A MIXED PROBLEM FOR THE WAVE EQUATION IN COORDINATE DOMAINS. II. OBTAINING OF A PRIORI ESTIMATES PROTECT IN MIXED PROBLEMS OF PROTECT MULTIDIMENSIONAL WAVE EQUATION*

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In the second part of the article (the first one has been published in the previous issue of journal) there is suggested a new and simple method to obtain the known a priori estimation of solution from W_2^1 of the mixed problem for the multidimensional wave equation in a half plane with boundary conditions of an oblique derivative type. The method combines elements of the Fourier–Laplace transform technique as well as the energy integrals technique. For the case of the mixed problem for the multidimensional wave equation in a coordinate corner, a domain of values of the boundary conditions parameters is selected out where the a priori estimation of solution from W_2^1 without loss of generality is valid.

Total list of the literature has been adduced in the first part of the article.

1. Mixed problem for the wave equation (the real coefficients case)

In this section, the following mixed problem for the wave equation in the domain $t > 0$, $(x, y, \mathbf{z}) \in R_+^{n+2}$, $n \geq 1$ is considered:

$$\mathcal{L}(\tau, \xi, \eta, \zeta_1, \dots, \zeta_n)u = u_{tt} - u_{xx} - u_{yy} - \Delta_z u = 0, \quad t > 0, \ x > 0, \tag{1.1}$$

$$u_t - au_x - bu_y - (\mathbf{c}, \zeta u) = 0, \quad x = 0, \tag{1.2}$$

$$u = \varphi(x, y, \mathbf{z}), \ u_t = \psi(x, y, \mathbf{z}), \quad t = 0. \tag{1.3}$$

Here

$$\mathbf{z} = (z_1, \dots, z_n), \quad \tau = \frac{\partial}{\partial t}, \quad \xi = \frac{\partial}{\partial x}, \quad \eta = \frac{\partial}{\partial y},$$

$$\zeta = (\zeta_1, \dots, \zeta_n)^T, \quad \zeta_k = \frac{\partial}{\partial z_k}, \quad k = \overline{1, n},$$

$$\Delta_z = (\zeta, \zeta) = \sum_{k=1}^{n} \zeta_k^2 = \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k^2},$$

$$\mathbf{c} = (c_1, \ \dots, \ c_n)^T, \quad (\mathbf{c}, \zeta u) = \sum_{k=1}^{n} c_k \frac{\partial u}{\partial z_k};$$

$a, b, c_k, \ k = \overline{1,n}$ are real numbers. Without loss of generality we will assume that $c_n \neq 0$, $R_+^{n+2} = \{(x, y, \mathbf{z}); x > 0, (y, \mathbf{z}) \in R^{n+1}\}$.

Remark 1.1. We will assume that for the problem (1.1)–(1.3) *the uniform Lopatinski condition* (ULC) holds. Mixed problem (1.1)–(1.3) is said to satisfy ULC on a boundary if:

$$\hat{\tau} + a\sqrt{\hat{\tau}^2 + |\gamma|^2} - ib\gamma_0 - i\sum_{k=1}^{n} c_k \gamma_k \neq 0$$

when $\mathrm{Re}\hat{\tau} \geq 0$, $|\hat{\tau}|^2 + |\gamma|^2 \neq 0$ (for more details about ULC see [17, 27]).

Here $\hat{\tau}$ is a complex number.

$$\gamma = (\gamma_0, \gamma_1, \ \dots, \ \gamma_n), \quad |\gamma|^2 = (\gamma, \gamma),$$

$\gamma_\alpha, \ \alpha = \overline{0,n}$ are real numbers. It can be shown that in the case when ULC fails there exist *the examples of ill-posedness of Hadamard's type* for problem (1.1)–(1.3) or the problems close to it. In terms of the boundary condition coefficients from (1.2), ULC can be written as (see [27]):

$$\begin{cases} a > 0, \\ b^2 + |\mathbf{c}|^2 < 1. \end{cases} \tag{1.4}$$

Remark 1.2. By straightfoward manipulations problem (1.1)–(1.3) can be put in a more simple way. The essence of these manipulations is the following. Let T be a real orthogonal matrix of order $(n+1)$. Then by replacement of the initial differential operators (i.e., by passing from the initial differential operators to their linear combinations):

$$\mu = \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = T^* \cdot \begin{pmatrix} \eta \\ \zeta \end{pmatrix}, \tag{1.5}$$

we obtain the following relations:

$$\tau^2 - \xi^2 - \eta^2 - \sum_{k=1}^{n} \zeta_k^2 = \tau^2 - \xi^2 - \sum_{\alpha=0}^{n} \mu_\alpha^2,$$

$$\tau - a\xi - b\eta - \sum_{k=1}^{n} c_k \zeta_k = \tau - a\xi - \left(T^* \cdot \begin{pmatrix} b \\ \mathbf{c} \end{pmatrix}, \mu\right) = \tau - a\xi - \tilde{b}\mu_0.$$

The latter relation is true since there exists such an orthogonal matrix T that

$$(b, c_1, \ \dots, \ c_n) \cdot T = (\tilde{b}, 0, \ \dots, \ 0),$$

where $\tilde{b} = \mathrm{sign}(c_n)\sqrt{b^2 + |\mathbf{c}|^2}$. With the above-mentioned relations taken into account, initial problem (1.1)–(1.3) can be reduced to the following so called canonical form.

Problem I. We seek the solution of the wave equation

$$u_{tt} - u_{xx} - u_{yy} - \Delta_z u = 0, \quad t > 0,\ (x, y, \mathbf{z}) \in R^{n+2}_{+}. \tag{1.1$'$}$$

which satisfies at $x = 0$ the boundary condition

$$u_t - a u_x - b u_y = 0, \quad t > 0,\ (y, \mathbf{z}) \in R^{n+1}, \tag{1.2$'$}$$

and at $t = 0$ the initial data (1.3) (while formulating Problem I, we return to the former notations). The uniform Lopatinski condition for Problem I is formulated as

$$a > 0, \quad b^2 < 1. \tag{1.4$'$}$$

The study of Problem I is the matter the present section.

In Problem I we carry out the Fourier transform with respect to the variables z_k, $k = \overline{1, n}$. Then Problem I looks like

$$\hat{u}_{tt} - \hat{u}_{xx} - \hat{u}_{yy} + 4\pi^2 |\xi|^2 \hat{u} = 0, \quad t > 0,\ (x, y) \in R^2_{+}, \tag{1.1$''$}$$

$$\hat{u}_t - a\hat{u}_x - b\hat{u}_y = 0, \quad t > 0,\ y \in R^1,\ x = 0, \tag{1.2$''$}$$

$$\hat{u} = \hat{\varphi}(x, y, \xi), \quad \hat{u}_t = \hat{\psi}(x, y, \xi), \quad (x, y) \in R^2_{+}. \tag{1.3$'$}$$

Here

$$\hat{u} = \hat{u}(t, x, y, \xi) = \int\limits_{R^n} e^{-2\pi i(\mathbf{z}, \xi)} u(t, x, y, \mathbf{z}) d\mathbf{z}$$

is Fourier transform of the function u, $\xi = (\xi_1, \ \ldots, \ \xi_n) \in R^n$, $\hat{\varphi}$, $\hat{\psi}$ are Fourier transforms of the functions φ and ψ (see (1.3)).

Following [5, 17, 27], for the vector

$$\mathbf{U} = \begin{pmatrix} \hat{u}_t \\ \hat{u}_x \\ \hat{u}_y \end{pmatrix}$$

we write the symmetric system (its validity on the solutions of (1.1$''$) is easily verified):

$$\{A_0\tau - B_0\xi - C_0\eta\}\mathbf{U} + 4\pi^2|\xi|^2 \hat{u}\mathbf{F} = 0, \tag{1.6}$$

where

$$\begin{aligned} A_0 &= \begin{pmatrix} k & l & m \\ l & k & in \\ m & -in & k \end{pmatrix} = T_0^* \cdot \begin{pmatrix} H & O_2 \\ O_2 & H \end{pmatrix} \cdot T_0, \\ B_0 &= \begin{pmatrix} l & k & in \\ k & l & m \\ -in & m & -l \end{pmatrix} = T_0^* \cdot \begin{pmatrix} O_2 & -H \\ -H & O_2 \end{pmatrix} \cdot T_0, \\ C_0 &= \begin{pmatrix} m & -in & k \\ in & -m & l \\ k & l & m \end{pmatrix} = T_0^* \cdot \begin{pmatrix} -H & O_2 \\ O_2 & H \end{pmatrix} \cdot T_0, \end{aligned}$$

$$\mathbf{F} = \begin{pmatrix} k \\ l \\ m \end{pmatrix}, \quad T_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$H = \begin{pmatrix} k-m & -l-in \\ -l+in & k+m \end{pmatrix},$$

O_2 is the zero matrix of order 2; $k,\ l,\ m,\ n$ are certain real constants. $A_0,\ B_0,\ C_0,\ H$ are the Hermitian matrices. System (1.6) can also be rewritten in the form

$$\{\bar{A}_0\tau - \bar{B}_0\xi - \bar{C}_0\eta\}\bar{\mathbf{U}} + 4\pi^2|\xi|^2\bar{\hat{u}}\mathbf{F} = 0. \tag{1.6$'$}$$

Here the bar means the complex conjugation.

Let us multiply system (1.6) scalarly by $\bar{\mathbf{U}}$ and system (1.6′) by $\mathbf{U}$ and sum up the expressions. Finally we obtain the following identity:

$$(\bar{\mathbf{U}}, A_0\mathbf{U})_t - (\bar{\mathbf{U}}, B_0\mathbf{U})_x - (\bar{\mathbf{U}}, C_0\mathbf{U})_y +$$
$$+ 4\pi^2|\xi|^2\{k(|\hat{u}|^2)_t + l(|\hat{u}|^2)_x + m(|\hat{u}|^2)_y\} = 0. \tag{1.7}$$

Here $|\hat{u}|^2 = \hat{u}\cdot\bar{\hat{u}}$. Let integrate (1.7) over the domain R^2_+, assuming that

$$|\mathbf{U}|^2 = (\bar{\mathbf{U}}, \mathbf{U}) \to 0,\ |\hat{u}|^2 \to 0 \text{ as } r \to \infty,$$

where $r = \sqrt{x^2+y^2}$. As a consequence, we obtain

$$\frac{d}{dt}\hat{J}_1(t) + \int\limits_{R^1} \{(\bar{\mathbf{U}}, B_0\mathbf{U}) - 4\pi^2|\xi|^2 l|\hat{u}|^2\}\Big|_{x=0} dy = 0. \tag{1.8}$$

Here

$$\hat{J}_1(t) = \iint\limits_{R^2_+} \{(\bar{\mathbf{U}}, A_0\mathbf{U}) + 4\pi^2|\xi|^2 k|\hat{u}|^2\} dxdy.$$

Now let us consider the forms $(\bar{\mathbf{U}}, A_0\mathbf{U})$ and $(\bar{\mathbf{U}}, B_0\mathbf{U})\big|_{x=0}$. The form

$$(\bar{\mathbf{U}}, A_0\mathbf{U}) = \left(\bar{\mathbf{V}}, \begin{pmatrix} H & O_2 \\ O_2 & H \end{pmatrix} \cdot \mathbf{V}\right) =$$
$$= (\bar{\mathbf{V}}^I, H\cdot\mathbf{V}^I) + (\bar{\mathbf{V}}^{II}, H\cdot\mathbf{V}^{II}) > 0$$

if $H > 0$, i.e. $k > 0,\ k^2 - m^2 - l^2 - n^2 > 0$. Here

$$\mathbf{V} = T_0\cdot\mathbf{U} = \begin{pmatrix} \mathbf{V}^I \\ \mathbf{V}^{II} \end{pmatrix},$$

$$\mathbf{V}^I = \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{u}_t - \hat{u}_y \\ -\hat{u}_x \end{pmatrix}, \quad \mathbf{V}^{II} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\hat{u}_x \\ \hat{u}_t + \hat{u}_y \end{pmatrix}.$$

Preparatory to considering the form $(\bar{\mathbf{U}}, B_0\mathbf{U})\big|_{x=0}$, let us rewrite the boundary conditions (1.2″) in the way

$$\mathbf{V}^I = S\cdot\mathbf{V}^{II}, \quad x = 0, \tag{1.2$'''$}$$

where

$$S = \begin{pmatrix} -\dfrac{2a}{1+b} & -\dfrac{1-b}{1+b} \\ 1 & 0 \end{pmatrix}.$$

Note that the matrix S is the Hurwitz one if ULC is true (i.e. its eigen-values lie on the left complex semi-plane).

The form

$$(\bar{\mathbf{U}}, B_0\mathbf{U})\big|_{x=0} = \left(\bar{\mathbf{V}}, \begin{pmatrix} O_2 & -H \\ -H & O_2 \end{pmatrix} \cdot \mathbf{V}\right)\Bigg|_{x=0} =$$

$$= -(\bar{\mathbf{V}}^I, H\cdot\mathbf{V}^{II})\big|_{x=0} - (\bar{\mathbf{V}}^{II}, H\cdot\mathbf{V}^I)\big|_{x=0} = -(\bar{\mathbf{V}}^{II}, [S^*H + HS]\cdot\mathbf{V}^{II})\big|_{x=0}.$$

Let

$$S^*H + HS = -G,$$

where $G = G^* > 0$ is a certain matrix. Then

$$(\bar{\mathbf{U}}, B_0\mathbf{U})\big|_{x=0} = (\bar{\mathbf{V}}^{II}, G\cdot\mathbf{V}^{II})\big|_{x=0} > 0. \tag{1.9}$$

Remark 1.3. If S is the Hurwitz matrix, then the Lapunov matrix equation

$$S^*H + HS = -G \tag{1.10}$$

is uniquely solvable with respect to H for any Hermitian matrix $G = G^*$. As this takes place (if $G > 0$), then $H = H^* > 0$ (about the solution of the Lapunov matrix equation see, for example, [25]).

Presenting (1.10) as

$$\begin{pmatrix} -s_1 & 1 \\ -s_2 & 0 \end{pmatrix}\begin{pmatrix} k-m & -l-in \\ -l+in & k+m \end{pmatrix} + \begin{pmatrix} k-m & -l-in \\ -l+in & k+m \end{pmatrix}\begin{pmatrix} -s_1 & -s_2 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} g_1 & -g_2 - ig_3 \\ -g_2 + ig_3 & g_4 \end{pmatrix}, \quad g_{1,4} > 0, \quad g_1g_4 - g_2^2 - g_3^2 > 0,$$

we easily find that

$$l = -g_4\frac{1+b}{2(1-b)} < 0. \tag{1.11}$$

Here $s_1 = \dfrac{2a}{1+b}$, $s_2 = \dfrac{1-b}{1+b}$.

In view of (1.9), (1.11), it follows from (1.8):

$$\frac{d}{dt}\hat{J}_1(t) \le 0. \tag{1.12}$$

Since

$$\frac{d}{dt}\left\{\iint\limits_{R_+^2} |\hat{u}|^2 dxdy\right\} \le \hat{J}_0(t), \tag{1.13}$$

summing up (1.12) and (1.13), we can obtain

$$\frac{d}{dt}\left\{\hat{J}_1(t)+\iint\limits_{R^2_+}|\hat{u}|^2dxdy\right\}\le \hat{J}_0(t)\le M_1\cdot\left\{\hat{J}_1(t)+\iint\limits_{R^2_+}|\hat{u}|^2dxdy\right\}. \tag{1.14}$$

Here

$$\hat{J}_0(t)=\iint\limits_{R^2_+}\{|\mathbf{U}|^2+(4\pi^2|\xi|^2+1)|\hat{u}|^2\}dxdy,$$

$$M_1=\max\left\{1,\frac{1}{k-\sqrt{l^2+m^2+n^2}}\right\}.$$

(1.14) yields:

$$\hat{J}_0(t)\le M_1M_2\hat{J}_0(0)e^{M_1t},\quad t>0, \tag{1.15}$$

where $M_2=\max\{1,k+\sqrt{l^2+m^2+n^2}$. In view of the Parceval equality, we obtain the *desired a priori estimate* for the solutions of Problem I at last:

$$\int\limits_{R^n}\hat{J}_0(t)d\xi=\int\limits_{R^{n+2}_+}\{u^2(t,x,y,\mathbf{z})+u^2_x(t,x,y,\mathbf{z})+$$

$$+u^2_y(t,x,y,\mathbf{z})+u^2_t(t,x,y,\mathbf{z})+|\zeta u(t,x,y,\mathbf{z})|^2\}dxdyd\mathbf{z}\le$$

$$\le M_1M_2e^{M_1t}\int\limits_{R^{n+2}_+}\{\varphi^2(x,y,\mathbf{z})+\varphi^2_x(x,y,\mathbf{z})+$$

$$+\psi^2(x,y,\mathbf{z})+\varphi^2_y(t,x,y,\mathbf{z})+|\zeta\varphi(x,y,\mathbf{z})|^2\}dxdyd\mathbf{z},\quad t>0$$

or

$$\|u(t)\|^2_{W^1_2(R^{n+2}_+)}+\|u_t(t)\|^2_{L_2(R^{n+2}_+)}\le$$

$$\le M_1M_2e^{M_1t}\{\|\varphi\|^2_{W^1_2(R^{n+2}_+)}+\|\psi\|^2_{L_2(R^{n+2}_+)}\},\quad t>0 \tag{1.16}$$

Here W^1_2 is the Sobolev space (see [5]).

Remark 1.4. Replacement (1.5) and Fourier transform are just an auxiliary expedient. Therefore estimate (1.16) also holds true for initial problem (1.1)–(1.3) provided that ULC (1.4) is fulfilled.

2. Mixed problem for the wave equation the complex coefficients case

Let in boundary condition (1.2) the coefficients $a,b,c_1,\ \dots,\ c_n$ be complex numbers: $a=a'+ia''$, $b=b'+ib''$, $c_k=c'_k+ic''_k$, $k=\overline{1,n}$. Without loss of generality, we will assume that $c''_n\ne 0$. In terms of coefficients of boundary condition (1.2) ULC can be formulated in rather sophisticated way (see [27]), and this situation is not discussed here.

Once again we will try to simplify initial problem (1.1)–(1.3) by a replacement of operators. To this end, first we turn the vector $(b'',c''_1,\ \dots,\ c''_n)$ to make all its components, except for the first one, equal to zero and then turn the vector $(\tilde{c}'_1,\ \dots,\ \tilde{c}'_n)$ (its components are defined below). As a consequence, problem (1.1)–(1.3) can be reduced to the following *canonical form.*

Problem II. We seek the solution of the wave equation

$$u_{tt} - u_{xx} - u_{yy} - \Delta_z u = 0, \quad t > 0,\ (x, y, \mathbf{z}) \in R^{n+2}_+, \tag{1.1}$$

which satisfy at $x = 0$ the boundary condition

$$u_t - au_x - \tilde{b}u_y - \tilde{\tilde{c}}_1 u_{z_1} = 0, \quad t > 0,\ (y, \mathbf{z}) \in R^{n+1}, \tag{2.1}$$

and at $t = 0$ the initial data (1.3′). Here $\tilde{b} = \tilde{b}' + i\tilde{b}''$, and

$$\tilde{b}'' = \operatorname{sign}(c''_n)\sqrt{(b'')^2 + |\mathbf{c}''|^2}, \quad \tilde{b}' = \frac{b'b'' + (\mathbf{c}', \mathbf{c}'')}{\tilde{b}''},$$

$$\mathbf{c}' = (c'_1,\ \dots,\ c'_n), \quad \mathbf{c}'' = (c''_1,\ \dots,\ c''_n),$$

$$\tilde{\tilde{c}}_1 = \operatorname{sign}(\tilde{c}'_n)|\tilde{\mathbf{c}}'|, \quad \tilde{\mathbf{c}}' = (\tilde{c}'_1,\ \dots,\ \tilde{c}'_n),$$

$$\tilde{c}'_n = \operatorname{sign}(c''_n)\frac{c''_{n-1}c'_n - c'_{n-1}c''_n}{\sqrt{(c''_{n-1})^2 + (c''_n)^2}},$$

$$\tilde{c}'_k = \frac{c'_{k-1}\sum\limits_{j=k}^{n}(c'_j c''_j) - c'_{k-1}\sum\limits_{j=k}^{n}(c''_j)^2}{\sqrt{\sum\limits_{j=k-1}^{n}(c''_j)^2\sum\limits_{j=k}^{n}(c''_j)^2}}, \quad k = \overline{1, n-1},$$

$(c_0 = b)$. In what follows we will again denote $\tilde{b}$, $\tilde{\tilde{c}}_1$ by b and c_1. Note that u, φ, ψ are complex-valued functions.

Remark 2.1. Following [27], we formulate ULC for Problem II as the requirement of the positive definiteness for the matrix made up from the coefficients of boundary condition (2.1):

$$\begin{pmatrix} a' & 0 & -\operatorname{Re}(a\bar{b}) & -a'c_1 \\ 0 & a' & ib'' & 0 \\ -\operatorname{Re}(a\bar{b}) & -ib'' & a' & 0 \\ -a'c_1 & 0 & 0 & a' \end{pmatrix} > 0,$$

i.e.

$$\begin{cases} a' > 0, \\ (a')^2 - (b'')^2 - (\operatorname{Re}(a\bar{b}))^2 > 0, \\ (1 - c_1^2)[(a')^2 - (b'')^2] - (\operatorname{Re}(a\bar{b}))^2 > 0. \end{cases} \tag{2.2}$$

It follows from (2.2), for example, that $c_1^2 < 1$.

Remark 2.2. In Problem II we make the following replacement of operators τ, ζ_1:

$$\tau = \frac{1}{\sqrt{1 - c_1^2}}\{\tau' + c_1\zeta'_1\}, \quad \zeta_1 = \frac{1}{\sqrt{1 - c_1^2}}\{c_1\tau' + \zeta'_1\},$$

or

$$\tau' = \frac{1}{\sqrt{1 - c_1^2}}\{\tau - c_1\zeta_1\}, \quad \zeta'_1 = \frac{1}{\sqrt{1 - c_1^2}}\{\zeta_1 - c_1\tau\},$$

Then the relations

$$\tau^2 - \xi^2 - \eta^2 - \sum_{k=1}^{n} \zeta_k^2 = (\tau')^2 - \xi^2 - \eta^2 - (\zeta_1')^2 - \sum_{k=2}^{n} \zeta_k^2,$$

$$\tau - a\xi - b\eta - c_1\zeta_1 = \sqrt{1-c_1^2}\,\tau' - a\xi - b\eta$$

are valid. With these relations taken into account, Problem II can be formulated as

$$\{(\tau')^2 - \xi^2 - \eta^2 - (\zeta_1')^2 - \sum_{k=2}^{n} \zeta_k^2\}u = 0, \quad t > 0, \ (x, y, \mathbf{z}) \in R_+^{n+2},$$

$$\{\tau' - \hat{a}\xi - \hat{b}\eta\}u = 0, \quad t > 0, \ x = 0, \ (y, \mathbf{z}) \in R^{n+1}. \tag{2.1$'$}$$

Here

$$\hat{a} = \frac{a}{\sqrt{1-c_1^2}}, \quad \hat{b} = \frac{b}{\sqrt{1-c_1^2}}.$$

In Problem II we carry out the Fourier transform with respect to the variables z_k, $k = \overline{2,n}$. Then we obtain the problem

$$\{(\tau')^2 - \xi^2 - \eta^2 - (\zeta_1')^2 + 4\pi^2|\xi'|^2\}\hat{u} = 0, \tag{1.1$''$}$$

$$t > 0, \ (x, y, z_1) \in R_+^3,$$

$$\{\tau' - \hat{a}\xi - \hat{b}\eta\}\hat{u} = 0, \quad t > 0, \ x = 0, \ (y, z_1) \in R^2, \tag{2.1$''$}$$

$$\hat{u} = \hat{\varphi}(x, y, z_1, \xi'), \quad \hat{u}_t = \hat{\psi}(x, y, z_1, \xi'), \quad (x, y, \mathbf{z}) \in R_+^3. \tag{1.3$''$}$$

Here

$$\hat{u} = \hat{u}(t, x, y, z_1, \xi') = \int_{R^{n-1}} e^{-2\pi i(\mathbf{z}', \xi')} u(t, x, y, z_1, \mathbf{z}') d\mathbf{z}'$$

are Fourier transform of the function u, $\xi' = (\xi_2, \ \dots, \ \xi_n)$, $\mathbf{z}' = (z_2, \ \dots, \ z_n)$, $\hat{\varphi}, \hat{\psi}$ are Fourier transforms of the functions φ and ψ (see (1.3)).

Following [5, 17, 27], for the vector

$$\mathbf{U} = \begin{pmatrix} \tau'\hat{u} \\ \xi\hat{u} \\ \eta\hat{u} \\ \zeta_1'\hat{u} \end{pmatrix}$$

we write the corresponding symmetric system (by direct calculations one can easily obtain the validity of this system on the solutions (1.1)):

$$\{\widetilde{A}\tau' - B\xi - C\eta - \widetilde{C}_1\zeta_1'\}\mathbf{U} + 4\pi^2|\xi'|^2\hat{u}\vec{\mathcal{F}} = 0, \tag{2.3}$$

or

$$\{A\tau - B\xi - C\eta - C_1\zeta_1\}\mathbf{U} + 4\pi^2|\xi'|^2\hat{u}\mathcal{F} = 0, \tag{2.3$'$}$$

where

$$\widetilde{A} = \left(\begin{array}{c|c} A_0 & 0 \\ \hline 0 & k \end{array}\right), \quad B = \left(\begin{array}{c|c} B_0 & 0 \\ \hline 0 & -l \end{array}\right), \quad C = \left(\begin{array}{c|c} C_0 & 0 \\ \hline 0 & -m \end{array}\right),$$

$$\widetilde{C}_1 = \left(\begin{array}{c|c} O_3 & F \\ \hline F^* & 0 \end{array}\right), \quad \mathcal{F} = \begin{pmatrix} F \\ 0 \end{pmatrix};$$

the matrices A_0, B_0, C_0, the vector $\mathbf{F}$ are described above (see the section 1).

$$A = \frac{1}{\sqrt{1-c_1^2}}\{c_1\widetilde{C}_1 + \widetilde{A}\} > 0 \text{ if } H > 0,$$

i.e., $k > 0$, $k^2 - l^2 - m^2 - n^2 > 0$ ($c_1^2 < 1!$);

$$C_1 = \frac{1}{\sqrt{1-c_1^2}}\{c_1\widetilde{A} + \widetilde{C}_1\}.$$

System (2.3′) can also be rewritten as follows:

$$\{\bar{A}\tau - \bar{B}\xi - \bar{C}\eta - \bar{C}_1\zeta_1\}\bar{\mathbf{U}} + 4\pi^2|\xi'|^2\hat{\mathbf{u}}\mathcal{F} = 0, \tag{2.3″}$$

Multiply system (2.3′) scalarly by $\bar{\mathbf{U}}$ and system (2.3″) by $\mathbf{U}$ and sum up the expressions:

$$(\bar{\mathbf{U}}, A\mathbf{U})_t - (\bar{\mathbf{U}}, B\mathbf{U})_x - (\bar{\mathbf{U}}, C\mathbf{U})_y - (\bar{\mathbf{U}}, C_1\mathbf{U})_{z_1} +$$

$$+4\pi^2|\xi'|^2\Big\{\frac{k}{\sqrt{1-c_1^2}}(|\hat{u}|^2)_t - \frac{kc_1}{\sqrt{1-c_1^2}}(|\hat{u}|^2)_{z_1} + l(|\hat{u}|^2)_x + m(|\hat{u}|^2)_y\Big\} = 0. \tag{2.4}$$

We integrate (2.4) over the domain R^3_+, assuming that $|\mathbf{U}|^2 = (\bar{\mathbf{U}}, \mathbf{U}) \to 0$, $|\hat{u}|^2 \to 0$ as $r \to \infty$ where $r = \sqrt{x^2+y^2+z_1^2}$. In the end, we obtain

$$\frac{d}{dt}\hat{J}_1(t) + \int\limits_{R^2}\{(\bar{\mathbf{U}}, B\mathbf{U}) - 4\pi^2|\xi'|^2 l|\hat{u}|^2\}\big|_{x=0}dydz_1 = 0. \tag{2.5}$$

Here

$$\hat{J}_1(t) = \int\limits_{R^3_+}\{(\bar{\mathbf{U}}, A\mathbf{U}) + \frac{k}{\sqrt{1-c_1^2}}4\pi^2|\xi'|^2 l|\hat{u}|^2\}dxdydz_1.$$

The form $(\bar{\mathbf{U}}, A\mathbf{U}) > 0$, if $H > 0$. The form

$$(\bar{\mathbf{U}}, B\mathbf{U})\big|_{x=0} = (\bar{\mathbf{U}}', B_0\mathbf{U}')\big|_{x=0} - l|\zeta_1'\hat{u}|^2\big|_{x=0},$$

where

$$\mathbf{U}' = \begin{pmatrix} \tau'\hat{u} \\ \xi\hat{u} \\ \eta\hat{u} \end{pmatrix}.$$

And the form

$$(\bar{\mathbf{U}}', B_0\mathbf{U}')\big|_{x=0} = -(\bar{\mathbf{V}}^{II}, [S^*H + HS]\mathbf{V}^{II})\big|_{x=0},$$

where

$$S = \begin{pmatrix} -\dfrac{2\hat{a}}{1+\hat{b}} & -\dfrac{1-\hat{b}}{1+\hat{b}} \\ 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \mathbf{V}^I \\ \mathbf{V}^{II} \end{pmatrix} = T_0\mathbf{U}'.$$

Note that if ULC (2.2) is true, the matrix S is the Hurwitz one. Hence the Lyapunov matrix equation

$$S^*H + HS = -G \tag{2.6}$$

is uniquely solvable with respect to H for any Hermitian matrix $G = G^* > 0$, and $H = H^* > 0$. Then

$$(\bar{\mathbf{U}}', B_0\mathbf{U}')\big|_{x=0} = -(\bar{\mathbf{V}}^{II}, G\mathbf{V}^{II}) > 0. \tag{2.7}$$

Remark 2.3. As in the section 1, we can show that $l > 0$. Let

$$H^{-1} = \begin{pmatrix} h_1 & h_2 \\ \bar{h}_2 & h_3 \end{pmatrix}.$$

Then equation (2.6) can be rewritten as

$$H^{-1}S^* + SH^{-1} = -H^{-1}GH^{-1}.$$

Since

$$H^{-1}S^* + SH^{-1} = \begin{pmatrix} -2h_1\mathrm{Re}s_1 - 2\mathrm{Re}(h_2\bar{s}_2) & h_1 - s_1h_2 - s_2h_3 \\ h_1 - \bar{s}_1\bar{h}_2 - \bar{s}_2h_3 & h_2 + \bar{h}_2 \end{pmatrix} < 0,$$

where $s_1 = \dfrac{2\hat{a}}{1+\hat{b}}$, $s_2 = \dfrac{1-\hat{b}}{1+\hat{b}}$, the inequality $2\mathrm{Re}h_2 < 0$ is true. It is easy to show that $\mathrm{Re}h_2 = \dfrac{l}{k^2 - l^2 - m^2 - n^2}$, i.e. $l < 0$.

In view of (2.7) and Remark 2.3, (2.5) yields

$$\frac{d}{dt}\hat{J}_1(t) \le 0. \tag{2.8}$$

Further reasonings are analogous to that presented at the end of section 1. That leads us to estimate (1.16) again. Remark 1.4 from section 1 remains also true.

In fact, since

$$(\bar{\mathbf{U}}, A\mathbf{U}) = (\bar{\mathbf{W}}, \hat{A}\mathbf{W}),$$

where

$$\mathbf{U} = \Gamma\mathbf{W}, \quad \Gamma = \frac{1}{\hat{d}}\begin{pmatrix} 1 & 0 & 0 & -c_1 \\ 0 & \hat{d} & 0 & 0 \\ 0 & 0 & \hat{d} & 0 \\ -c_1 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} \tau\hat{u} \\ \xi\hat{u} \\ \eta\hat{u} \\ \zeta_1\hat{u} \end{pmatrix}, \quad \hat{d} = \sqrt{1-c_1^2}, \quad \hat{A} = \Gamma A\Gamma;$$

then

$$\lambda_{\min}(\hat{A})(\bar{\mathbf{W}}, \mathbf{W}) \le (\bar{\mathbf{U}}, A\mathbf{U}) \le \lambda_{\max}(\hat{A})(\bar{\mathbf{W}}, \mathbf{W}). \tag{2.9}$$

Here $\lambda_{\min}(\hat{A})$, $\lambda_{\max}(\hat{A})$ are minimal and maximal eigen-values of matrix $\hat{A}$. Since

$$\frac{d}{dt}\Big\{\iiint\limits_{R^3_+} |\hat{u}|^2 dx dy dz_1\Big\} \le \hat{J}_0(t), \tag{2.10}$$

summing up (2.8) and (2.10), in view of (2.9), we finally obtain

$$\frac{d}{dt}\Big\{\hat{J}_1(t) + \iiint\limits_{R^3_+} |\hat{u}|^2 dxdydz_1\Big\} \leq \hat{J}_0(t) \leq$$

$$\leq M_3\Big\{\hat{J}_1(t) + \iiint\limits_{R^3_+} |\hat{u}|^2 dxdydz_1\Big\}. \tag{2.11}$$

Here

$$\hat{J}_0(t) = \iiint\limits_{R^3_+} \{|\mathbf{W}|^2 + (4\pi^2|\xi'|^2 + 1)|\hat{u}|^2\} dxdydz_1,$$

$$M_3 = \max\Big\{1, \frac{1}{\lambda_{\min}(\hat{A})}\Big\}.$$

It follows from (2.11):

$$\hat{J}_0(t) \leq M_3 M_4 \hat{J}_0(0) e^{M_3 t}, \quad t > 0, \tag{2.12}$$

where $M_4 = \max\{1, \lambda_{\max}(\hat{A})\}$. Regarding the Parceval equality, from (2.12) we obtain the desired a priori estimate for the solutions of Problem II in the form of (1.6).

Remark 2.4. The matrix $\hat{A}$ looks like

$$\hat{A} = \begin{pmatrix} \tilde{k} & l & m & -\tilde{k}c_1 \\ l & \tilde{k} & i\tilde{n} & 0 \\ m & -i\tilde{n} & \tilde{k} & 0 \\ -\tilde{k}c_1 & 0 & 0 & \tilde{k} \end{pmatrix}, \quad \tilde{k} = \frac{k}{\hat{d}},\ \tilde{n} = \frac{n}{\hat{d}}\ (\hat{A} > 0).$$

Then

$$\lambda_{\min}(\hat{A}) = \tilde{k} - Q, \quad \lambda_{\max}(\hat{A}) = \tilde{k} + Q,$$

where

$$Q = \sqrt{\frac{q_0 + \sqrt{q_0^2 - 4\tilde{n}^2\tilde{k}^2c_1^2}}{2}}, \quad q_0 = m^2 + \tilde{n}^2 + l^2 + \tilde{k}^2c_1^2.$$

3. Mixed problem for the wave equation in a domain with a corner

In this section, the mixed problem for the wave equation in the domain

$$t > 0, \quad (x, y, \mathbf{z}) \in R^{n+2}_{++}, \quad n \geq 1$$

is studied (the case $n = 0$ has been considered in [4, 5]):

$$u_{tt} - u_{xx} - u_{yy} - \Delta_z u = 0 \text{ at } t > 0,\ x > 0,\ y > 0, \tag{3.1}$$

$$u_t - au_x - bu_y - (\mathbf{c}, \zeta u) = 0 \text{ at } x = 0, \tag{3.2}$$

$$u_t - \alpha u_y - \beta u_x - (\mathbf{d}, \zeta u) = 0 \text{ at } y = 0, \tag{3.3}$$

$$u = \varphi(x, y, \mathbf{z}), \quad u_t = \psi(x, y, \mathbf{z}) \text{ at } t = 0. \tag{3.4}$$

Here $\mathbf{d} = (d_1, \ \dots, \ d_n)^T$; $a, b, c_k, \alpha, \beta, d_k$, $k = \overline{1, n}$ are real numbers. Without loss of generality, we will assume $c_n \neq 0$; $R^{n+2}_{++} = \{(x, y, \mathbf{z}) : x > 0, y > 0, \mathbf{z} \in R^n\}$.

Following the idea of section 1, we simplify problem (3.1)–(3.4) by an appropriate replacement of the initial differential operators. For this purpose let us first turn the vector $\mathbf{c}$, making all its components equal to zero except for the first one. Then we make the analogous turn of the vector $(\tilde{d}_2, \ \dots, \ \tilde{d}_n)$ (the components of this vector will be defined below). As a consequence, the problem (3.1)–(3.4) can be put in the following canonical form.

Problem III. We seek the solution of the wave equation

$$u_{tt} - u_{xx} - u_{yy} - \Delta_z u = 0, \quad t > 0, (x, y, \mathbf{z}) \in R^{n+2}_{++}, \tag{3.1$'$}$$

satisfying at $x = 0$ and $y = 0$ the boundary conditions

$$u_t - a u_x - b u_y - \tilde{c}_1 u_{z_1} = 0, \quad t > 0, \ (y, \mathbf{z}) \in R^{n+1}_{+}, \tag{3.2$'$}$$

$$u_t - \alpha u_y - \beta u_x - \tilde{d}_1 u_{z_1} - \tilde{d}_2 u_{z_2} = 0, \quad t > 0, \ (y, \mathbf{z}) \in R^{n+1}_{+}, \tag{3.3$'$}$$

and at $t = 0$ the initial data (3.1). Here

$$\tilde{c}_1 = \operatorname{sign}(c_n)|\mathbf{c}|, \quad \tilde{d}_1 = \operatorname{sign}(c_n)\frac{(\mathbf{c}, \mathbf{d})}{|\mathbf{c}|}, \quad \tilde{d}_n = \operatorname{sign}(c_n)\frac{d_n c_{n-1} - d_{n-1} c_n}{\sqrt{c^2_{n-1} + c^2_n}},$$

$$\tilde{d}_k = \frac{c_{k-1} \sum\limits_{j=k}^{n} (c_j d_j) - d_{k-1} \sum\limits_{j=k}^{n} c_j^2}{\sqrt{\sum\limits_{j=k}^{n} c_j^2 \cdot \sum\limits_{j=k-1}^{n} c_j^2}}, \quad k = \overline{2, n-1},$$

$$\tilde{\tilde{d}}_2 = \operatorname{sign}(\tilde{d}_n)|\tilde{\mathbf{d}}'|, \quad \tilde{\mathbf{d}}' = (\tilde{d}_2, \ \dots, \ \tilde{d}_n).$$

In the subsequent discussion $\tilde{c}_1, \tilde{d}_1, \tilde{\tilde{d}}_2$ will be again denoted as c_1, d_1, d_2. In addition, we will assume $d_2 \neq 0$.

Remark 3.1. ULC on the boundary $x = 0$ for Problem III can be written as (see (1.4)):

$$a > 0, \quad b^2 + c_1^2 < 1, \tag{3.5}$$

and for the boundary $y = 0$ as follows:

$$\alpha > 0, \quad \beta^2 + d_1^2 + d_2^2 < 1. \tag{3.6}$$

Remark 3.2. In Problem III we make the following replacement of the operators $\tau, \zeta_{1,2}$:

$$\begin{pmatrix} \tau \\ \zeta_1 \\ \zeta_2 \end{pmatrix} = T \begin{pmatrix} \tau' \\ \zeta_1' \\ \zeta_2' \end{pmatrix},$$

where $T = (t_{kj})$, $k, j = \overline{1, 3}$ is the matrix of real coefficients t_{kj}. We will assume that the following relations hold:

$$\tau^2 - \zeta_1^2 - \zeta_2^2 = (\tau')^2 - (\zeta_1')^2 - (\zeta_2')^2,$$

$$\tau - c_1\zeta_1 = r\tau' - r_1\zeta_1',$$
$$\tau - d_1\zeta_1 - d_2\zeta_2 = r\tau' - r_2\zeta_1',$$

where $r, r_{1,2}$ are certain constants which will be defined below. Then for the coefficients of matrix T and the quantities $r, r_{1,2}$ we obtain the following system:

$$t_{11}^2 - t_{21}^2 - t_{31}^2 = 1,$$
$$t_{12}^2 - t_{22}^2 - t_{32}^2 = -1,$$
$$t_{13}^2 - t_{23}^2 - t_{33}^2 = -1,$$
$$t_{11}t_{12} - t_{21}t_{22} - t_{31}t_{32} = 0,$$
$$t_{11}t_{13} - t_{21}t_{23} - t_{31}t_{33} = 0,$$
$$t_{12}t_{13} - t_{22}t_{23} - t_{32}t_{33} = 0,$$
$$t_{11} - c_1t_{21} = r,$$
$$t_{12} - c_1t_{22} = -r_1,$$
$$t_{13} - c_1t_{23} = 0,$$
$$t_{11} - d_1t_{21} - d_2t_{31} = r,$$
$$t_{12} - d_1t_{22} - d_2t_{32} = -r_2,$$
$$t_{13} - d_1t_{23} - d_2t_{33} = 0.$$

Solving this system, we find:

$$t_{13} = c_1t_{23}, \quad t_{33} = \rho t_{23}, \quad t_{23}^2 = \frac{1}{\Delta}, \quad \Delta = 1 - c_1^2 + \rho^2, \quad \rho = \frac{c_1 - d_1}{d_2};$$

$$t_{12} = \frac{c_1 r}{\Delta}, \quad t_{11} = r\frac{1+\rho^2}{\Delta}, \quad t_{31} = \frac{c_1 r\rho}{\Delta}, \quad r^2 = \frac{\Delta}{1+\rho^2};$$
$$t_{21} = 0, \quad t_{22} = -\frac{g\rho}{1+\rho^2}, \quad t_{32} = \frac{g}{1+\rho^2},$$
$$r_1 = -\frac{c_1\rho g}{1+\rho^2}, \quad g = \frac{r_2 - r_1}{d_2}, \quad g^2 = 1 + \rho^2.$$

The matrix T^{-1} looks like

$$T^{-1} = \begin{pmatrix} \dfrac{1}{r} & -\dfrac{c_1}{r(1+\rho^2)} & -\dfrac{c_1\rho}{r(1+\rho^2)} \\ 0 & -\dfrac{\rho}{g} & \dfrac{1}{g} \\ -\dfrac{c_1}{\Delta t_{23}} & \dfrac{1}{\Delta t_{23}} & \dfrac{\rho}{\Delta t_{23}} \end{pmatrix}.$$

Due to all the above stated, Problem III can be reformulated as

$$\{(\tau')^2 - \xi^2 - \eta^2 - (\zeta_1')^2 - (\zeta_2')^2 - \sum_{k=3}^{n} \zeta_k^2\}u = 0, \quad t > 0, \quad (x, y, \mathbf{z}) \in R_{++}^{n+2}; \qquad (3.1'')$$

$$\{r\tau' - a\xi - b\eta - r_1\zeta_1'\}u = 0, \quad t > 0, \quad x = 0, \quad (y, \mathbf{z}) \in R_+^{n+1}; \tag{3.2''}$$

$$\{r\tau' - \alpha\xi - \beta\eta - r_2\zeta_1'\}u = 0, \quad t > 0, \quad y = 0, \quad (x, \mathbf{z}) \in R_+^{n+1}. \tag{3.3''}$$

We carry out the Fourier transform with respect to the variables z_k, $k = \overline{3, n}$ in Problem III. Then we obtain

$$\{(\tau')^2 - \xi^2 - \eta^2 - (\zeta_1')^2 - (\zeta_2')^2 + 4\pi^2|\xi''|^2\}\hat{u} = 0, \quad t > 0, \quad (x, y, z_1, z_2) \in R_{++}^4; \tag{3.1'''}$$

$$\{r\tau' - a\xi - b\eta - r_1\zeta_1'\}\hat{u} = 0, \quad t > 0, \quad x = 0, \quad (y, z_1, z_2) \in R_+^3; \tag{3.2'''}$$

$$\{r\tau' - \alpha\eta - \beta\eta - r_2\zeta_1'\}\hat{u} = 0, \quad t > 0, \quad y = 0, \quad (x, z_1, z_2) \in R_+^3. \tag{3.3'''}$$

$$\hat{u} = \hat{\varphi}(x, y, z_1, z_2, \xi''), \quad \hat{u}_t = \hat{\psi}(x, y, z_1, z_2, \xi''), \quad (x, y, z_1, z_2) \in R_{++}^4. \tag{3.4'}$$

Here

$$\hat{u} = \hat{u}(t, x, y, z_1, z_2, \xi'') = \int\limits_{R^{n-2}} e^{-2\pi i(\mathbf{z}'', \xi'')} u(t, x, y, z_1, z_2, \mathbf{z}'') d\mathbf{z}''$$

is Fourier transform of the function u, $\xi'' = (\xi_3, \ldots, \xi_n)$, $\mathbf{z}'' = (z_3, \ldots, z_n)$.

For the vector

$$\mathbf{U} = \begin{pmatrix} \tau'\hat{u} \\ \xi\hat{u} \\ \eta\hat{u} \\ \zeta_1'\hat{u} \\ \zeta_2'\hat{u} \end{pmatrix}$$

we write down the following symmetric system (its validity can be easily verified by the straightforward calculations):

$$\{\widetilde{A}\tau' - B\xi - C\eta - \widetilde{C}_1\zeta_1' - \widetilde{C}_2\zeta_2'\}\mathbf{U} + 4\pi^2|\xi''|^2\hat{u}\mathcal{F} = 0, \tag{3.7}$$

or

$$\{A\tau - B\xi - C\eta - C_1\zeta_1 - C_2\zeta_2\}\mathbf{U} + 4\pi^2|\xi''|^2\hat{u}\mathcal{F} = 0, \tag{3.7'}$$

where

$$\widetilde{A} = \left(\begin{array}{c|c} A_0 & 0 \\ \hline 0 & k \end{array}\right), \quad B = \left(\begin{array}{c|c} B_0 & 0 \\ \hline 0 & -l \end{array}\right), \quad C = \left(\begin{array}{c|c} C_0 & 0 \\ \hline 0 & -m \end{array}\right),$$

$$\widetilde{C}_1 = \left(\begin{array}{c|c} D_0 & 0 \\ \hline 0 & -n \end{array}\right), \quad \widetilde{C}_2 = \left(\begin{array}{c|c} O_4 & F \\ \hline F^* & 0 \end{array}\right), \quad \mathcal{F} = \begin{pmatrix} F \\ 0 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} k & l & m & n \\ l & k & 0 & 0 \\ m & 0 & k & 0 \\ n & 0 & 0 & k \end{pmatrix}, \quad B_0 = \begin{pmatrix} l & k & 0 & 0 \\ k & l & m & n \\ 0 & m & -l & 0 \\ 0 & n & 0 & -l \end{pmatrix},$$

$$C_0 = \begin{pmatrix} m & 0 & k & 0 \\ 0 & -m & l & 0 \\ k & l & m & n \\ 0 & 0 & n & m \end{pmatrix}, \quad D_0 = \begin{pmatrix} n & 0 & 0 & k \\ 0 & -n & 0 & l \\ 0 & 0 & -n & m \\ k & l & m & n \end{pmatrix},$$

$$F = \begin{pmatrix} k \\ l \\ m \\ n \end{pmatrix}, \quad A = \frac{1}{r}\widetilde{A} + \frac{c_1}{\Delta t_{23}}\widetilde{C}_2,$$

$$C_1 = \frac{c_1}{r(1+\rho^2)}\widetilde{A} - \frac{\rho}{g}\widetilde{C}_1 + \frac{1}{\Delta t_{23}}\widetilde{C}_2, \quad C_2 = \frac{c_1\rho}{r(1+\rho^2)}\widetilde{A} + \frac{1}{g}\widetilde{C}_1 + \frac{\rho}{\Delta t_{23}}\widetilde{C}_2;$$

$k, l, m, n,$ are real constants. Note that the matrix $A > 0$ if $k > 0$, $k^2 - l^2 - -m^2 - n^2 > 0$ and $r = \sqrt{\frac{\Delta}{1+\rho^2}}$, $r > 0$. Rewrite system (3.7′) as

$$\{A\tau - B\xi - C\eta - C_1\zeta_1 - C_2\zeta_2\}\bar{\mathbf{U}} + 4\pi^2|\xi''|^2\bar{\hat{u}}\mathcal{F} = 0. \tag{3.7″}$$

Multiply scalarly system (3.7′) by $\bar{\mathbf{U}}$, and system (3.7″) by U, and sum up the expressions:

$$(\bar{\mathbf{U}}, A\mathbf{U})_t - (\bar{\mathbf{U}}, B\mathbf{U})_x - (\bar{\mathbf{U}}, C\mathbf{U})_y - (\bar{\mathbf{U}}, C_1\mathbf{U})_{z_1} - (\bar{\mathbf{U}}, C_2\mathbf{U})_{z_2} +$$
$$+4\pi^2|\xi''|^2\Big\{\frac{k}{r}\Big[(|\hat{u}|^2)_t - \frac{c_1}{1+\rho^2}(|\hat{u}|^2)_{z_1} - \frac{c_1\rho}{1+\rho^2}(|\hat{u}|^2)_{z_2}\Big] +$$
$$+l(|\hat{u}|^2)_x + m(|\hat{u}|^2)_y + \frac{n}{g}[-\rho(|\hat{u}|^2)_{z_1} + (|\hat{u}|^2)_{z_2}]\Big\} = 0. \tag{3.8}$$

Integrate (3.8) over the domain R^4_{++}, assuming that $|\mathbf{U}|^2 \to 0$, $|\hat{u}|^2 \to 0$ at $\tilde{r} \to \infty$, where $\tilde{r} = \sqrt{x^2 + y^2 + z_1^2 + z_2^2}$. In the end, we obtain

$$\frac{d}{dt}\hat{J}_1(t) + \iiint\limits_{R^3_+} \{(\bar{\mathbf{U}}, B\mathbf{U}) - 4\pi^2|\xi''|^2 l|\hat{u}|^2\}\big|_{x=0} dy dz_1 dz_2 +$$
$$+ \iiint\limits_{R^3_+} \{(\bar{\mathbf{U}}, C\mathbf{U}) - 4\pi^2|\xi''|^2 m|\hat{u}|^2\}\big|_{y=0} dx dz_1 dz_2 = 0. \tag{3.9}$$

Here

$$\hat{J}_1(t) = \int\limits_{R^4_{++}} \Big\{(\bar{\mathbf{U}}, A\mathbf{U}) + \frac{k}{r}4\pi^2|\xi''|^2|\hat{u}|^2\Big\}\big|_{x=0} dx dy dz_1 dz_2.$$

The form $(\bar{\mathbf{U}}, A\mathbf{U}) > 0$ if $k > 0$, $k^2 - l^2 - m^2 - n^2 > 0$. The form

$$(\bar{\mathbf{U}}, B\mathbf{U})\big|_{x=0} = (\bar{\mathbf{U}}', B_0\mathbf{U}')\big|_{x=0} - l|\zeta_2'\hat{u}|^2\big|_{x=0}.$$

Analogously, the form

$$(\bar{\mathbf{U}}, C\mathbf{U})\big|_{y=0} = (\bar{\mathbf{U}}', C_0\mathbf{U}')\big|_{y=0} - m|\zeta_2'\hat{u}|^2\big|_{y=0}.$$

Here

$$\mathbf{U}' = \begin{pmatrix} \tau'\hat{u} \\ \xi\hat{u} \\ \eta\hat{u} \\ \zeta_1'\hat{u} \end{pmatrix}.$$

In view of (3.2‴), (3.3‴), the conditions

$$(\bar{\mathbf{U}}', B_0\mathbf{U}')\big|_{x=0} > 0,$$

$$(\bar{\mathbf{U}}', C_0\mathbf{U}')\big|_{y=0} > 0$$

can be reformulated as the requirement of the positive definiteness for the following matrices:

$$\begin{pmatrix} 2k\hat{a}+l(\hat{a}^2+1) & \hat{b}(l\hat{a}+k)+m & \hat{c}(l\hat{a}+k)+n \\ \hat{b}(l\hat{a}+k)+m & -l(1-\hat{b}^2) & l\hat{b}\hat{c} \\ \hat{c}(l\hat{a}+k)+n & l\hat{b}\hat{c} & -l(1-\hat{c}^2) \end{pmatrix} > 0 \tag{3.10}$$

$$\hat{a}=\frac{a}{r}, \quad \hat{b}=\frac{b}{r}, \quad \hat{c}=\frac{r_1}{r},$$

$$\begin{pmatrix} -m(1-\hat{\beta}^2) & \hat{\beta}(m\hat{\alpha}+k)+l & m\hat{\beta}\hat{\gamma} \\ \hat{\beta}(m\hat{\alpha}+k)+l & 2k\hat{\alpha}+m(\hat{\alpha}^2+1) & \hat{\gamma}(m\hat{\alpha}+k)+n \\ m\hat{\beta}\hat{\gamma} & \hat{\gamma}(m\hat{\alpha}+k)+n & -m(1-\hat{\gamma}^2) \end{pmatrix} > 0 \tag{3.11}$$

$$\hat{\alpha}=\frac{\alpha}{r}, \quad \hat{\beta}=\frac{\beta}{r}, \quad \hat{\gamma}=\frac{r_2}{r}.$$

Besides, we require

$$k>0, \quad m<0, \quad l<0, \quad k^2-m^2-l^2-n^2>0. \tag{3.12}$$

Then with regard to (3.10), (3.11), (3.12) it follows from (3.9) that

$$\frac{d}{dt}\hat{J}_1(t) \le 0. \tag{3.13}$$

Further reasonings are analogous to that at the end of the sections 1, 2. Actually, since

$$(\bar{\mathbf{U}}, A\mathbf{U}) = (\bar{\mathbf{W}}, \hat{A}\mathbf{W}),$$

where

$$\mathbf{U}=\Gamma\mathbf{W}, \quad \hat{A}=\Gamma^* A\Gamma,$$

$$\Gamma = \begin{pmatrix} \frac{1}{r} & 0 & 0 & -\frac{c_1}{r(1+\rho^2)} & -\frac{c_1\rho}{r(1+\rho^2)} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{\rho}{g} & \frac{1}{g} \\ -\frac{c_1}{\Delta t_{23}} & 0 & 0 & \frac{1}{\Delta t_{23}} & \frac{\rho}{\Delta t_{23}} \end{pmatrix},$$

$$\mathbf{W} = \begin{pmatrix} \tau\hat{u} \\ \xi\hat{u} \\ \eta\hat{u} \\ \zeta_1\hat{u} \\ \zeta_2\hat{u} \end{pmatrix},$$

then

$$\lambda_{\min}(\hat{A})(\bar{\mathbf{W}}, \mathbf{W}) \le (\bar{\mathbf{U}}, A\mathbf{U}) \le \lambda_{\max}(\hat{A})(\bar{\mathbf{W}}, \mathbf{W}). \tag{3.14}$$

Since

$$\frac{d}{dt}\left\{\int\limits_{R^4_{++}} |\hat{u}|^2 dx dy dz_1 dz_2\right\} \le \hat{J}_0(t), \tag{3.15}$$

summing up (3.13), (3.15) and using (3.14), we finally obtain

$$\frac{d}{dt}\Big\{\hat{J}_1(t)+\int\limits_{R^4_{++}}|\hat{u}|^2dxdydz_1dz_2\Big\}\le\hat{J}_0(t)\le$$

$$\le M_5\Big\{\hat{J}_1(t)+\int\limits_{R^4_{++}}|\hat{u}|^2dxdydz_1dz_2\Big\}. \tag{3.16}$$

Here

$$\hat{J}_0(t)=\int\limits_{R^4_{++}}\{|\mathbf{W}|^2+4\pi^2|\hat{u}|^2|\xi''|^2+|\hat{u}|^2\}dxdydz_1dz_2,$$

$$M_5=\max\Big\{1,\frac{1}{\lambda_{\min}(\hat{A})},\frac{k}{r}\Big\}.$$

From (3.16) it follows that

$$\hat{J}_0(t)\le M_5M_6\hat{J}_0(0)e^{M_5t},\quad t>0, \tag{3.17}$$

where $M_6=\max\{1,\lambda_{\max}(\hat{A}),\frac{k}{r}\}$. Regarding the Parceval equality, from (3.17) we obtain in the end the desired a priori estimate for the solutions of Problem III from (1.16). Remark 1.4 from section 1 remains true.

Remark 3.3. A priori estimate (3.17) was obtained under conditions that matrices (3.10), (3.11) are positive definite and unequalities (3.12) hold. It has not been shown yet that ULC (3.5), (3.6) are sufficient for the existance of such real numbers k,l,m,n that (3.12) is true and the matrices (3.10), (3.11) are positive definite. The special examples have been considered in the diploma thesis of A. A. Beljaev, a student of the Novosibirsk University.

4. Mixed problem for the vector wave equation

In this section, the mixed problem for the wave equation in the domain $t>0$, $(x,y,\mathbf{z})\in R^{n+2}_+$, $n\ge1$ is briefly discussed (the case $n=0$ has been considered in [18]):

$$\mathcal{L}(\tau,\xi,\eta,\zeta_1,\ \ldots,\ \zeta_n)\mathbf{U}=\mathbf{U}_{tt}-\mathbf{U}_{xx}-\mathbf{U}_{yy}-\Delta_z\mathbf{U}=0 \text{ at } t>0,\ x>0; \tag{4.1}$$

$$J_1\mathbf{U}_t-A_1\mathbf{U}_x-B_1\mathbf{U}_y=0,\ \text{at } x=0; \tag{4.2}$$

$$\mathbf{U}=\mathbf{\Phi}(x,y,\mathbf{z}),\ \mathbf{U}_t=\mathbf{\Psi}(x,y,\mathbf{z}) \text{ at } t=0. \tag{4.3}$$

Here $J_1,\ A_1,\ B_1$ are the constant complex matrices of order N.

Remark 4.1. We will assume that for problem (4.1)–(4.3) ULC holds. In terms of the coefficients of boundary conditions (4.2) ULC can be written as follows (see [18]):

a) the matrix J_1+B_1 is not degenerate,

b) all the eigen-values of the matrix

$$S=\left(\begin{array}{c|c}-S_1 & -S_2\\ \hline I_N & O_N\end{array}\right)$$

lie strictly in the left semi-plane, i.e.

$$\mathrm{Re}\lambda_i(S) < 0, \quad i = \overline{1, 2N}.$$

Here $S_1 = 2(J_1 + B_1)^{-1} A_1$, $S_2 = (J_1 + B_1)^{-1}(J_1 - B_1)$. Note that the characteristic equation for S

$$\det(S - \lambda I_{2N}) = 0$$

can be written as

$$\det(\lambda^2 I_N + \lambda S_1 + S_2) = 0.$$

In problem (4.1)–(4.3) we make the Fourier transform with respect to the variables z_k, $k = \overline{1, n}$. In the end, we obtain:

$$\widehat{\mathbf{U}}_{tt} - \widehat{\mathbf{U}}_{xx} - \widehat{\mathbf{U}}_{yy} + 4\pi^2|\xi|^2\widehat{\mathbf{U}} = 0, \quad t > 0,\ (x, y) \in R^2_+, \tag{4.1$'$}$$

$$J_1\widehat{\mathbf{U}}_t - A_1\widehat{\mathbf{U}}_x - B_1\widehat{\mathbf{U}}_y = 0, \quad t > 0,\ y \in R^1,\ x = 0, \tag{4.2$'$}$$

$$\widehat{\mathbf{U}} = \widehat{\mathbf{\Phi}}(x, y, \xi),\ \widehat{\mathbf{U}}_t = \widehat{\mathbf{\Psi}}(x, y, \xi), \quad (x, y) \in R^2_+. \tag{4.3$'$}$$

For the vector

$$\mathbf{W} = \begin{pmatrix} \widehat{\mathbf{U}}_t \\ \widehat{\mathbf{U}}_x \\ \widehat{\mathbf{U}}_y \end{pmatrix}$$

we write the symmetric system

$$\{A_0\tau - B_0\xi - C_0\eta\}\mathbf{W} + 4\pi^2|\xi|^2\mathbf{F}\widehat{\mathbf{U}} = 0. \tag{4.4}$$

Here

$$A_0 = \begin{pmatrix} K & L & M \\ L & K & i\mathcal{N} \\ M & -i\mathcal{N} & K \end{pmatrix} = T_0^* \cdot \begin{pmatrix} H & O_{2N} \\ O_{2N} & H \end{pmatrix} \cdot T_0,$$

$$B_0 = \begin{pmatrix} L & K & i\mathcal{N} \\ K & L & M \\ -i\mathcal{N} & M & -L \end{pmatrix} = T_0^* \cdot \begin{pmatrix} O_{2N} & -H \\ -H & O_{2N} \end{pmatrix} \cdot T_0,$$

$$C_0 = \begin{pmatrix} M & -i\mathcal{N} & K \\ i\mathcal{N} & -M & L \\ K & L & M \end{pmatrix} = T_0^* \cdot \begin{pmatrix} -H & O_{2N} \\ O_{2N} & H \end{pmatrix} \cdot T_0,$$

$$\mathbf{F} = \begin{pmatrix} K \\ L \\ M \end{pmatrix}, \quad T_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} I_N & O_N & -I_N \\ O_N & -I_N & O_N \\ O_N & -I_N & O_N \\ I_N & O_N & I_N \end{pmatrix},$$

$$H = \begin{pmatrix} K - M & -L - i\mathcal{N} \\ -L + i\mathcal{N} & K + M \end{pmatrix},$$

$K, L, M, \mathcal{N}$ are some arbitrary for the present Hermitian matrices of order N. System (4.4) can also be written in the form

$$\{\bar{A}_0\tau - \bar{B}_0\xi - \bar{C}_0\eta\}\bar{\mathbf{W}} + 4\pi^2|\xi|^2\mathbf{F}\bar{\widehat{\mathbf{U}}} = 0. \tag{4.4$'$}$$

Let multiply scalarly system (4.4) by $\bar{\mathbf{W}}$ and system (4.4′) by $\mathbf{W}$. Sum up the expressions obtained. In the end, we have:

$$(\bar{\mathbf{W}}, A_0\mathbf{W})_t - (\bar{\mathbf{W}}, B_0\mathbf{W})_x - (\bar{\mathbf{W}}, C_0\mathbf{W})_y +$$

$$+4\pi^2|\xi|^2\{(\hat{\bar{\mathbf{U}}}, K\hat{\mathbf{U}})_t + (\hat{\bar{\mathbf{U}}}, L\hat{\mathbf{U}})_x + (\hat{\bar{\mathbf{U}}}, M\hat{\mathbf{U}})_y\} = 0. \tag{4.5}$$

Integrate (4.5) over R^2_+, assuming that $|\mathbf{W}|^2 = (\bar{\mathbf{W}}, \mathbf{W}) \to 0$, $|\widehat{\mathbf{U}}|^2 \to 0$ as $r \to \infty$ where $r = \sqrt{x^2+y^2}$. Thus, we obtain

$$\frac{d}{dt}\hat{J}_1(t) + \int\limits_{R^1} \{(\bar{\mathbf{W}}, B_0\mathbf{W}) - 4\pi^2|\xi|^2(\hat{\bar{\mathbf{U}}}, L\hat{\mathbf{U}})\}\Big|_{x=0} dy = 0. \tag{4.6}$$

Here

$$\hat{J}_1(t) = \iint\limits_{R^2_+} \{(\bar{\mathbf{W}}, A_0\mathbf{W}) + 4\pi^2|\xi|^2(\hat{\bar{\mathbf{U}}}, K\hat{\mathbf{U}})\} dxdy.$$

Now we consider the forms $(\bar{\mathbf{W}}, A_0\mathbf{W})$ and $(\bar{\mathbf{W}}, B_0\mathbf{W})\big|_{x=0}$. The form

$$(\bar{\mathbf{W}}, A_0\mathbf{W}) = \left(\bar{\mathbf{V}}, \begin{pmatrix} H & O_{2N} \\ O_{2N} & H \end{pmatrix} \cdot \mathbf{V}\right) =$$

$$= (\bar{\mathbf{V}}^I, H\cdot\mathbf{V}^I) + (\bar{\mathbf{V}}^{II}, H\cdot\mathbf{V}^{II}) > 0,$$

if $H > 0$. Here

$$\mathbf{V} = T_0 \cdot \mathbf{W} = \begin{pmatrix} \mathbf{V}^I \\ \mathbf{V}^{II} \end{pmatrix},$$

$$\mathbf{V}^I = \frac{1}{\sqrt{2}}\begin{pmatrix} \widehat{\mathbf{U}}_t - \widehat{\mathbf{U}}_y \\ -\widehat{\mathbf{U}}_x \end{pmatrix}, \quad \mathbf{V}^{II} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\widehat{\mathbf{U}}_x \\ \widehat{\mathbf{U}}_t + \widehat{\mathbf{U}}_y \end{pmatrix}.$$

We rewrite boundary conditions (4.2′) as:

$$\mathbf{V}^I = S\cdot\mathbf{V}^{II}, \quad x = 0, \tag{4.2″}$$

The form

$$(\bar{\mathbf{W}}, B_0\mathbf{W})\big|_{x=0} = \left(\bar{\mathbf{V}}, \begin{pmatrix} O_{2N} & -H \\ -H & O_{2N} \end{pmatrix} \cdot \mathbf{V}\right)\Big|_{x=0} =$$

$$= -(\bar{\mathbf{V}}^I, H\cdot\mathbf{V}^{II})\big|_{x=0} - (\bar{\mathbf{V}}^{II}, H\cdot\mathbf{V}^I)\big|_{x=0} = -(\bar{\mathbf{V}}^{II}, [S^*H + HS]\cdot\mathbf{V}^{II})\big|_{x=0}.$$

Let

$$S^*H + HS = -G,$$

where $G = G^* > 0$ is a certain matrix. Then

$$(\bar{\mathbf{W}}, B_0\mathbf{W})\big|_{x=0} = (\bar{\mathbf{V}}^{II}, G\cdot\mathbf{V}^{II})\big|_{x=0} > 0. \tag{4.7}$$

Now we assume that

$$L < 0. \tag{4.8}$$

Then from (4.6), in view of (4.7), (4.8), we obtain

$$\frac{d}{dt}\hat{J}_1(t) \le 0. \tag{4.9}$$

Since

$$\frac{d}{dt}\left\{\iint\limits_{R^2_+} (\bar{\widehat{\mathbf{U}}}, \widehat{\mathbf{U}})dxdy\right\} \le \hat{J}_0(t), \tag{4.10}$$

summing up (4.9) and (4.10), in the end we obtain

$$\frac{d}{dt}\left\{\hat{J}_1(t) + \iint\limits_{R^2_+} |\widehat{\mathbf{U}}|^2 dxdy\right\} \le \hat{J}_0(t) \le M_7 \cdot \left\{\hat{J}_1(t) + \iint\limits_{R^2_+} |\widehat{\mathbf{U}}|^2 dxdy\right\}. \tag{4.11}$$

Here

$$\hat{J}_0(t) = \iint\limits_{R^2_+} \{|\mathbf{W}|^2 + (4\pi^2|\xi|^2 + 1)|\widehat{\mathbf{U}}|^2\}dxdy,$$

$$M_7 = \max\left\{1, \frac{1}{\lambda_{\min}(A_0)}, \frac{1}{\lambda_{\min}(K)}\right\}.$$

(4.11) yields:

$$\hat{J}_0(t) \le M_7 M_8 \hat{J}_0(0) e^{M_7 t}, \quad t > 0, \tag{4.12}$$

where $M_8 = \max\{1, \lambda_{\max}(A_0)\}$. With regard to the Parceval equality, from (4.12) we obtain the desired a priori estimate for the solutions of problem (4.1)–(4.3):

$$\|\mathbf{U}(t)\|^2_{W^1_2(R^{n+2}_+)} + \|\mathbf{U}_t(t)\|^2_{L_2(R^{n+2}_+)} \le$$

$$\le M_7 M_8 e^{M_7 t}\big\{\|\mathbf{\Phi}\|^2_{W^1_2(R^{n+2}_+)} + \|\mathbf{\Psi}_t\|^2_{L_2(R^{n+2}_+)}\big\}, \quad t > 0.$$

Now we discuss the question of validity of equality (4.8). By virtue of the Lapunov theorem (see [25]), ULC being true (see the Remark 4.1), the matrix equation

$$S^* H + HS = -G \tag{4.13}$$

has the unique solution $H = H^* > 0$ for any right-hand side $G = G^* > 0$. Here

$$G = \begin{pmatrix} G_1 & G_2 \\ G_2^* & G_3 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & H_2 \\ H_2^* & H_3 \end{pmatrix},$$

$$G_{1,3} = G^*_{1,3} > 0, \quad H_{1,3} = H^*_{1,3} > 0,$$

and

$$K = K^* = \frac{1}{2}(H_1 + H_3),$$

$$M = M^* = \frac{1}{2}(H_3 - H_1),$$

$$L = L^* = -\frac{1}{2}(H_2 + H_2^*),$$

$$\mathcal{N} = \mathcal{N}^* = -\frac{i}{2}(H_2^* - H_2).$$

Rewrite the matrix equation (4.13) as

$$H_1S_1 + S_1^*H_1 = G_1 + H_2 + H_2^*,$$

$$H_2^*S_2 + S_2^*H_2 = G_3,$$
$$H_3 - H_1S_2 - S_1^*H_2 = -G_2,$$
$$H_3 - S_2^*H_1 - H_2^*S_1 = -G_2^*. \qquad (4.13')$$

On the whole, the question of validity of (4.8) remains open. However, it is possible to point out such boundary conditions (4.2) for which this question can be easily solved. Indeed, let $B_1 = O_N$. Then $S_2 = I_N$, and second subsystem (4.13′) can be written as

$$H_2^* + H_2 = G_3,$$

i.e.

$$L = -\frac{1}{2}(H_2 + H_2^*) = -\frac{1}{2}G_3 < 0.$$

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