(cr-s)_+^{\lambda}$ and $(cr+s)_-^{\lambda}$

We define

$$(cr-s)_+^{\lambda}=\begin{cases}(cr-s)^{\lambda}, \text{ for } cr\geq s,\\ 0, \text{ for } cr<s,\end{cases} \tag{3.1}$$

and

$$(cr+s)_-^{\lambda}=\begin{cases}(-(cr+s))^{\lambda}, \text{ for } cr+s<0,\\ 0, \text{ for } cr+\geq 0\end{cases} \tag{3.2}$$

where λ is a complex number.

The generalized function $(c-s)_+^{\lambda}$, where λ is a complex number, is defined by

$$\langle(cr-s)_+^{\lambda},\varphi(x)\rangle=\int_{cr-s\geq 0}(cr-s)^{\lambda}\varphi(x)dx.$$

Since

$$dx=dy_1dy_2...dy_pdx_{p+1}dx_{p+2}...dx_{p+q}=c^pdx_1dx_2...dx_pdx_{p+1}...dx_{p+q},$$

we have

$$\langle(cr-s)_+^{\lambda},\varphi\rangle=c^p\int_{cr-s\geq 0}(cr-s)^{\lambda}\varphi(x)r^{p-1}drs^{q-1}dsd\Omega^{(p)}d\Omega^{(q)}$$

or

$$\left\langle\frac{(cr-s)_+^{\lambda}}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}},\varphi\right\rangle=c^p\int_{r=0}^{\infty}\int_{s=0}^{cr}(cr-s)^{\lambda}\Psi(r,s)r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}dsdr$$

where $\Psi(r,s)$ is defined by (2.5).

Let $u=cr$ and $v=s$, then

$$\left\langle\frac{(cr-s)_+^{\lambda}}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}},\varphi\right\rangle=c^p\int_{r=0}^{\infty}\int_{s=0}^{u}(u-v)^{\lambda}\Psi\left(\frac{u}{c},v\right)\left(\frac{u}{c}\right)^{\frac{p-1}{2}}v^{\frac{q-1}{2}}dv\frac{du}{c}$$

put $v = ut$, we obtain

$$\left\langle \frac{(cr-s)_+^\lambda}{r^{\frac{p-1}{2}} s^{\frac{q-1}{2}}}, \varphi \right\rangle = c^p \int_0^\infty \int_0^1 (u-ut)^\lambda \Psi\left(\frac{u}{c}, ut\right) \frac{u^{\frac{p-1}{2}}}{c^{\frac{p-1}{2}}} u^{\frac{q-1}{2}} t^{\frac{q-1}{2}} u dt \frac{du}{c} =$$

$$= c^{\frac{p-1}{2}} \int_0^\infty \int_0^1 u^\lambda (1-t)^\lambda \Psi\left(\frac{u}{c}, ut\right) u^{\frac{p+q}{2}} t^{\frac{q-1}{2}} dt du =$$

$$= c^{\frac{p-1}{2}} \int_0^\infty u^{\lambda+\frac{n}{2}} du \int_0^1 (1-t)^\lambda t^{\frac{q-1}{2}} \Psi\left(\frac{u}{c}, ut\right) dt = c^{\frac{p-1}{2}} \int_0^\infty u^{\lambda+\frac{n}{2}} G(\lambda,u) du \tag{3.3}$$

where

$$G(\lambda,u) = \int_0^1 (1-t)^\lambda t^{\frac{q-1}{2}} \Psi(\frac{u}{c}, ut) dt.$$

Now $G(\lambda,u)$ have poles at $\lambda = -k (k = 1, 2, ...)$, by I. M. Gelfand and G. E. Shilov [1, p. 49],

$$\operatorname*{Res}_{\substack{\lambda=-k \\ k=1,2,\ldots}} \langle x_+^\lambda, \varphi \rangle = \frac{\varphi^{(k-1)}(0)}{(k-1)!}.$$

Thus

$$\operatorname*{Res}_{\lambda=-k} G(\lambda,u) = \frac{(-1)^{k-1}}{(k-1)!} \left[\frac{\partial^{k-1}}{\partial t^{k-1}} \left\{ t^{\frac{q-1}{2}} \Psi(\frac{u}{c}, ut) \right\} \right]_{t=1}. \tag{3.4}$$

Since $G(\lambda,u)$ have poles at $\lambda = -k$, we write

$$G(\lambda,u) = \frac{G_0(u)}{\lambda+k} + G_1(\lambda,u)$$

in the neighborhood of $\lambda = -k$ where

$$G_0(u) = \operatorname*{Res}_{\lambda=-k} G(\lambda,u) \tag{3.5}$$

and $G_1(\lambda,u)$ is reguler at $\lambda = -k$.

Thus (3.3) can be written in the form

$$\left\langle \frac{(cr-s)_+^\lambda}{r^{\frac{p-1}{2}} s^{\frac{q-1}{2}}}, \varphi \right\rangle = c^{\frac{p-1}{2}} \int_0^\infty u^{\lambda+n/2} \left[\frac{G_0(u)}{\lambda+k} + G_1(\lambda,u) \right] du =$$

$$= \frac{c^{\frac{p-1}{2}}}{\lambda+k} \int_0^\infty u^{\lambda+n/2} G_0(u) du + c^{\frac{p-1}{2}} \int_0^\infty G_1(\lambda,u) du.$$

Thus $\operatorname*{Res}_{\lambda=-k} \left\langle \frac{(cr-s)_+^\lambda}{r^{\frac{p-1}{2}} s^{\frac{q-1}{2}}}, \varphi \right\rangle = c^{\frac{p-1}{2}} \int_0^\infty u^{-k+n/2} G_0(u) du.$

By (3.4) and (3.5) we obtain

$$\operatorname*{Res}_{\lambda=-k} \left\langle \frac{(cr-s)_+^\lambda}{r^{\frac{p-1}{2}} s^{\frac{q-1}{2}}}, \varphi \right\rangle = c^{\frac{p-1}{2}} \int_0^\infty u^{-k+n/2} \frac{(-1)^{k-1}}{(k-1)!} \left[\frac{\partial^{k-1}}{\partial t^{k-1}} \left\{ t^{\frac{q-1}{2}} \Psi\left(\frac{u}{c}, ut\right) \right\} \right]_{t=1} du.$$

Let $s = ut$. Then $\frac{\partial}{\partial t} = u \frac{\partial}{\partial s}$.

Thus, we have

$$\operatorname*{Res}_{\lambda=-k}\left\langle \frac{(cr-s)_+^\lambda}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}},\varphi\right\rangle = \frac{c^{\frac{p-1}{2}}(-1)^{k-1}}{(k-1)!}\int_0^\infty u^{-k+n/2}u^{k-1}\left[\frac{\partial^{k-1}}{\partial s^{k-1}}\left\{s^{\frac{q-1}{2}}u^{\frac{1-q}{2}}\Psi\left(\frac{u}{c},s\right)\right\}\right]_{\substack{t=1\\ s=u}}du =$$

$$= \frac{(-1)^{k-1}c^{\frac{p-1}{2}}}{(k-1)!}\int_0^\infty \left[\frac{\partial^{k-1}}{\partial s^{k-1}}\left\{s^{\frac{q-1}{2}}\Psi\left(\frac{u}{c},s\right)\right\}\right]_{\substack{t=1\\ s=u}} u^{\frac{p-1}{2}}du =$$

$$= \frac{(-1)^{k-1}}{(k-1)!}c^{\frac{p-1}{2}}c^{\frac{p+1}{2}}\int_0^\infty \left[\frac{\partial^{k-1}}{\partial s^{k-1}}\left\{s^{\frac{q-1}{2}}\Psi(r,s)\right\}\right]_{s=cr} r^{\frac{p-1}{2}}dr$$

for $u = cr$. Putting $k = 1$. Thus

$$\operatorname*{Res}_{\lambda=-1}\left\langle \frac{(cr-s)_+^\lambda}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}},\varphi\right\rangle = c^p\int_0^\infty \left[s^{\frac{q-1}{2}}\Psi(r,s)\right]_{s=cr} r^{\frac{p-1}{2}}dr. \tag{3.6}$$

From (2.6) and (3.6) we obtain

$$\frac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}} = \operatorname*{Res}_{\lambda=-1}\frac{(cr-s)_+^\lambda}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}. \tag{3.7}$$

Now consider the generalized function $(cr+s)_-^\lambda$. We have

$$\langle (cr+s)_-^\lambda,\varphi\rangle = c^p\int_{cr+s<0}(-cr-s)^\lambda\varphi(r,s)r^{p-1}s^{q-1}dsdrd\Omega^{(p)}d\Omega^{(q)},$$

$$\left\langle \frac{(cr+s)_-^\lambda}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}},\varphi\right\rangle = c^p\int_{-\infty}^0\left[\int_0^{-cr}(-cr-s)^\lambda\Psi(r,s)s^{\frac{q-1}{2}}ds\right]r^{\frac{p-1}{2}}dr.$$

Let $u = cr$ and $v = s$. We have for $v = -ut$

$$\left\langle \frac{(cr+s)_-^\lambda}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}},\varphi\right\rangle = c^p\int_{-\infty}^0\left[\int_0^{-u}(-(u+v))^\lambda\Psi\left(\frac{u}{c},v\right)v^{\frac{q-1}{2}}dv\right]\left(\frac{u}{c}\right)^{\frac{p-1}{2}}d(\frac{u}{c}) =$$

$$= c^p\int_{-\infty}^0\left[\int_0^1(-(u-ut))^\lambda\Psi\left(\frac{u}{c},-ut\right)(-ut)^{\frac{q-1}{2}}(-u)dt\right]\frac{u^{\frac{p-1}{2}}}{c^{\frac{p-1}{2}}}d\left(\frac{u}{c}\right) =$$

$$= \frac{c^p}{(-c)^{\frac{p-1}{2}}c}\int_{-\infty}^o(-u)^{\lambda+n/2}du\int_0^1(1-t)^\lambda t^{\frac{q-1}{2}}\Psi\left(\frac{u}{c},-ut\right)dt =$$

$$= \frac{c^p}{(-c)^{\frac{p-1}{2}}c}\int_{-\infty}^o(-u)^{\lambda+n/2}G(\lambda,-u)du$$

where

$$G(\lambda,-u) = \int_0^1(1-t)^\lambda t^{\frac{q-1}{2}}\Psi\left(\frac{u}{c},-ut\right)dt.$$

Since $G(\lambda,-u)$ have poles at $\lambda = -j$ $(j = 1,2,...)$ we have

$$\operatorname*{Res}_{\lambda=-j}G(\lambda,u) = \frac{(-1)^{j-1}}{(j-1)!}\left[\frac{\partial^{j-1}}{\partial t^{j-1}}\left\{t^{\frac{q-1}{2}}\Psi\left(\frac{u}{c},-ut\right)\right\}\right]_{t=1}.$$

Now, in the neighborhood of $\lambda = -j$ we write

$$G(\lambda, -u) = \frac{1}{x+j} G_0(-u) + G_1(\lambda, -u)$$

where $G_0(-u) = \operatorname*{Res}_{\lambda=-j} G(\lambda, -u)$ and $G_1(\lambda, -u)$ is regular function.

The same as before, for $s = -ut$ and $\dfrac{\partial}{\partial t} = -u \dfrac{\partial}{\partial s}$

$$\operatorname*{Res}_{\lambda=-j} \left\langle \frac{(cr+s)_-^\lambda}{r^{\frac{p-1}{2}} s^{\frac{q-1}{2}}}, \varphi \right\rangle = \frac{c^p}{(-c)^{\frac{p-1}{2}} c} \int_{-\infty}^0 (-u)^{-j+\frac{n}{2}} \frac{(-1)^{j-1}}{(j-1)!} \left[\frac{\partial^{j-1}}{\partial t^{j-1}} \left\{ t^{\frac{q-1}{2}} \Psi\left(\frac{u}{c}, -ut\right) \right\} \right]_{t=1} du =$$

$$= \frac{(-1)^{j-1}}{(j-1)!} \frac{c^p}{(-c)^{\frac{p-1}{2}} c} (-c)^{\frac{p-1}{2}} c \int_{-\infty}^0 \left[\frac{\partial^{j-1}}{\partial s^{j-1}} \left\{ s^{\frac{q-1}{2}} \Psi\left(\frac{u}{c}, s\right) \right\} \right]_{s=-u} (-u)^{\frac{p-1}{2}} du =$$

$$= \frac{(-1)^{j-1}}{(j-1)!} c^p \int_{-\infty}^0 \frac{\partial^{j-1}}{\partial s^{j-1}} \left[s^{\frac{q-2}{2}} \Psi(r,s) \right] r^{\frac{p-1}{2}} dr.$$

Putting $j = 1$ we obtain

$$\operatorname*{Res}_{\lambda=-1} \left\langle \frac{(cr+s)_-^\lambda}{r^{\frac{p-1}{2}} s^{\frac{q-1}{2}}}, \varphi \right\rangle = c^p \int_{-\infty}^o \left[s^{\frac{q-1}{2}} \Psi(r,s) \right]_{s=-cr} r^{\frac{p-1}{2}} dr. \tag{3.8}$$

From (2.7) and (3.8) we obtain

$$\frac{\delta(cr+s)}{r^{\frac{p-1}{2}} s^{\frac{q-1}{2}}} = \operatorname*{Res}_{\lambda=-1} \frac{(cr+s)_-^\lambda}{r^{\frac{p-1}{2}} s^{\frac{q-1}{2}}}. \tag{3.9}$$

4. The generalized function $(c^2r^2 - s^2)_-^\lambda$

We have

$$\langle (c^2r^2 - s^2)_-^\lambda, \varphi \rangle = \int\limits_{c^2r^2 - s^2 \le 0} (-(c^2r^2 - s^2))^\lambda \varphi(x) dx =$$

$$= c^p \int\limits_{c^2r^2 - s^2 \le 0} (-(c^2r^2 - s^2))^\lambda \varphi(r,s) r^{p-1} dr d\Omega^{(p)} s^{q-1} ds d\Omega^{(q)} =$$

$$= c^p \int_{s=0}^\infty \int_{r=0}^{\frac{s}{c}} (s^2 - c^2r^2)^\lambda \Psi(r,s) r^{p-1} s^{q-1} dr\, ds$$

where $\Psi(r,s)$ is defined by (2.5).

Let $u = c^2r^2$ and $v = s^2$. Thus

$$\left\langle \frac{(c^2r^2 - s^2)_-^\lambda}{r^{p-1} s^{q-1}}, \varphi \right\rangle = \frac{1}{4} \frac{c^p}{c^2} \int_0^\infty \int_0^v (v-u)^\lambda \Psi_1(u,v) r^{-1} s^{-1} du\, dv =$$

$$= \frac{1}{4} c^{p-1} \int_0^\infty \int_0^v (v-u)^\lambda \Psi_1(u,v) u^{\frac{-1}{2}} v^{\frac{-1}{2}} du\, dv.$$

Let $u=vt$ we obtain

$$\left\langle \frac{(c^2r^2-s^2)^{\lambda}_{-}}{r^{p-1}s^{q-1}},\varphi\right\rangle=\frac{1}{4}c^{p-1}\int_0^{\infty}v^{\lambda}\int_0^1(1-t)^{\lambda}t^{\frac{-1}{2}}\Psi_1(vt,v)dt=\frac{1}{4}c^{p-1}\int_0^{\infty}v^{\lambda}\Phi(\lambda,v)dv$$

where

$$\Phi(\lambda,v)=\int_0^1(1-t)^{\lambda}t^{\frac{-1}{2}}\Psi(vt,v)dt. \tag{4.1}$$

Now $\Phi(\lambda,v)$ have poles at $\lambda=-j$ $(j=1,2,3,...)$, we write

$$\Phi(\lambda,v)=\frac{\Phi_0(v)}{\lambda+j}+\Phi_1(\lambda,v).$$

Thus $\operatorname*{Res}_{\lambda=-j}\Phi(\lambda,v)=\Phi_0(v)$ and $\Phi_1(\lambda,v)$ is regular.

We have

$$\left\langle \frac{(c^2r^2-s^2)^{\lambda}_{-}}{r^{p-1}s^{q-1}},\varphi\right\rangle=\frac{1}{4}c^{p-1}\int_0^{\infty}v^{\lambda}\left[\frac{\Phi_0(v)}{\lambda+j}+\Phi_1(\lambda,v)\right]dv. \tag{4.2}$$

We see that $\left\langle \frac{(c^2r^2-s^2)^{\lambda}_{-}}{r^{p-1}s^{q-1}},\varphi\right\rangle$ has a pole of order two at $\lambda=-j$ $(j=1,2,3,...)$.

In the neighborhood of such λ we expand $\frac{(c^2r^2-s^2)^{\lambda}_{-}}{r^{p-1}s^{q-1}}$ in the Laurent series

$$\frac{(c^2r^2-s^2)^{\lambda}_{-}}{r^{p-1}s^{q-1}}=\frac{A^j}{(\lambda+j)^2}+\frac{B^j}{\lambda+j}+... \tag{4.3}$$

From (4.2) and (4.3), we have

$$\lim_{\lambda\to-j}<(\lambda+j)^2\frac{(c^2r^2-s^2)^{\lambda}_{-}}{r^{p-1}s^{q-1}},\varphi>=<A^j,\varphi>=\frac{1}{4}c^{p-1}\lim_{\lambda\to-j}(\lambda+j)\int_0^{\infty}v^{\lambda}\Phi_0(v)dv=$$

$$=\frac{1}{4}c^{p-1}\operatorname*{Res}_{\lambda=-j}\int_0^{\infty}v^{\lambda}\Phi_0(v)dv=\frac{1}{4}c^{p-1}\frac{\Phi_0^{(j-1)}(0)}{(j-1)!}$$

by [1, p. 49]. Let $j=1$, we have $\langle A^1,\varphi\rangle=\frac{1}{4}c^{p-1}\Phi_0(0)$. By (4.1)

$$\Phi_0(0)=\operatorname*{Res}_{\lambda=-j}\Phi(\lambda,0)=\operatorname*{Res}_{\lambda=-j}\int_0^1(1-t)^{\lambda}t^{\frac{-1}{2}}\Psi_1(0,0)dt.$$

Since

$$\Psi_1(u,v)=\Psi(r,s)=\int_{\Omega}\varphi(r,s)d\Omega^{(p)}d\Omega^{(q)},$$

we have $\Psi_1(0,0)=\Psi(0,0)=\varphi(0)\Omega^{(p)}\Omega^{(q)}$.

Thus

$$\lim_{\lambda\to-1}<(\lambda+1)^2\frac{(c^2r^2-s^2)^{\lambda}_{-}}{r^{p-1}s^{q-1}},\varphi>=\frac{1}{4}c^{p-1}\varphi(0)\Omega^{(p)}\Omega^{(q)}\operatorname*{Res}_{\lambda=-1}\int_0^1(1-t)^{\lambda}t^{\frac{-1}{2}}dt=$$

$$=\frac{1}{4}c^{p-1}\varphi(0)\Omega^{(p)}\Omega^{(q)}\operatorname*{Res}_{\lambda=-1}\left(\frac{\Gamma(\lambda+1)\Gamma(\frac{1}{2})}{\Gamma(\lambda+1+\frac{1}{2})}\right).$$

We have

$$\operatorname*{Res}_{\lambda=-1}\left(\frac{\Gamma(\lambda+1)\Gamma(\frac{1}{2})}{\Gamma(\lambda+1+\frac{1}{2})}\right)=\operatorname*{Res}_{\lambda=-1}\left(\frac{\Gamma(\lambda+2)}{\lambda+1}\frac{\Gamma(\frac{1}{2})}{\Gamma(\lambda+1+\frac{1}{2})}\right)=$$

$$=\lim_{\lambda\to-1}\left[(\lambda+1)\frac{\Gamma(\lambda+2)\Gamma(\frac{1}{2})}{(\lambda+1)\Gamma(\lambda+1+\frac{1}{2})}\right]=\frac{\Gamma(1)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2})}=1.$$

Thus

$$\lim_{\lambda\to-1}\left\langle(\lambda+1)^2\frac{(c^2r^2-s^2)_-^\lambda}{r^{p-1}s^{q-1}},\varphi\right\rangle=\frac{1}{4}c^{p-1}\varphi(0)\Omega^{(p)}\Omega^{(q)}=\frac{1}{4}c^{p-1}\frac{2\pi^{p/2}}{\Gamma(\frac{p}{2})}\frac{2\pi^{q/2}}{\Gamma(\frac{q}{2})}\varphi(0)$$

where $\Omega^{(p)}$ and $\Omega^{(q)}$ are surfaces of unit sphere in R^p and R^q and equal to $\frac{2\pi^{p/2}}{\Gamma(\frac{p}{2})}$ and $\frac{2\pi^{q/2}}{\Gamma(\frac{q}{2})}$ respectively.

It follows that

$$\lim_{\lambda\to-1}\left\langle(\lambda+1)^2\frac{(c^2r^2-s^2)_-^\lambda}{r^{p-1}s^{q-1}},\varphi\right\rangle=\left\langle\frac{c^{p-1}}{4}\frac{(2\pi^{\frac{p}{2}})(2\pi^{\frac{q}{2}})\delta}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})},\varphi\right\rangle. \tag{4.4}$$

5. The multiplicative product of $\frac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}$ and $\frac{\delta(cr+s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}$

We define

$$(cr-s)_+^\lambda=\begin{cases}(cr-s)^\lambda, \text{ if } cr-s\geq 0,\\ 0, \text{ if } cr-s<0\end{cases}$$

and

$$(cr+s)_-^\lambda=\begin{cases}(-(cr+s))^\lambda, \text{ if } cr+s\leq 0,\\ 0, \text{ if } cr+s>0.\end{cases}$$

From the definition, it follows that

$$(cr-s)_+^\lambda(cr+s)_-^\lambda=(c^2r^2-s^2)_-^\lambda. \tag{5.1}$$

Theorem. *Given the nondegenerated quadratic form*

$$u=c^2(x_1^2+x_2^2+...+x_p^2)-(x_{p+1}^2+x_{p+2}^2+...+x_{p+q}^2)$$

where $p+q=n$ is the dimension of the space R^n and c is a real number. Write $r=\sqrt{x_1^2+x_2^2+...+x_p^2}$ and $s=\sqrt{x_{p+1}^2+x_{p+2}^2+...+x_{p+q}^2}$. Then the following formula is valid.

$$\frac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}\frac{\delta(cr+s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}=\frac{c^{p-1}\pi^{\frac{n}{2}}\delta(x)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$$

where δ is the Dirac-delta function, $x=(x_1,x_2,...,x_n)\in R^n$ and $p+q=n$.

Proof: From (3.7) and (3.9), we have by (5.1)

$$\frac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}\frac{\delta(cr+s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}=\operatorname*{Res}_{\lambda=-1}\frac{(cr-s)_+^\lambda}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}\operatorname*{Res}_{\lambda=-1}\frac{(cr+s)^\lambda}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}=$$

$$= \lim_{\lambda\to-1}(\lambda+1)\frac{(cr-s)_+^\lambda}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}\lim_{\lambda\to-1}(\lambda+1)\frac{(cr+s)^\lambda}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}} = \lim_{\lambda\to-1}\left[(\lambda+1)^2\frac{(c^2r^2-s^2)_-^\lambda}{r^{p-1}s^{q-1}}\right].$$

Thus by (4.4)

$$\left\langle \frac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}\ \frac{\delta(cr+s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}},\varphi\right\rangle = \lim_{\lambda\to-1}\left\langle(\lambda+1)^2\frac{(c^2r^2-s^2)_-^\lambda}{r^{p-1}s^{q-1}},\varphi\right\rangle = \left\langle\frac{c^{p-1}\pi^{\frac{n}{2}}\delta(x)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})},\varphi\right\rangle.$$

It follows that

$$\frac{\delta(cr-s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}}\ \frac{\delta(cr+s)}{r^{\frac{p-1}{2}}s^{\frac{q-1}{2}}} = c^{p-1}\frac{\pi^{\frac{n}{2}}\delta(x)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}.$$

for $p>1$ and $q>1$ with $p+q=n$.

In particular, with the same process as before, we obtain the formula in (see [2], p. 158, eq. (4.4)) $p=1$, $c=1$. Then it follows that $q=n-1$, $r=x_1$ and $s=\sqrt{x_2^2+x_3^2+...+x_{p+q}^2}$, that is

$$\frac{\delta(x_1-s)}{s^{\frac{n-2}{2}}}\ \frac{\delta(x_1+s)}{s^{\frac{n-2}{2}}} = \frac{1}{2}\frac{\pi^{\frac{n-1}{2}}\delta(x)}{\Gamma(\frac{n-1}{2})}. \tag{5.2}$$

If $n=4$, from (5.2) we obtain

$$\frac{\delta(x_1-s)}{s}\frac{\delta(x_1+s)}{s} = \frac{1}{2}\pi^{\frac{3}{2}}\frac{\delta(x)}{\Gamma(\frac{3}{2})} = \frac{\pi^{\frac{3}{2}}\delta(x)}{\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{1}{2}\pi^{\frac{3}{2}}\frac{\delta(x)}{\pi^{\frac{1}{2}}} = \pi\delta(x)$$

or

$$\frac{\delta(x_1-s)\delta(x_1+s)}{s^2} = \pi\delta(x) \tag{5.3}$$

where

$$s=\sqrt{x_2^2+x_3^2+x_4^2}.$$

The formula (5.3) is used for a perturbative calculation of Green function in quantum field theories (see [2], p. 160, and [3]).

References

[1] GELFAND I. M., SHILOV G. E. *Generalized Functions*, vol. 1, Academic Press, New York and London.

[2] TELLEZ M. AGUIRRE The multiplication product $\frac{\delta(x_0-|x|)}{|x|^{\frac{(n-2)}{2}}}\ \frac{\delta(x_0+|x|)}{|x|^{\frac{(n-2)}{2}}}$. *J. of Math. Chem.*, **22**, 1997, 149–160.

[3] BOLLINI C., ROCCA M. Convolution of ultradistribution. *Preprint*, La Plata, 1997, to appear.

Received for publication May 24, 1999