The millennium-problem of fluid mechanics — the solution of the Navier — Stokes equations

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The paper sets out from the inclusion of the analysis of existence and regularity of solutions of the Navier — Stokes equations in the seven unsolved problems of mathematics by the Clay Mathematics Institute in the USA in 2000. It then mentions d'Alembert's early paradoxical attempts to determine the aerodynamic drag and the insolvability of the Euler equations at the time when they were first published. After the discussion of the derivation of the Navier — Stokes equations early results as for example Helmholtz's vorticity theorems and Reynolds' approach to turbulent flows are introduced, followed by Prandtl's revolutionary boundary-layer theory and lifting-line theory. The next section sketches the rapid development of modern computing machines, enabling the introduction of numerical methods into fluid mechanics. Arrangement of computational grids and solution techniques are briefly discussed. The results of a recent international workshop on drag prediction and an example showing the use of numerical methods in aerodynamic design are used to demonstrate the state of the art. The summary concludes with a look on future problems.

Keywords: the millennium-problem, fluid mechanics, Navier — Stokes equations.

Introduction

It was in the year 2000 that the Clay Mathematics Institute in Cambridge (USA) selected the analysis of existence and regularity of solutions of the Navier — Stokes equations for three-dimensional incompressible flows, the millennium-problem of fluid mechanics, as one of the seven unsolved problems of mathematics [1]. One million US dollars were offered as prize money for the solution. Although more than ten years have passed in the meantime successful solution approaches did not become known as yet, a disappointing result, especially, if one realizes that a large body of literature pertaining to the subject exists. The understanding of the mathematical nature of the Navier — Stokes equations is still rather limited, as Charles L. Fefferman remarks in the official description of the problem [2]. But disappointments have always accompanied research in fluid mechanics. For example the former Secretary General of the Académie Francaise and member of the Académie des Sciences and also of the Preußische Akademie der Wissenschaften, Jean-Baptiste le Rond d'Alembert, writes 1752 in [3] in the translation of [4]:

"*I do not see then, I admit, how one can explain the resistance of fluids by theory in a satisfactory manner. It seems to me on the contrary that this theory, dealt with profound attention, gives, at least in most cases, resistance absolutely zero; a singular paradox which I leave to geometricians to explain.*"

The last part of the statement, the reference to the geometricians remains vague and indeed discouraging. It is not easy to see, how the geometricians could possibly take care of the paradoxical result, unless one wants to imply that a very large amount of numerical work is required for an accurate description of complex aerodynamic configurations. But most likely it was not this problem which d'Alembert meant, when he described his non-satisfying result of his carefully worked-out theory, yielding no resistance at all. Almost another century was necessary to find a suitable answer.

Fig. 1. Jean-Baptiste le Rond d'Alembert, (1718–1783), Secretary General of the Académie Francaise, member of the Académie des Sciences and of the Prussian Academy of Sciences, exchanged letters with Catharine II. the Great and retired on a pension of Frederick II. the Great of Prussia. The above remark can be found in J. le R. d'Alembert: "Essai d'une nouvelle théorie de la résistance des fluides", Paris, 1752, Opuscules mathematiques, Paris, 1768, V, 132–138.

Also Leonhard Euler offers certain skepticism only a short time later, after his successful formulation of the equations of motion for inviscid fluid flow in 1755, later named after him, he remarks in [5]:

"*I hope to reach the goal with some luck, so that the remaining difficulties are only of analytical but not of mechanical nature.*"

Euler's remark is best be understood if one remembers, that Johann Bernoulli was the first who applied the fundamental laws of mechanics to describe one-dimensional fluid motion, published in his Hydraulica in 1742 [6].

Fig. 2. Leonhard Euler, (1707–1783), Professor at the University in Saint Petersburg, was appointed 1741 by Frederick II. to the Royal Prussian Academy of Sciences in Berlin; returned to Saint Petersburg in 1766. Catharine II. the Great strongly supported him, even when he lost his sight completely in 1771. His equations of motion for inviscid flows were first published in Principe's géneraux du mouvement des fluides, Memoires de l'Acad. des Sciences de Berlin 11, 274–315, also in Opera Omnia, II 12. 54–91, 1755.

It was only thirteen years later that Euler had expanded Johann Bernoulli's new approach to describe fluid motion to general incompressible, unsteady three-dimensional flow, a gigantic step forward at that time. But also Euler's skepticism was justified: About 200 years had to pass, until finally solutions of the Euler equations could be constructed for the description of flows about aerodynamic configurations. The non-linearities appearing in them prohibited direct applications for a long time.

Equations of motion for viscous flows

Almost another seventy years went by, until the complete equations of motions for viscous flows, today called the Navier — Stokes equations, could be formulated. They were first published in 1823 by Claude-Louis-Marie-Henry Navier in [7]. The equations derived by him describe the conservation of mass and momentum for an infinitesimally small volume element of incompressible fluid in motion.

Fig. 3. Claude-Louis-Marie-Henry Navier (1785–1836), Professor of mechanics at the École des Ponts et Chaussées and later Professor of calculus and mechanics at the École Polytechnic, brought the theory of elasticity into a usable form; he is considered to be the founder of modern structural mechanics, his main contribution is the derivation of the Navier — Stokes equations, first published in Mémoire sur les lois du mouvement des fluides, Mémoires de léAcademie des Sciences 6, 389–416, 1823.

With the usual notation, v denoting the velocity, p the pressure, ρ the density, and η the viscosity, the equations read as follows:

$$\Delta \cdot v = 0,$$

$$\rho \left(\frac{\partial v}{\partial t} + (v \cdot \Delta) v \right) = -\Delta p + \eta \Delta^2 v + f.$$

The naming of the equations is not clear, if it is remembered that Adhémer Barré de Saint-Venant already in 1843 published the equations in the form given in [8], two years before the publication of Sir Georg Gabriel Stokes [9].

Fig. 4. Adhémar Jean Claude Barré de Saint-Venant (1797–1886), Professor of mathematics at the École des Ponts et Chaussées in Paris, member of the Académie des Sciences; introduced the vector calculus in France; in 1843 he published a correct derivation of the Navier — Stokes equations, two years prior to Stokes, in his Mémoire sur le équations générales de léequilibre et du mouvement des corps solides élastiques et des fluides in Journal de École Polytechnique 13, pp. 1–174, Cahier XX, 1831.

De Saint-Venant was also first to recognize in the derivation that the viscosity coefficient could replace the shear modulus and serve as a multiplicative factor of the velocity gradients. But still the above equations were not named after him. Perhaps the reason is, that Stokes provided solutions for two problems of the equations of motion simplified for very slow motion — that is the second term of the left-hand side could be left out — in [10] in 1851. In the so-called Stokes' first problem he analyzed a flow situation, which is generated, when a plane wall is suddenly accelerated. In dealing with his second problem he provided a solution describing the flow near an oscillating flat plate. These two problems could be solved with

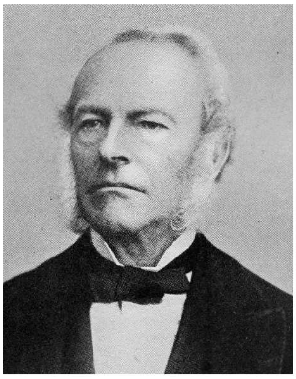

Fig. 5. Sir George Gabriel Stokes (1819–1903), Lucasian Professor of mathematics at Cambridge University, member, secretary, and president of the Royal Society, worked in pure mathematics, mathematical and experimental physics, his theoretical works were mainly in hydrodynamics. His derivation of the equations of motion was first published under the title "On the Theories of the Internal Friction of Fluids in Motion", Transactions of the Cambridge Philosophical Society, 8, 287–305, 1845.

the solution techniques available at that time. Two noteworthy results had already been obtained along these lines, but without the use of the Navier — Stokes equations: In 1839 Gotthilf Heinrich Ludwig Hagen in [11], and about the same time Jean Louis Marie Poiseuille in [12] published a simple relation describing the flow of water in a pipe. The results were obtained from experimental studies.

Early findings

Since the integration of the complete Navier — Stokes equations was not yet possible, further success concerning their solution could not be reported. However, already in 1858, 35 years after their derivation, Hermann von Helmholtz, at that time professor for physiology in Heidelberg, was able to derive a new relation from the equations of motion, which later became known as the vorticity transport equation. In his derivation he used the Euler equations — that is the equations given above without the last but one term. By introducing the definition of the notion of vorticity in [13] he obtained a vector equation with which the motion of vortices in inviscid fluid flows could be described.

Helmholtz was also able to derive three theorems, which characterize the behavior of vortex filaments in inviscid flows. As formulated in his original paper [13], they read in the translation:

1. A water particle, which does not rotate from the beginning on, cannot begin to rotate at a later time.

2. Water particles which belong to a vortex filament at an arbitrary time will always belong to that same filament, even when the particles are in motion.

3. The vortex filaments must therefore form closed loops in the fluid or can end only at its boundaries.

Fig. 6. Hermann von Helmholtz (1821–1894), Professor of pathology and physiology in Königsberg, Bonn, and Heidelberg; universal scholar, was ennobled 1883 and elected first president of the newly founded Physikalisch-Technische Reichsanstalt in 1888. His work on the vorticity transport equation appeared under the title "Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen", Celles J. 55, 25, 1858.

The vortex theorems played an important role in the developments that followed. The realization, that a vortex ring cannot be cut is a direct consequence of Helmholtz's third theorem. It also became the starting point in the development of the theory of lift. Needless to say that the vorticity transport equation could not be solved either because of the non-linearities of the terms describing the convective acceleration.

Another fundamental cornerstone of the nineteenth century was laid five years later in 1883 by Osborne Reynolds, professor of mechanics in Manchester [14]. He showed in an experiment that under certain conditions — today characterized by a dimensionless similarity parameter, called Reynolds number — that an originally plain laminated pipe flow would generate pressure and velocity fluctuations, later on termed laminar-turbulent transition, eventually turning into fully turbulent flow.

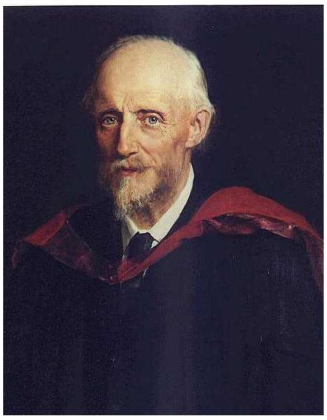

Fig. 7. Osborne Reynolds (1842–1912), Professor of civil and mechanical engineering at Owens College in Manchester, 1877 Fellow of the Royal Society; worked in fluid mechanics, electrical engineering, magnetism, and astrophysics; most important similarity parameter in fluid dynamics is named after Reynolds. His 1883 experiments were published in "An Experimental Investigation of the Circumstances whether the Motion of Water Shall Be Direct or Sinuous and of the Law of Resistance in Parallel Channels", Philosophical Transactions of the Royal Society of London, series A, 174, 1883, 935–982.

Reynolds also introduced the concept of turbulent fluctuations into the Navier — Stokes equations and postulated that after time-averaging the equations they could be used for describing turbulent flows. Since information is lost in the averaging process new unknown terms appeared, which nowadays are called Reynolds stresses. From the time of Reynolds' experiment on the construction of suitable closure relations is regarded as the central problem of turbulent flow research.

New theories

In the year 1904 Ludwig Prandtl presented a paper at the III. International Mathematics Congress in which he showed with an order-of-magnitude analysis that in fluids with small viscosity the frictional forces come into play only in the vicinity of solid boundaries [15].

In his derivation, which soon became known as boundary-layer theory, Prandtl was able to simplify the Navier — Stokes equations to the boundary-layer equations, which for special cases could be solved with the aid of the similar-solutions technique. The first of such solutions was given by Prandtl's student Hermann Blasius in 1908. He determined the skin friction of a flat plate at zero incidence in laminar flow with his solution of the boundary-layer equations [16].

Another fundamental theory followed ten years later in 1918, when Prandtl was able to formulate the lifting-line theory with the aid of a Helmholtz vortex filament, shaped into the form of a horseshoe [17]. This theory enabled aerodynamicists from then on to determine the lift and the induced drag of wings of finite span. The lifting-line theory became the fundament of all the later developments of wing theories that followed.

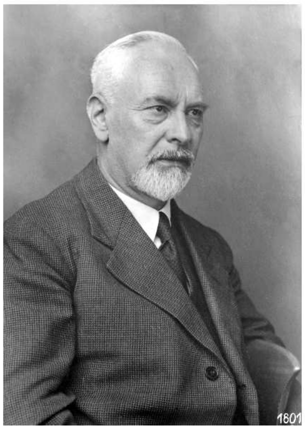

Fig. 8. Ludwig Prandtl, (1875–1953), Professor at the Universities of Hanover and Göttingen, Director of the Institute for Technical Physics, built first wind tunnel in 1909; President of the Aerodynamic Research Laboratory at Gottingen; developed the concept of boundary-layer theory, first published in “Über Fléssigkeitsbewegung bei sehr kleiner Reibung”, Verhandl. III. Intern. Math. Kongr. Heidelberg, 484–491, 1904; his fundamental work on wing theory was published in “Tragflügeltheorie I. u. II.” Mitt., Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl., 451–477, 1918; 107–137, 1919.

Prandtl also foresaw the development of modern computing. One of his former students, formerly professor of mathematics at Freiburg University, Henry Görtler reports in [18], that before and during World War II Prandtl developed a mechanical computing machine, he had dreamed of to solve the initial-two-point boundary-value problem of the boundary-layer theory with. The machine never worked, and only a design drawing dating back to the year 1941, just a short time before the advent of the electronic computing machines, was saved.

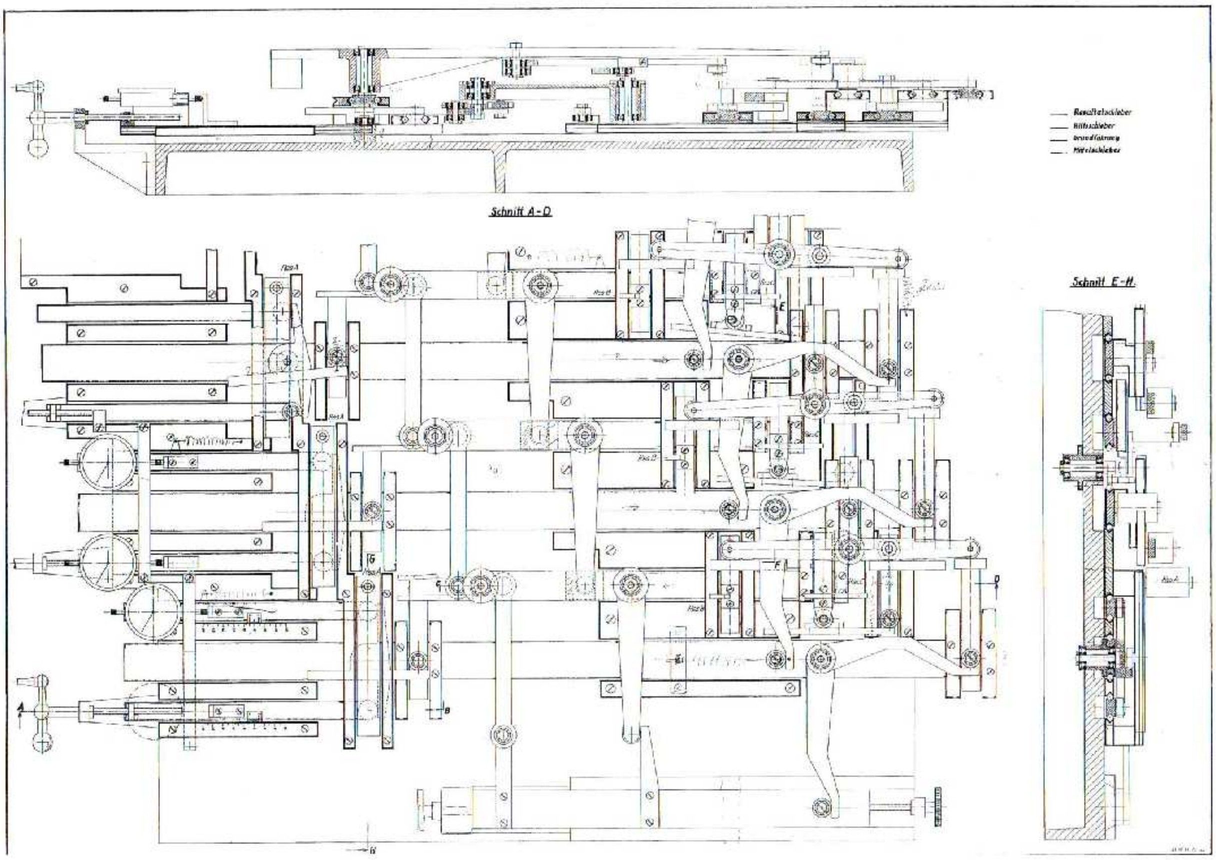

Fig. 9. Ludwig Prandtl’s unfulfilled dream: A mechanical computing machine, he thought he could solve the initial-two-point boundary-value problem of his boundary-layer theory with. The design draft shown dates back to 1941, only a few years before the advent of the electronic computing machines. The machine was never completed, as reported in [18], only the draft was saved.

The ascent of numerical solutions

Even in the twenties solutions of the Navier — Stokes equations could not be availed for the description of viscous flows. If flows with large Reynolds numbers were to be described, the flow field would be divided into an inviscid part and the boundary layer. The inviscid flow was then computed with the potential-flow theory and the flow in the boundary layer with new integral methods for the solution of Prandtl's boundary-layer equations, derived by Theodore von Kármán [19].

Since the wind-tunnel technique was substantially advanced by Prandtl and his students in Göttingen from the turn of the century on, extensive measurements of pressure distributions were used for flow investigations. This scenario prevailed until the sixties. Astonishingly enough this situation was confirmed by one of the fathers of informatics, John von Neumann, after whom the architecture of modern computing machines is named and whom we owe the scientification of modern computing. Von Neumann strongly shaped and pushed forward the unforeseen development of modern computers and computational techniques. He developed the first computer of the Institute for Advanced Studies in Princeton. In 1963 von Neumann remarked in [20]:

"*Thus, wind tunnels are, for example, used at present, at least in large part, as computing devices of the so-called analogy type (.....) to integrate the nonlinear partial differential equations of fluid dynamics.*"

This remark offered by one of the leading mathematicians of his time came as a surprise, in as much as von Neumann emphasized that wind tunnels were used as computing machines and not so much as instruments to determine flow characteristics with by measuring, for example, certain quantities, like pressure or velocities.

Fig. 10. John von Neumann (1903–1958), after appointments as university lecturer in Göttingen, Berlin, and Hamburg, followed an invitation to Princeton University in 1930, and joined the newly founded Institute for Advanced Studies in 1933. He is known as one of the fathers of informatics. The architecture of modern computing machines is named after him. The above remark may be found in an article entitled: "On the Principles of Large Scale Computing Machines", Collected Works, Pergamon Press, Oxford, 1–34, 1963, by J.H. Goldstine, and J. von Neumann.

The following fifty years witnessed a development of the information and computing technology that not only affected all branches of science and technology but also our entire life. For example in 1991 the US Congress promulgated the High Performance Computing and Communication Act, thereby proposing high-speed communications systems in order to enhance science and education in the USA. In [21] it is reported that the Japanese government in 2006 declared the supercomputer technology as one of the key technologies of national importance:

"*The Japanese government selected the supercomputing technology as one of the key technologies of national importance... and launched the Next Generation Supercomputer project in 2006. ... The system with 10 Petaflops class performance is planned to be completed in 2012. ... One of the goals of this project is to develop ... the grand challenge applications.*"

The aim of this project is to increase the computational performance of presently available machines so that solutions of the “Grand Challenges” can be provided. The Grand Challenges were formulated in the 1980 s in the USA as those fundamental problems in science and technology with many possibilities of applications, but with the expectation that their solutions can only be obtained in the near future with the aid of supercomputers. Turbulent flow research also belongs to these problems. The Nobel laureate Richard Feynman remarked already in the 1960 s: “*Turbulence is the most important unsolved problem in classical physics*”.

From the mid 1960 s on the rapid development of the computer technology enabled the construction of numerical solutions of the equations of motion in fluid dynamics. This requires the choice of a suitable mathematical model for the problem to be analyzed, as either the complete Navier — Stokes equations, the Euler equations, or Prandtl’s boundary-layer equations. The non-linear partial differential equations have to be discretized and replaced by algebraic equations, which can be solved on supercomputers. A computational grid has to be designed for the discretisation, which must be arranged in such a way, that the local flow structures can properly be resolved. Because of limited storage capacity this is not always possible. The last step of the numerical solution requires the presentation of the results.

In the discretisation of the equations of motion a certain condition has to be met, which guarantees that the domain of dependence of the differential equations is always included by the domain of dependence of the discretisation scheme chosen, or else the convergence of the numerical solution cannot always be ensured. It surprises one today that this condition, the domain-of-dependence condition, was already derived in 1928 by Richard Courant, Kurt Otto Friedrich, and Hans Lewy in Göttingen, published in [22], in a time, when numerical solutions of the Navier — Stokes equations were not in sight.

There are several ways in which the numerical solutions of the Navier — Stokes equations can be constructed. For example in the direct numerical simulation (DNS) the Navier —

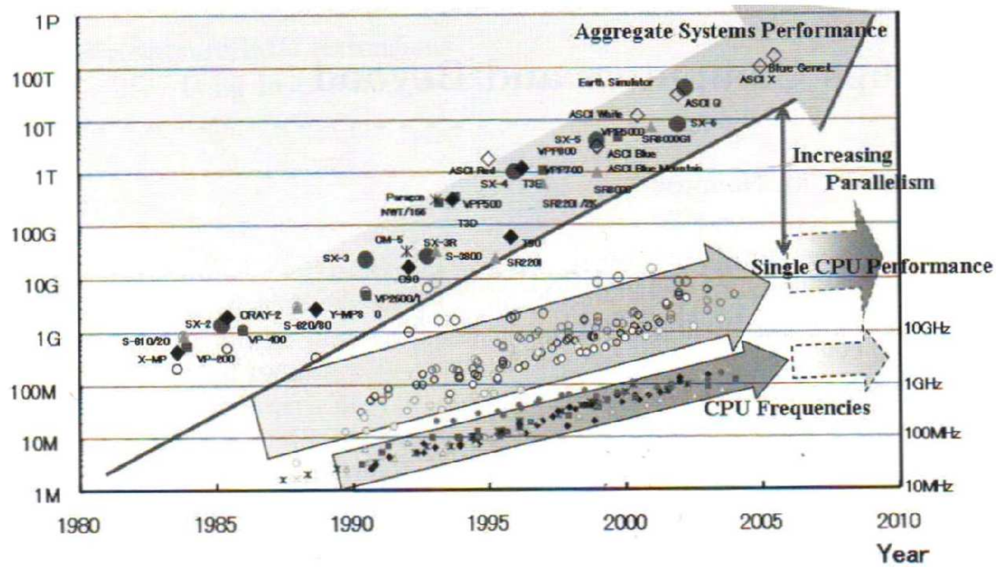

Fig. 11. Increase of the performance of supercomputers during the past 25 years. Data were published by T. Watanabe and M. Nomura in their article “Petaflops Computers and Beyond”, Notes on Numerical Fluid Mechanics and Multidisciplinary Design, 100, 481–490, 2009.

Stokes equations are discretized and solved on a grid, which guarantees that the local flow structures are adequately resolved. This method is suited for the simulation of flows characterized by relatively small Reynolds numbers. Another method of solution is the large-eddy-simulation technique (LES). In this approach the large vortex structures are directly computed, and the smaller vortex structures are described with modeled approximations. The large-eddy-simulation technique is used when flows with moderate Reynolds numbers are to be described. The third technique mentioned here consists in the solution of the Reynolds-averaged Navier — Stokes equations (RANS), as introduced by Reynolds in 1883. Prior to their solution these equations have to be closed with closure relations for the description of the Reynolds' stress tensor.

As for the numerical solutions there also exist several possibilities for the discretisation of the Navier — Stokes equations. If straight-forward finite differences are used, the derivatives in the differential equations are substituted by difference approximations, and the resulting difference equations are solved with explicit or implicit algorithms. In the finite-volume technique the integral forms of the conservation equations are discretized. A third method is the finite-element method: The flow field is subdivided into finitely large elements. In a second step shape functions are defined for the elements, which when inserted in the conservation equations supply a system of algebraic equations that can be solved on computers.

The computational grids are generated by discrete decomposition of the flow field into area or volume elements. The grids can be structured, following certain prescribed regularities, as for example Cartesian volume elements. Also unstructured grids are used: They are generated by prescribing the coordinates of the vertices of the elements. The unstructured grids offer good adaptation possibilities for the resolution of local flow structures.

The following Fig. 12 shows an example for the generation of a structured hybrid Cartesian grid for the computation of the flow around an aerodynamic shape. The grid is composed of rectangular cells, based on an octree-data structure, with an imbedded grid with triangular-prismatic cells for the computation of the flow in the boundary layer as described

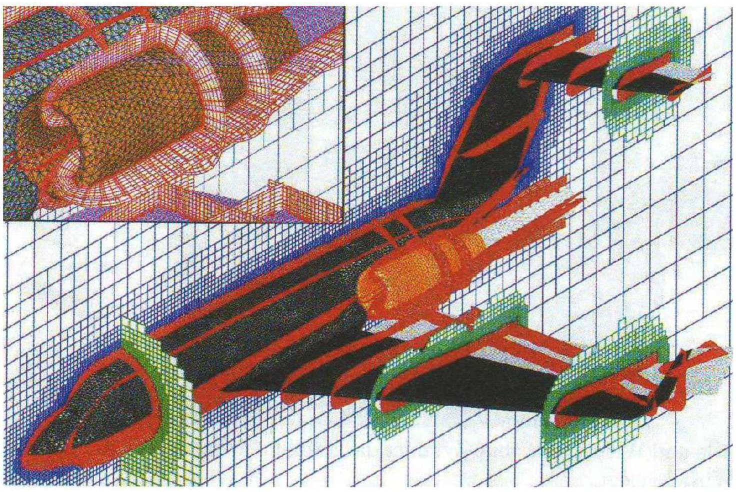

Fig. 12. Example for a generation of a structured hybrid Cartesian grid for the computation of the flow around an aerodynamic shape. The grid consists of rectangular cells of different size, generated with an octree-data structure. A second grid with triangular–prismatic cells is imbedded near the surface for the computation of the boundary-layer flow as described in [23]. A large computational effort is necessary for the generation of the grid.

in [23]. It is clear from Fig. 12 that a large effort is necessary for the grid generation. Perhaps it was this difficulty d'Alembert was hinting at in [3], that could not be overcome in the eighteenth century with no computing machines available.

Many technical flows are characterized by extraordinary large Reynolds numbers, as for example, flows about airplanes Re: $(O10^8)$. A direct simulation of such flows on supercomputers presently available is not possible, since according to [24] a computer performance of about 10^{23} flops would be required for the solution. The large-eddy simulations today allow investigations of modeled flow problems.

The Reynolds-number restriction was already discovered in 1963 by Jacob E. Fromm. He remarked in [25], that in his simulation of the flow about a rectangular block the solution would fail at a Reynolds number of $\mathrm{Re} = 6 \times 10^3$:

"*It is believed, however, that the Re* = 6000 *case is about as far as the calculational method can be extended in its current form, since here the instability discussed earlier tends to confuse the display patterns.*"

The instability of Fromm's numerical solution of the Navier — Stokes equations can also be recognized in a comparison with experimental flow visualization provided by A.M. Lippich in 1958, shown in Fig. 13.

Although actual flow conditions are still difficult to simulate on supercomputers, today flow computations, for example to determine lift and drag of airplane configurations can be carried out at much larger Reynolds numbers, see [27], than those chosen in [25] and [26]. Nevertheless it remains a laborious task for several reasons. One, of course, is the difficulty of correctly predicting the Reynolds stresses, with often little or no reliable information available. Turbulent flow research still has to depend on judicious assumptions. Another reason is the approximate nature of the numerical solutions due to their dependence on grid spacing and orders of approximation of the discretisation procedure chosen.

For these and other reasons the American Institute of Aeronautics and Astronautics organizes "Drag Prediction Workshops" in order to determine the accuracy of predictions of numerical solutions of the conservation equations, so far restricted to aerodynamic problems. The fourth workshop took place in 2009. Test computations were carried out for the transonic

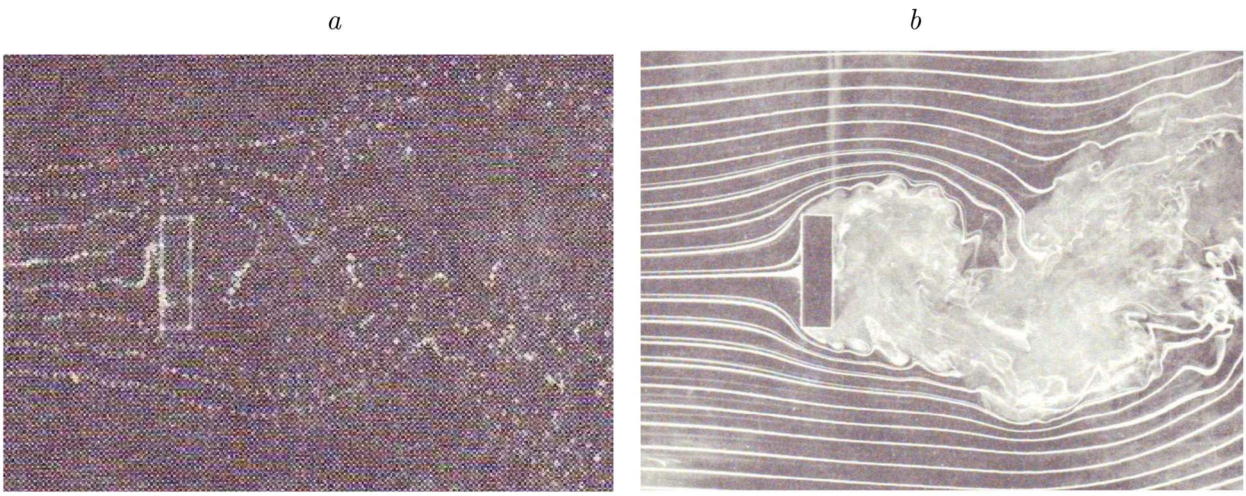

Fig. 13. Fromm's numerical visualization of the flow around a rectangular block, obtained from the Navier — Stokes equations in [25] (*a*). Lippich's experimental flow visualization of the flow around a rectangular block obtained by smoke injection described in [26] (*b*).

flow about the "NASA Common Research Transonic Wing-Body-Tail Model" for Reynolds numbers of the order Re $= 2 \times 10^7$, about one order of magnitude smaller than those of flight conditions, but four orders of magnitude larger than the Reynolds number in Fromm's 1963 computation.

The workshop was organized as an international meeting: Nineteen groups from various countries participated with 29 solutions. Researchers from the USA and Europe provided eleven solutions each and seven were submitted from Asia and Russia. Industry presented results of seven solutions, research establishments and venders nine each, and academia four. The presentations showed that grid generation still occupies a large portion of the work, with structured grids used in nine solutions, but unstructured in twenty.

An example for the results obtained is pictured in Fig. 15, taken from [27]. Shown is the Lilienthal polar diagram, which gives the dimensionless lift coefficient as a function of the dimensionless drag coefficient. The computed data are compared with experimental results.

Figure 15 contains data for several test conditions. For example, the influence of the grid spacing on the accuracy of the solution was investigated by varying the number of grid

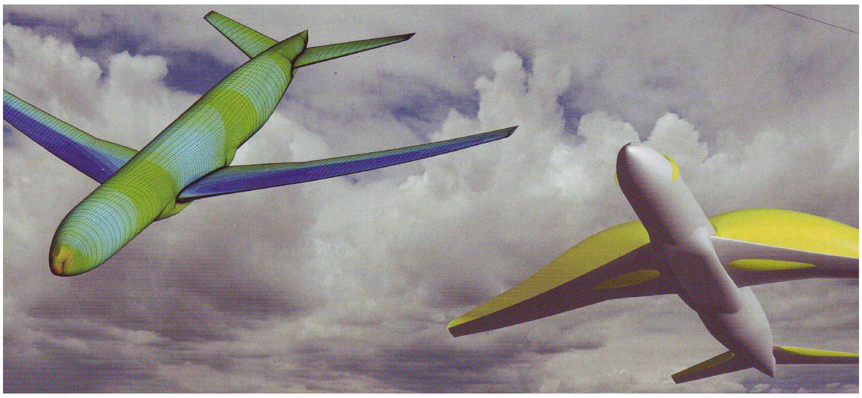

Fig. 14. The NASA Common Transonic Research Model, consisting of a wing-body-tail configuration, used in the AIAA "Drag Prediction Workshops" for comparison calculations for a Reynolds number Re $= 2 \times 10^7$, see, for example [27].

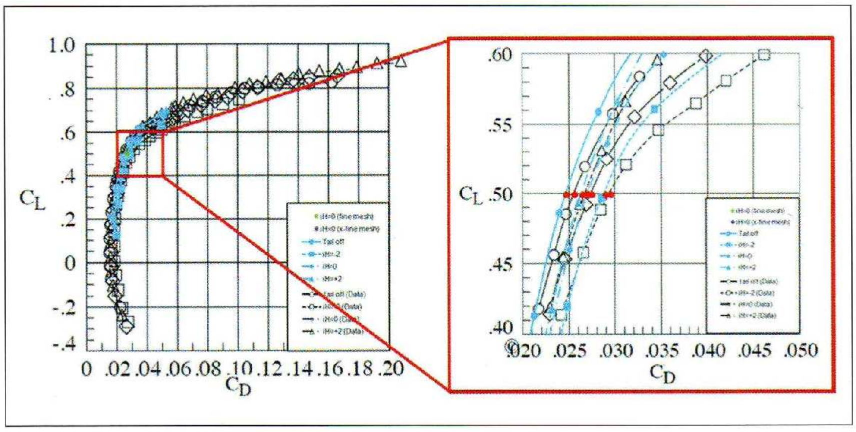

Fig. 15. Comparison of numerical and experimental results of the fourth AIAA "Drag Prediction Workshop" of 2009. Shown is the dimensionless lift coefficient as a function of the dimensionless drag coefficient in the form of Lilienthal's polar diagram for several test conditions, discussed in [27] and elsewhere.

Fig. 16. Present and future applications of numerical methods in industrial design and optimization of aerodynamic configurations. Solutions of Reynolds-averaged Navier — Stokes equations are preferably used as described in an article by K. Becker, J. Vassberg: "Numerical Aerodynamics in Transport Aircraft Design", in Notes on Numerical Fluid Mechanics and Multidisciplinary Design, Vol. 100, 2009.

cells between 3.5 to 10^5 million cells. A second test case was focused on the influence of the Reynolds number on the drag coefficient. In these studies the Reynolds number was varied between $\mathrm{Re} = 2 \times 10^6$ and $\mathrm{Re} = 2 \times 10^7$. It is reported in [27] that the computed results matched the experimental data best at the higher Reynolds number. The workshop also included the investigation of other problems, fore example, studies of the influence of the stabilizer on the drag and lift coefficients.

Altogether the results of the fourth workshop were summarized in [28]. In the general conclusion it is manifested, that a number of numerical solutions of the Navier — Stokes equations is available, which for all test cases yield good agreement. The authors state in [28], that

…"*there is a set of CFD codes whose members … agree relatively well with each other … over all the test cases. …it is comprised of flow solvers that are based on all types of grids. Hence, several structured, unstructured, and hybrid mesh solvers have matured sufficiently to be useful CFD tools for accurate drag prediction.*"

In the past fifty years numerical solutions of the Navier — Stokes equations have matured to such a degree that the accurate determination of the drag for the flow conditions mentioned is possible. According to [29] numerical methods are already intensively being used in analyses of aerodynamic characteristics of airplane configurations in the design and optimization phase. These investigations mainly prefer solutions of the Reynolds-averaged Navier — Stokes equations. Figure 16 published in [29] shows the many applications of numerical methods in industrial applications.

Concluding remarks

The enormous progress in the development of modern computing machines and numerical solutions of the Navier — Stokes equations during the past fifty years makes it possible today to determine lift and drag of aerodynamic configurations for flows up to moderate Reynolds numbers. Counting from d'Alembert's first attempts in 1752 it took more than

200 years until the drag of a body moving in a fluid could be computed with presently available numerical simulation techniques, although with some severe restrictions: Transition of laminar to turbulent flow and fully turbulent flows still belong to the most important unsolved problems of fluid mechanics. They pose problems in the construction of numerical solutions for flow simulation, as approximations have to be introduced.

Flows with high Reynolds numbers still remain inaccessible for direct simulation. The validity of results of numerical flow computations can therefore only be checked by comparing with experimental results, because of the approximations that have to be introduced in the solutions. Presently available computational speeds and storage capacities, although large, pose another problem: They restrict the temporal and spatial resolution of the flow motion.

Nevertheless, the new computational methods in fluid mechanics in the mean time have become an indispensible tool for fluid flow research and development. Many applications demonstrate the usefulness of modern simulation techniques, how incomplete they still may be. Some of the most difficult fundamental problems still wait for their solution. For these reasons the millennium problem formulated by the Clay Mathematics Institute, the analysis of existence and regularity of solutions of the Navier — Stokes equations for the description of three-dimensional flows remains of outmost importance.

Acknowledgement

Names, titles, and citations of references [3, 5–10] can be found in István Szabó: Geschichte der mechanischen Prinzipien, Birkhäuser Verlag, Basel und Stuttgart, 1976. Wikipedia is acknowledged for the pictures shown in Figs. 1–8 and 10.

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