RICHARDSON EXTRAPOLATION FOR EIGENVALUE OF DISCRETE SPECTRAL PROBLEM ON GENERAL MESH*

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Использовано разложение собственного числа и экстраполяция Ричардсона для улучшения аппроксимации первого собственного числа в спектральной проблеме. К тому же выведенное разложение собственного числа не зависит от триангуляции. Это позволяет доказать эффективность экстраполяции Ричардсона для произвольной триангуляции.

Introduction

It is well known that the extrapolation method, which was established by Richardson in 1926, is an efficient procedure for improving an accuracy of a solution for many problems in numerical analysis. The effectiveness of this technique relies heavily on the existence of an asymptotic expansion for the error. The application of this approach in the finite difference method can be found in the book of Marchuk and Shaidurov [1]. This technique has been well demonstrated in the framework of the finite element method [2–8, 9].

An application of the extrapolation method to the eigenvalue problem was first proposed by Q. Lin and T. Lü [3], and was analyzed in [2–5, 7].

Usually in the finite element method, we first need to get an error expansion for a solution approximation such as [4, 5, 8]

$$u_h(x) - u_I(x) = c_1(u)h^{k_1} + O(h^{k_1+\delta_1}) \tag{1}$$

or for an eigenvalue approximation [2, 3, 10]

$$\lambda_h - \lambda = c_2(u)h^{k_2} + O(h^{k_2+\delta_2}), \tag{2}$$

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where c_1, c_2 are independent of h, $\delta_1 > 0$ and $\delta_2 > 0$. Then, we can use the extrapolation method.

Our final goal is to get higher order convergence. In this paper, we directly analyze the effectiveness of the eigenvalue extrapolation for the general mesh.

For simplicity, we consider the following eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega, \tag{3}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{4}$$

$$\int_\Omega u^2 dxdy = 1, \tag{5}$$

where Ω is a convex polygonal domain in $\mathbf{R}^2$. The equations (3)–(5) can be written in a weak formulation:

to seek $(\lambda, u) \in \mathbf{R} \times H_0^1(\Omega)$ *such that* $(u, u) = 1$ *and*

$$a(u, v) = \lambda(u, v) \qquad \forall v \in H_0^1(\Omega), \tag{6}$$

where $H_0^1(\Omega) = \{v | v \in H^1(\Omega), v|_{\partial\Omega} = 0\}$,

$$a(u, v) = \int_\Omega \nabla u \nabla v, \tag{7}$$

$$(u, v) = \int_\Omega uv. \tag{8}$$

Note that eigenvalues satisfy the following properties:

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Let $\mathbf{T}_h$ be a consistent triangulation of the domain Ω which satisfies the following quasi-uniform condition:

$$\exists\sigma > 0 \text{ such that } \; h_e/\rho_e < \sigma, \quad \forall e \in \mathbf{T}_h$$

and

$$\exists\gamma > 0, \text{ such that } \; \max\{h/h_e, \; e \in \mathbf{T}_h\} \le \gamma,$$

where h_e is the diameter of e; ρ_e is the maximum diameter of the inscribed circle in e; and $h = \max\{h_e, e \in \mathbf{T}_h\}$.

The linear finite element space V_h on $\mathbf{T}_h$ is defined as follows:

$$V_h = \{v \in H^1(\Omega), v|_e \in P_1 \quad \forall e \in \mathbf{T}_h\} \cap H_0^1(\Omega),$$

where $P_1 = \text{span}\{1, x, y\}$. If $u \in H^2(\Omega)$, then the interpolation u_I on $e \in \mathbf{T}_h$ is defined by equalities

$$u_I(\mathbf{p}_i) = u(\mathbf{p}_i), \quad i = 1, 2, 3,$$

where $\mathbf{p}_i$ are three vertices of the element e.

The corresponding discrete finite element equation is:

to seek $(\lambda_h, u_h) \in \mathbf{R} \times V_h$ *such that* $(u_h, u_h) = 1$ *and*

$$a(u_h, v) = \lambda_h(u_h, v) \quad \forall v \in V_h. \tag{9}$$

We also need to define the finite element projection $R_h u$ as

$$a(R_h u, v) = a(u, v) \qquad \forall v \in V_h. \tag{10}$$

It is known about the convergence rate that [11–13]

$$|\lambda_h - \lambda| + \|u_h - u\|_0 + \|R_h u - u\|_0 \leq ch^2, \tag{11}$$

where $\|\cdot\|_0$ denotes the L^2-norm.

Other notations for Sobolev spaces and the corresponding norms (including those with a fractional order) are standard and can be found in many sources like [14].

The rest of the paper is organized in the following way. In section 2 we give some useful preliminary lemmas. An eigenvalue expansion is obtained in section 3. Section 4 is devoted to eigenvalue extrapolation and analysis of its effectiveness. Two numerical examples are given to illustrate the validity of our analysis.

1. Some useful notations and preliminary lemmas

We first need to define some notations and give some geometric identities for an arbitrary element e. Let e have vertices $\mathbf{p}_i = (x_i, y_i)$ $(1 \leq i \leq 3)$ oriented counterclockwise. Let s_i $(1 \leq i \leq 3)$ denote the edges of the element e; $\mathbf{n}_i$ $(1 \leq i \leq 3)$ are the unit outward normal vectors; $\mathbf{t}_i = (\cos\theta_i, \sin\theta_i)$ $(1 \leq i \leq 3)$ are the unit tangent vectors with the counterclockwise orientation, and θ_i are its corresponding angles with the x-axis; h_i $(1 \leq i \leq 3)$ are the edge lengths; H_i $(1 \leq i \leq 3)$ are the perpendicular heights (see Fig. 1). We also need to define the following constants of the element e:

$$l_i = h_i/h, \quad i = 1, 2, 3, \qquad \alpha = |e|/h^2.$$

We also use the periodic relation for the subscripts: $i + 3 = i$. Let $\partial_i = \partial/\partial\mathbf{t}_i$.

Now we give some lemmas. Similar constructions can been found in some papers (see [2] and references in it), but are poorly known in Russian literature.

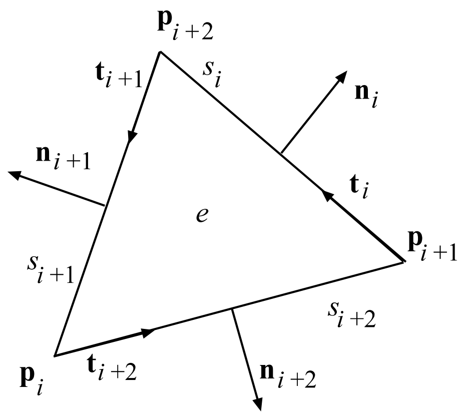

Fig. 1. The main features of an element e

Lemma 1.

$$\begin{aligned}\mathbf{t}_i\cdot\mathbf{n}_{i+1} &= \frac{2|e|}{h_ih_{i+1}}, &(12)\\ \mathbf{n}_i\cdot\mathbf{t}_{i+1} &= -\frac{2|e|}{h_ih_{i+1}}, &(13)\\ \mathbf{n}_i &= \frac{h_ih_{i+1}}{2|e|}[(\mathbf{n}_i\cdot\mathbf{n}_{i+1})\mathbf{t}_i-\mathbf{t}_{i+1}],\quad i=1,2,3. &(14)\end{aligned}$$

Proof. First, we have

$$\frac{1}{2}h_iH_i=|e|,\quad (h_i\mathbf{t}_i)\cdot\mathbf{n}_{i+1}=H_{i+1},$$

then

$$\mathbf{t}_i\cdot\mathbf{n}_{i+1}=\frac{1}{h_i}H_{i+1}=\frac{2|e|}{h_ih_{i+1}}.$$

So, we obtain (12). Similarly we can obtain (13).

Since $\mathbf{t}_i$ and $\mathbf{t}_{i+1}$ are two linearly independent vectors, we have two constants β_i and β_{i+1} such that

$$\mathbf{n}_i=\beta_i\mathbf{t}_i+\beta_{i+1}\mathbf{t}_{i+1}.$$

Using the equality $\mathbf{t}_{i+1}\cdot\mathbf{n}_{i+1}=0$ and (12), we have

$$\mathbf{n}_i\cdot\mathbf{n}_{i+1}=\beta_i\mathbf{t}_i\cdot\mathbf{n}_{i+1}=\frac{2\beta_i|e|}{h_ih_{i+1}}.$$

So,

$$\beta_i=\frac{h_ih_{i+1}}{2|e|}\mathbf{n}_i\cdot\mathbf{n}_{i+1}.$$

Similarly, using the equality $\mathbf{t}_i\cdot\mathbf{n}_i=0$ and (13), we have $\beta_{i+1}=-h_ih_{i+1}/2|e|$. This completes the proof. □

Using (14), we can get the following differential property.

Lemma 2.

$$\frac{\partial v}{\partial\mathbf{n}_i} = \frac{l_il_{i+1}}{2\alpha}[(\mathbf{n}_i\cdot\mathbf{n}_{i+1})\partial_iv-\partial_{i+1}v]. \tag{15}$$

Proof. From (14) we have the equality

$$\begin{aligned}\frac{\partial v}{\partial\mathbf{n}_i}=\nabla v\cdot\mathbf{n}_i &= \frac{l_il_{i+1}}{2\alpha}\nabla v\cdot[(\mathbf{n}_i\cdot\mathbf{n}_{i+1})\mathbf{t}_i-\mathbf{t}_{i+1}]=\\ &= \frac{l_il_{i+1}}{2\alpha}[(\mathbf{n}_i\cdot\mathbf{n}_{i+1})\partial_iv-\partial_{i+1}v].\end{aligned}$$

□

We also need the following integration formula.

Lemma 3. *Assume that $v\in C^1(\bar{e})$, then we have*

$$h_{i+1}\int\limits_{s_i}vds-h_i\int\limits_{s_{i+1}}vds=\frac{h_1h_2h_3}{2|e|}\int\limits_e\partial_{i+2}vdxdy. \tag{16}$$

Proof. With the Green formula, we have

$$\int_e \partial_{i+2} v dx dy = \int_{\partial e} v \mathbf{t}_{i+2} \cdot \mathbf{n} ds,$$

where ∂e is the boundary of the element e. Using the equality $\mathbf{t}_{i+2} \cdot \mathbf{n}_{i+2} = 0$, (12), and (13), we have

$$\begin{aligned} \int_e \partial_{i+2} v dx dy &= \int_{s_i} v \mathbf{t}_{i+2} \cdot \mathbf{n}_i ds + \int_{s_{i+1}} v \mathbf{t}_{i+2} \cdot \mathbf{n}_{i+1} ds = \\ &= \frac{2h_{i+1}|e|}{h_1 h_2 h_3} \int_{s_i} v ds - \frac{2h_i|e|}{h_1 h_2 h_3} \int_{s_{i+1}} v ds. \end{aligned}$$

Multiplying this by $h_1 h_2 h_3 / 2|e|$, we obtain (16). □

2. Eigenvalue expansion

In this section, we give the eigenvalue error expansion which is independent of a triangulation.

We use the linear finite elements to approximate the eigenvalue problem. Then we have the following eigenvalue error transform formula [2, 10]:

$$\lambda_h - \lambda = \lambda(u - u_I, u_h) - a(u - u_I, R_h u) + O(h^4). \tag{17}$$

So, in order to get the eigenvalue error expansion, we just need to compute the terms $(u - u_I, u_h)$ and $a(u - u_I, R_h u)$.

First, we need the following one-dimensional interpolation expansion which is derived by combining the Bramble-Hilbert Lemma with scaling argument [2].

Lemma 4. *Let u_I be the linear interpolant of u on e and s_i be an edge of the element e. Assume that $u \in H^4(s_i)$. Then we have*

$$\int_{s_i} (u - u_I) ds = -\frac{h_i^2}{12} \int_{s_i} \partial_i^2 u ds + O(h^4)|u|_{4,s_i}. \tag{18}$$

Proof. Let $\hat{s} = [0, 1]$ be the reference edge and define the affine transformation F from s_i to $\hat{s}$. Define the functions $\hat{u}(\hat{x}) = u(x)$, $\hat{u}_I(\hat{x}) = u_I(x)$.

Consider the following linear functional on $\hat{s}$

$$B(\hat{u}) = \int_{\hat{s}} (\hat{u} - \hat{u}_I) d\hat{s} + \frac{1}{12} \int_{\hat{s}} \partial_{\hat{x}}^2 \hat{u} d\hat{s}.$$

By the Sobolev embedding theorem, we know that the functional B is bounded:

$$|B(\hat{u})| \le C \|\hat{u}\|_{4,\hat{s}}.$$

A direct computation shows that

$$B(\hat{u}) = 0 \quad \forall \hat{u} \in P_3(\hat{s}).$$

Then the Bramble-Hilbert lemma gives

$$|B(\hat{u})| \leq C|\hat{u}|_{4,\hat{s}}.$$

With the inverse map of F, we obtain (18). □

Now, let's consider the interpolation error expansion of $a(u - u_I, R_h u)$.

Theorem 1. *Let u_I be the piecewise linear interpolant of u. If $u \in H^{4.5}(\Omega)$, we have the following expansion:*

$$a(u - u_I, v) = -\frac{h^2}{12} W(u, v, \mathbf{T}_h) + \frac{h^2}{12} K(u, v, \mathbf{T}_h) + O(h^3)\|u\|_{4.5}\|v\|_1, \tag{19}$$

where

$$\begin{aligned} W(u, v, \mathbf{T}_h) &= \sum_{e \in \mathbf{T}_h} \sum_{i=1}^{3} l_i^3 \frac{l_{i+1}(\mathbf{n}_i \cdot \mathbf{n}_{i+1})}{2\alpha} \Big((\partial_i^2 uv)(\mathbf{p}_{i+2}) - (\partial_i^2 uv)(\mathbf{p}_{i+1}) \Big) - \\ &\quad - \sum_{e \in \mathbf{T}_h} \sum_{i=1}^{3} \frac{l_{i+2}^4}{2\alpha} \Big((\partial_{i+2}^2 uv)(\mathbf{p}_{i+2}) - (\partial_{i+2}^2 uv)(\mathbf{p}_{i+1}) \Big), \end{aligned} \tag{20}$$

$$\begin{aligned} K(u, v, \mathbf{T}_h) &= \sum_{e \in \mathbf{T}_h} \sum_{i=1}^{3} l_i^3 \frac{l_{i+1}(\mathbf{n}_i \cdot \mathbf{n}_{i+1})}{2\alpha} \int_{s_i} \partial_i^3 uv ds - \sum_{e \in \mathbf{T}_h} \sum_{i=1}^{3} \frac{l_i^4}{2\alpha} \int_{s_{i+1}} \partial_i^2 \partial_{i+1} uv ds + \\ &\quad + \sum_{e \in \mathbf{T}_h} \sum_{i=1}^{3} l_i^3 \frac{l_1 l_2 l_3}{(2\alpha)^2} \int_e \partial_{i+2} \partial_i^2 u \partial_{i+1} v dxdy. \end{aligned} \tag{21}$$

Proof. We need the following inequality for the finite element space V_h and $e \in \mathbf{T}_h$:

$$|v_h|_{1,\partial e} \leq ch^{-1/2}\|v_h\|_{1,e}, \tag{22}$$

and the trace inequality

$$\|u\|_{0,\partial e} \leq ch^{-1/2}\|u\|_{1/2,e}, \quad \text{for } u \in H^{1/2}(e). \tag{23}$$

With the Green formula we have for $v \in V_h$ that

$$a(u - u_I, v) = \sum_{e \in \mathbf{T}_h} \int_e \nabla(u - u_I)\nabla v dxdy = \sum_{e \in \mathbf{T}_h} \sum_{i=1}^{3} \int_{s_i} (u - u_I)\frac{\partial v}{\partial \mathbf{n}} ds.$$

From Lemma 4, we can obtain

$$\begin{aligned} a(u - u_I, v) &= -\frac{h^2}{12} \sum_{e \in \mathbf{T}_h} \sum_{i=1}^{3} l_i^2 \frac{l_i l_{i+1}}{2\alpha} \int_{s_i} \partial_i^2 u((\mathbf{n}_i \cdot \mathbf{n}_{i+1})\partial_i v - \partial_{i+1} v) + \\ &\quad + O(h^3)\|u\|_{4.5}\|v\|_1. \end{aligned} \tag{24}$$

From Lemma 3, we get

$$\int_{s_i} \partial_i^2 u \partial_{i+1} v ds = \frac{l_i}{l_{i+1}} \int_{s_{i+1}} \partial_i^2 u \partial_{i+1} v ds + \frac{l_i l_{i+2}}{2\alpha} \int_e \partial_{i+2} \partial_i^2 u \partial_{i+1} v dxdy.$$

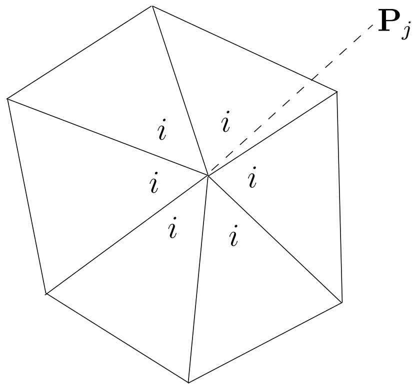

Fig. 2. The patch ω_j and the local numbers of $\mathbf{P}_j$ in each $e \in \omega_j$

Substituting it into (24), we obtain that

$$\begin{aligned} a(u-u_I,v) &= -\frac{h^2}{12}\sum_{e\in\mathbf{T}_h}\sum_{i=1}^{3} l_i^3\frac{l_{i+1}(\mathbf{n}_i\cdot\mathbf{n}_{i+1})}{2\alpha}\int\limits_{s_i}\partial_i^2 u\partial_i v ds + \\ &+\frac{h^2}{12}\sum_{e\in\mathbf{T}_h}\sum_{i=1}^{3}\frac{l_{i+2}^4}{2\alpha}\int\limits_{s_i}\partial_{i+2}^2 u\partial_i v ds + \\ &+\frac{h^2}{12}\sum_{e\in\mathbf{T}_h}\sum_{i=1}^{3} l_i^3\frac{l_1l_2l_3}{(2\alpha)^2}\int\limits_{e}\partial_{i+2}\partial_i^2 u\partial_{i+1} v dxdy + \\ &+O(h^3)\|u\|_{4.5}\|v\|_1. \end{aligned}$$

With the integration by parts on edge s_i, we obtain (19). □

Let N_h denote the set of vertices of the triangulation $\mathbf{T}_h$ and ω_j^h denote the patch around the node $\mathbf{P}_j$ (see Fig. 2). From (20) and the assumption that the local number of $\mathbf{P}_j$ in each triangle $e \in \omega_j^h$ is i (see Fig. 2), we have

$$\begin{aligned} W(u,v,\mathbf{T}_h) &= \sum_{e\in\mathbf{T}_h}\sum_{i=1}^{3} l_i^3\frac{l_{i+1}(\mathbf{n}_i\cdot\mathbf{n}_{i+1})}{2\alpha}\Big((\partial_i^2 uv)(\mathbf{p}_{i+2}) - (\partial_i^2 uv)(\mathbf{p}_{i+1})\Big) - \\ &- \sum_{e\in\mathbf{T}_h}\sum_{i=1}^{3}\frac{l_{i+2}^4}{2\alpha}\Big((\partial_{i+2}^2 uv)(\mathbf{p}_{i+2}) - (\partial_{i+2}^2 uv)(\mathbf{p}_{i+1})\Big) = \\ &= \sum_{e\in\mathbf{T}_h}\sum_{i=1}^{3}\Big(l_i^3\frac{l_{i+1}(\mathbf{n}_i\cdot\mathbf{n}_{i+1})}{2\alpha}(\partial_i^2 uv)(\mathbf{p}_{i+2}) - l_{i+1}^3\frac{l_{i+2}(\mathbf{n}_{i+1}\cdot\mathbf{n}_{i+2})}{2\alpha}(\partial_{i+1}^2 uv)(\mathbf{p}_{i+2})\Big) - \\ &- \sum_{e\in\mathbf{T}_h}\sum_{i=1}^{3}\Big(\frac{l_{i+2}^4}{2\alpha}(\partial_{i+2}^2 uv)(\mathbf{p}_{i+2}) - \frac{l_i^4}{2\alpha}(\partial_i^2 uv)(\mathbf{p}_{i+2})\Big) = \\ &= \sum_{e\in\mathbf{T}_h}\sum_{i=1}^{3}\Big(l_{i+1}^3\frac{l_{i+2}(\mathbf{n}_{i+1}\cdot\mathbf{n}_{i+2})}{2\alpha}\partial_{i+1}^2 u(\mathbf{p}_i) - l_{i+2}^3\frac{l_i(\mathbf{n}_{i+2}\cdot\mathbf{n}_i)}{2\alpha}\partial_{i+2}^2 u(\mathbf{p}_i)\Big)v(\mathbf{p}_i) - \end{aligned}$$

$$-\sum_{e\in\mathbf{T}_h}\sum_{i=1}^{3}\Big(\frac{l_i^4}{2\alpha}\partial_i^2 u(\mathbf{p}_i)-\frac{l_{i+1}^4}{2\alpha}\partial_{i+1}^2 u(\mathbf{p}_i)\Big)v(\mathbf{p}_i)=$$

$$=\sum_{\mathbf{P}_j\in N_h}\Big[\sum_{e\in\omega_j^h}\Big(l_{i+1}^3\frac{l_{i+2}(\mathbf{n}_{i+1}\cdot\mathbf{n}_{i+2})}{2\alpha}\partial_{i+1}^2 u(\mathbf{P}_j)-l_{i+2}^3\frac{l_i(\mathbf{n}_{i+2}\cdot\mathbf{n}_i)}{2\alpha}\partial_{i+2}^2 u(\mathbf{P}_j)\Big)\Big]v(\mathbf{P}_j)-$$

$$-\sum_{\mathbf{P}_j\in N_h}\Big[\sum_{e\in\omega_j^h}\Big(\frac{l_i^4}{2\alpha}\partial_i^2 u(\mathbf{P}_j)-\frac{l_{i+1}^4}{2\alpha}\partial_{i+1}^2 u(\mathbf{P}_j)\Big)\Big]v(\mathbf{P}_j). \tag{25}$$

Let $\mathbf{N}_i=(\cos^2\theta_i, 2\sin\theta_i\cos\theta_i, \sin^2\theta_i)$. Assume $\mathbf{T}_h$ has N nodes and let's define the matrices $\mathbf{Mes}(\mathbf{T}_h)\in\mathbf{R}^{N\times 3}$ and $\mathbf{d}_u\in\mathbf{R}^{N\times 3}$ as follows

$$\begin{aligned}\mathbf{Mes}(\mathbf{T}_h)(j,:) &= \sum_{e\in\omega_j^h}\Big(l_{i+1}^3\frac{l_{i+2}(\mathbf{n}_{i+1}\cdot\mathbf{n}_{i+2})}{2\alpha}\mathbf{N}_{i+1}-l_{i+2}^3\frac{l_i(\mathbf{n}_{i+2}\cdot\mathbf{n}_i)}{2\alpha}\mathbf{N}_{i+2}\Big)-\\ &\quad -\sum_{e\in\omega_j^h}\Big(\frac{l_i^4}{2\alpha}\mathbf{N}_i-\frac{l_{i+1}^4}{2\alpha}\mathbf{N}_{i+1}\Big),\end{aligned} \tag{26}$$

$$\mathbf{d}_u(j,:)=\Big(\partial_x^2 u(\mathbf{P}_j), \partial_x\partial_y u(\mathbf{P}_j), \partial_y^2 u(\mathbf{P}_j)\Big), \tag{27}$$

where $\mathbf{Mes}(\mathbf{T}_h)(j,:)$ and $\mathbf{d}_u(j,:)$ denote the j-th row of the corresponding matrix.

Corollary 1. *For $W(u,v,\mathbf{T}_h)$, we have*

$$|W(u,v,\mathbf{T}_h)| = |\sum_{\mathbf{P}_j\in N_h}\mathbf{Mes}(\mathbf{T}_h)(j,:)\cdot\mathbf{d}_u(j,:)v(\mathbf{P}_j)|\le \tag{28}$$

$$\le Ch^{-1}\|\mathbf{Mes}(\mathbf{T}_h)\|_F\|u\|_{3.5}\|v\|_0, \tag{29}$$

where the matrix $\mathbf{Mes}(\mathbf{T}_h)$ is defined by (26) *and $\|\cdot\|_F$ denotes the Frobenius matrix norm.*

Proof. From (25) and (26), we can easily obtain (28). And with the following relations

$$ch^{-1}\|v\|_0\le(\sum_{\mathbf{P}_j\in N_h}v(\mathbf{P}_j)^2)^{\frac12}\le Ch^{-1}\|v\|_0, \tag{30}$$

we can obtain (29). □

Now, let's expand the term $(u-u_I,u_h)$. With this aim, we need the following result [2].

Lemma 5. *Assume that $\hat e$ is the reference triangle (see Fig. 3), $\hat u\in H^3(\hat e)$, and $\hat u_I$ is the linear interpolant of $\hat u$ on $\hat e$. Then we have the following expansion:*

$$\begin{aligned}\int_{\hat e}(\hat u-\hat u_I)\hat v d\hat x d\hat y &= -\frac{1}{12}\int_{\hat e}(\partial_{\hat x}^2\hat u+\partial_{\hat y}^2\hat u-\partial_{\hat x}\partial_{\hat y}\hat u)\hat v d\hat x d\hat y+\frac{1}{360}\int_{\hat e}\partial_{\hat y}^2\hat u\partial_{\hat x}\hat v d\hat x d\hat y-\\ &\quad -\frac{1}{180}\int_{\hat e}\partial_{\hat y}^2\hat u\partial_{\hat y}\hat v d\hat x d\hat y-\frac{1}{180}\int_{\hat e}\partial_{\hat x}^2\hat u\partial_{\hat x}\hat v d\hat x d\hat y+\frac{1}{360}\int_{\hat e}\partial_{\hat x}^2\hat u\partial_{\hat y}\hat v d\hat x d\hat y+\\ &\quad +\frac{1}{180}\int_{\hat e}\partial_{\hat x}\partial_{\hat y}\hat u(\partial_{\hat x}\hat v+\partial_{\hat y}\hat v)d\hat x d\hat y+C|u|_{3,\hat e}\|v\|_{0,\hat e}\ \forall\hat v\in P_1(\hat e).\end{aligned} \tag{31}$$

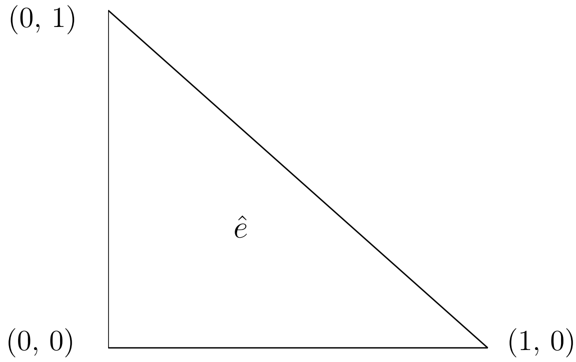

Fig. 3. The reference element $\hat{e}$

Proof. The proof is similar to that of Lemma 4. We just need to define the following bilinear functional on the reference element $\hat{e}$:

$$\begin{aligned} B(\hat{u},\hat{v}) \;=\; & \int_{\hat{e}} (\hat{u}-\hat{u}_I)\hat{v}\,d\hat{x}d\hat{y} + \frac{1}{12}\int_{\hat{e}} (\partial_{\hat{x}}^2\hat{u} + \partial_{\hat{y}}^2\hat{u} - \partial_{\hat{x}}\partial_{\hat{y}}\hat{u})\hat{v}\,d\hat{x}d\hat{y} - \frac{1}{360}\int_{\hat{e}} \partial_{\hat{y}}^2\hat{u}\partial_{\hat{x}}\hat{v}\,d\hat{x}d\hat{y} + \\ & + \frac{1}{180}\int_{\hat{e}} \partial_{\hat{y}}^2\hat{u}\partial_{\hat{y}}\hat{v}\,d\hat{x}d\hat{y} + \frac{1}{180}\int_{\hat{e}} \partial_{\hat{x}}^2\hat{u}\partial_{\hat{x}}\hat{v}\,d\hat{x}d\hat{y} - \frac{1}{360}\int_{\hat{e}} \partial_{\hat{x}}^2\hat{u}\partial_{\hat{y}}\hat{v}\,d\hat{x}d\hat{y} - \\ & - \frac{1}{180}\int_{\hat{e}} \partial_{\hat{x}}\partial_{\hat{y}}\hat{u}(\partial_{\hat{x}}\hat{v} + \partial_{\hat{y}}\hat{v})\,d\hat{x}d\hat{y} \quad \forall \hat{v} \in P_1(\hat{e}). \end{aligned}$$

By the Sobolev embedding theorem and the inverse inequality [14], we get the following:

$$|B(\hat{u},\hat{v})| \le C\|\hat{u}\|_{3,\hat{e}}\|v\|_{0,\hat{e}} \quad \forall \hat{v} \in P_1(\hat{e}).$$

Direct verification shows that

$$B(\hat{u},\hat{v}) = 0 \;\; \forall \hat{u} \in P_2(\hat{e}) \;\; \forall \hat{v} \in P_1(\hat{e}).$$

From the Bramble-Hilbert lemma, we have

$$|B(\hat{u},\hat{v})| \le C|\hat{u}|_{3,\hat{e}}\|v\|_{0,\hat{e}}.$$

So, we obtain (31) and complete the proof. □

Theorem 2. *Assume that $u \in H^3(\Omega)$. Let u_I be the piecewise linear interplant of u on Ω, then we have*

$$\int_{\Omega} (u-u_I)v\,dxdy \;=\; -\frac{h^2}{12}M(u,v,\mathbf{T}_h) + O(h^3)\|u\|_3\|v\|_1 \quad \forall v \in V_h, \tag{32}$$

where

$$M(u,v,\mathbf{T}_h) = \sum_{e\in\mathbf{T}_h}\int_e (l_{i+1}^2\partial_{i+1}^2 u + l_{i-1}^2\partial_{i-1}^2 u + l_{i+1}l_{i-1}\partial_{i+1}\partial_{i-1}u)v\,dxdy \tag{33}$$

$\forall i \in \{1,2,3\}$ in every element e.

Proof. First, we define an affine mapping F: $\hat{e} \to e$ by

$$(x, y) = (h_{i-1}\mathbf{t}_{i-1}, -h_{i+1}\mathbf{t}_{i+1}) \cdot (\hat{x}, \hat{y}) + \mathbf{p}_i \quad \forall i \in \{1, 2, 3\}.$$

So, we have

$$\partial_{\hat{x}}\hat{u} = h_{i-1}\partial_{i-1}u, \quad \partial_{\hat{y}}\hat{u} = -h_{i+1}\partial_{i+1}u$$

and

$$\partial^2_{\hat{x}}\hat{u} = h^2_{i-1}\partial^2_{i-1}u, \quad \partial_{\hat{x}}\partial_{\hat{y}}\hat{u} = -h_{i-1}h_{i+1}\partial_{i-1}\partial_{i+1}u, \quad \partial^2_{\hat{y}}\hat{u} = h^2_{i+1}\partial^2_{i+1}u.$$

From Lemma 5 and the mapping F, we obtain (32). □

So, from (17), Theorems 1 and 2, we give the following eigenvalue error expansion:

$$\lambda_h - \lambda = -\frac{\lambda h^2}{12}M(u, u_h, \mathbf{T}_h) + \frac{h^2}{12}W(u, R_h u, \mathbf{T}_h) - \frac{h^2}{12}K(u, R_h u, \mathbf{T}_h) + O(h^3). \tag{34}$$

3. Eigenvalue extrapolation

In this section, we give the eigenvalue extrapolation scheme and analyze its effectiveness.

In order to use the extrapolation method, we need to refine the mesh $\mathbf{T}_h$ in the regular way. Each element $e \in \mathbf{T}_h$ is subdivided into 4 congruent triangles by connecting the midpoints of its edges (see Fig. 4). Thus we get the finer mesh $\mathbf{T}_{h/2}$.

For the relation between $\mathbf{T}_h$ and $\mathbf{T}_{h/2}$, we have the following lemma.

Lemma 6. *If* $\mathbf{T}_{h/2}$ *is obtained from* $\mathbf{T}_h$ *by the regular refinement, we have*

$$\|\mathbf{Mes}(\mathbf{T}_h)\|_F = \|\mathbf{Mes}(\mathbf{T}_{h/2})\|_F, \tag{35}$$
$$\|\mathbf{Mes}(\mathbf{T}_h)\|_F \leq Ch^{-1}. \tag{36}$$

Proof. For any new node $\mathbf{P}_j$ produced by refining $\mathbf{T}_h$ in the regular way, from (26), we have

$$\mathbf{Mes}(\mathbf{T}_{h/2})(j, :) = (0, 0, 0).$$

And for the old nodes of $\mathbf{T}_h$, the corresponding rows don't change. Thus, we can obtain (35), (36) can be directly obtained from the quasi-uniform condition of $\mathbf{T}_h$. □

Let us solve the problem (9) twice, on the meshes $\mathbf{T}_h$ and $\mathbf{T}_{h/2}$. Then we have the following eigenvalue extrapolation formula.

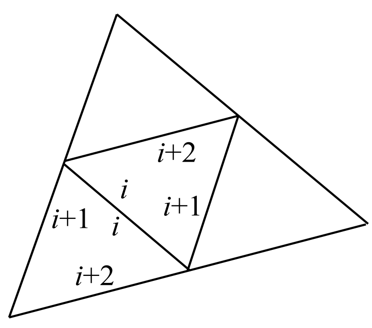

Fig. 4. The elements of $\mathbf{T}_{h/2}$ in an element $e \in \mathbf{T}_h$

Theorem 3. *Assume that* $u \in H^{4.5}(\Omega)$*. Let* λ_h *and* $\lambda_{h/2}$ *be the eigenvalue approximations of the problem* (9) *on the meshes* $\mathbf{T}_h$ *and* $\mathbf{T}_{h/2}$*, respectively. Then we have the following extrapolation equality:*

$$\frac{4\lambda_{h/2} - \lambda_h}{3} = \lambda + O(h^3)\|\mathbf{Mes}(\mathbf{T}_h)\|_F. \tag{37}$$

Proof. From Theorem 1, (34) and the relation between $\mathbf{T}_h$ and $\mathbf{T}_{h/2}$, we can know in $e \in \mathbf{T}_h$, the interior edge integration for the elements of $\mathbf{T}_{h/2}$ will cancel. And with Theorem 2, we have

$$\begin{aligned} 4(\lambda_{h/2} - \lambda) - (\lambda_h - \lambda) &= -\frac{\lambda h^2}{12} M(u, u_{h/2} - u_h, \mathbf{T}_h) + \frac{h^2}{12} W(u, R_{h/2}u - R_h u, \mathbf{T}_h) + \\ &\quad + \frac{h^2}{12} K(u, R_{h/2}u - R_h u, \mathbf{T}_h) + O(h^3). \end{aligned} \tag{38}$$

From (11), we have the following error estimates

$$h\|u_h - u_{h/2}\|_1 + \|u_h - u_{h/2}\|_0 \leq ch^2, \tag{39}$$

$$h\|R_h u - R_{h/2}u\|_1 + \|R_h u - R_{h/2}u\|_0 \leq ch^2. \tag{40}$$

Then, we have

$$|M(u, u_{h/2} - u_h, \mathbf{T}_h)| \leq ch^2, \tag{41}$$

$$|K(u, R_{h/2}u - R_h u, \mathbf{T}_h)| \leq ch. \tag{42}$$

And from Theorem 1, we have

$$\begin{aligned} |W(u, R_{h/2}u - R_h u, \mathbf{T}_h)| &\leq Ch^{-1}\|u\|_{3.5}\|\mathbf{Mes}(\mathbf{T}_h)\|_F \|R_{h/2}u - R_h u\|_0 \leq \\ &\leq Ch\|\mathbf{Mes}(\mathbf{T}_h)\|_F \|u\|_{3.5}^2. \end{aligned} \tag{43}$$

So, we can obtain the result (37). □

So from (35), (36), and Theorem 3, we can know that the eigenvalue extrapolation can achieve the $O(h^3)$ convergence rate if we refine the mesh in the regular way starting with any initial mesh $\mathbf{T}_{h_0}$. Especially, we have the following corollary.

Corollary 2. *Assume that* $u \in H^{4.5}(\Omega)$, $\mathbf{T}_h$ *is produced by refining the triangulation* $\mathbf{T}_{h_0}$ *in the regular way and* $h_0 = O(1)$*. Let* λ_h *and* $\lambda_{h/2}$ *be the eigenvalue approximations of the problem* (9) *on the meshes* $\mathbf{T}_h$ *and* $\mathbf{T}_{h/2}$*, respectively. Then we have the following extrapolation equality:*

$$\frac{4\lambda_{h/2} - \lambda_h}{3} = \lambda + O(h^3). \tag{44}$$

Proof. (44) can be obtained easily from (35)–(37). □

4. Numerical results

In this section we give some numerical results for the eigenvalue extrapolation. Let us take the domain $\Omega = [0, 1] \times [0, 1]$.

The initial mesh is generated by using the Delaunay triangulation algorithm without any optimization.

In order to illustrate the convergence rate, we define the notations:

$$\lambda_h^{\text{extra}} = \frac{4\lambda_{h/2} - \lambda_h}{3},$$

$$\text{err}_h = |\lambda_h - \lambda|,$$

$$\text{err}_h^{\text{extra}} = \lambda_h^{\text{extra}} - \lambda,$$

$$R_h = \frac{\log(\text{err}_h/\text{err}_{h/2})}{\log(2)},$$

$$R_h^{\text{extra}} = \frac{\log(\text{err}_h^{\text{extra}}/\text{err}_{h/2}^{\text{extra}})}{\log(2)}.$$

We know that R_h^{extra} indicates the convergence rate for the eigenvalue extrapolation method.

Here, we give two numerical results for the first eigenvalue $\lambda = 2\pi^2$. The first example is for the coarse mesh $\mathbf{T}_{1/8}$ (Fig. 5) and the second one is for the fine mesh $\mathbf{T}_{1/20}$ (Fig. 6).

Example 1:

T a b l e 1. The results for the initial mesh $\mathbf{T}_{1/8}$

Mesh	$\mathbf{T}_h$	$\mathbf{T}_{h/2}$	$\mathbf{T}_{h/4}$
$\|\mathbf{Mes}(\mathbf{T}_h)\|_F$	17.227	17.227	17.227
λ_h	20.062	19.821	19.760
λ_h^{extra}	/	19.741	19.739
err_h	0.322	0.082	0.021
$\text{err}_h^{\text{extra}}$	/	0.002	$2.310 \cdot 10^{-4}$
R_h	/	1.973	1.988
R_h^{extra}	/	/	3.154

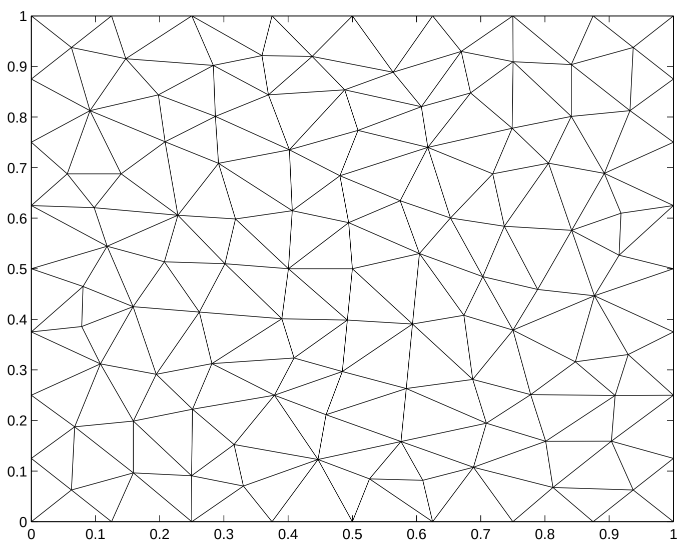

Fig. 5. The initial mesh $\mathbf{T}_{1/8}$

Example 2:

T a b l e 2. The results for the initial mesh $\mathbf{T}_{1/20}$

Mesh	$\mathbf{T}_h$	$\mathbf{T}_{h/2}$	$\mathbf{T}_{h/4}$
$\lVert\mathbf{Mes}(\mathbf{T}_h)\rVert_F$	40.267	40.267	40.267
λ_h	19.790	19.752	19.743
λ_h^{extra}	/	19.740	19.739
err_h	0.051	0.013	0.003
$\text{err}_h^{\text{extra}}$	/	$4.270 \cdot 10^{-4}$	$4.560 \cdot 10^{-5}$
R_h	/	1.964	1.985
R_h^{extra}	/	/	3.227

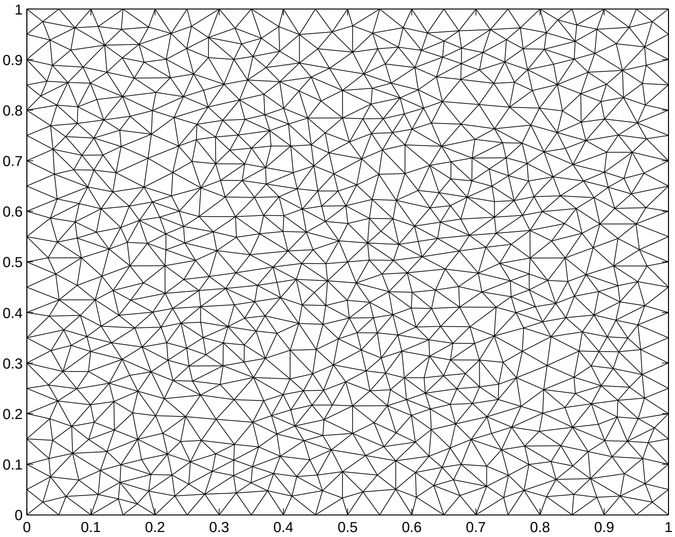

Fig. 6. The initial mesh $\mathbf{T}_{1/20}$

In both cases, one can see in Tabl. 1, 2 that R_h demonstrates the usual second order of convergence, whereas R_h^{extra} indeed confirms the third (and somewhat more) order like in Theorem 3.

Concluding Remarks

The matrix $\mathbf{Mes}(\mathbf{T}_h)$ can be used to measure the superconvergence of the triangulation $\mathbf{T}_h$ [4].

The method and the result can be extended to a more general case and as a by-product, we can use the approximations of higher accuracy to form a class of a posteriori error estimators [15, 16] for the eigenvalue approximations.

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