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ASYMPTOTIC STABILITY OF THE EQUILIBRIUM STATE FOR THE MACROSCOPIC BALANCE EQUATIONS OF CHARGE TRANSPORT IN SEMICONDUCTORS*

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В случае задачи о баллистическом диоде для моментных уравнений переноса заряда в полупроводниках, основанных на принципе максимума энтропии, доказывается устойчивость (по Ляпунову) состояния равновесия.

Introduction

Modelling modern submicron electron devices requires an accurate description of energy transport in order to cope with high-field phenomena such as hot electron propagation, impact ionization and heat generation in the bulk material. Also, for many applications in optoelectronics one needs to describe the transient interaction of electromagnetic radiation with carriers in complex semiconductor materials and since the characteristic times are of order of the electron momentum or energy flux relaxation times, some higher moments of the distribution function are necessarily involved. These phenomena cannot be described satisfactorily within the framework of the drift-diffusion equations (which do not comprise energy as a dynamical variable and also are valid only in the quasi-stationary limit) and the simplest hydrodynamic models.

As known in the hierarchy of approximate macroscopic models beyond the drift-diffusion equations one finds the hydrodynamical models which are obtained from the infinite set of moment equations of the Boltzmann transport equation (BTE) by a suitable truncation procedure. It is well-known too that moment systems require a closure assumption in order to lead to closed system of evolution equations. In [1] by using the maximum entropy ansatz for the

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closure one obtains explicit constitutive relations for the stress tensor and the flux of energy flux tensor.

In the present paper we consider a one dimensional problem, represented by a $n^+ - n - n^+$ ballistic diode [2 – 4]. Roughly speaking the physical situation is constituted by a semiconductor divided in three parts: two region of high doping (the n^+ regions) with inside a region of low doping (the n region). When the doping is uniform we will refer as bulk semiconductors.

The dynamics of the charge carriers depends on the applied potential (the bias voltage). When the applied voltage is negligible one expect that the situation of global thermodynamical equilibrium is reached: the charge are at rest with the same temperature of the crystal.

We shall prove that for the model under consideration the equilibrium solution is asymptotically stable in the sense of Lyapunov.

The plan of the paper is the following. In section 1 the basic equations are presented. The asymptotic stability of the equilibrium state is showed in sections 2 and 3.

1. Basic equations and formulation of the problem

Following [1] we can write such moment equations

$$\frac{\partial n}{\partial t} + \frac{\partial (nV^i)}{\partial x^i} = 0, \tag{1.1}$$

$$\frac{\partial (nP^i)}{\partial t} + \frac{\partial (nU^{ij})}{\partial x^j} + neE^i = n\mathbb{C}_P^i, \ i = 1, 2, 3, \tag{1.2}$$

$$\frac{\partial (nW)}{\partial t} + \frac{\partial (nS^i)}{\partial x^i} + neV_iE^i = n\mathbb{C}_W, \tag{1.3}$$

$$\frac{\partial (nS^i)}{\partial t} + \frac{\partial (nF^{ij})}{\partial x^j} + neE_jG^{ij} = n\mathbb{C}_W^i, \ i = 1, 2, 3, \tag{1.4}$$

where n is the electron density, V^i, $i = 1, 2, 3$ are the components of the average electron velocity, $P^i (= m^* V^i)$, $i = 1, 2, 3$ are the components of the average crystal momentum, m^* is the effective electron mass (for silicon $m^* = 0.32 m_e$ with m_e mass of one electron in the vacuum), $U^{ij} (= U^{(0)}\delta^{ij} = \frac{2}{3} W \delta^{ij})$, $i, j = 1, 2, 3$ is the flow of crystal momentum, W is the average electron energy, e is the absolute value of the electron charge, E^i, $i = 1, 2, 3$ are the components of the electric field, $\mathbb{C}_P^i$, $i = 1, 2, 3$ are the components of the production of the crystal momentum balance equations, S^i, $i = 1, 2, 3$ are the components of the energy flux, $\mathbb{C}_W$ is the production of the energy balance equation, $F^{ij} (= \frac{9}{10} \frac{W^2}{m^*} \delta^{ij})$, $i = 1, 2, 3$ is the flux of the energy flux, $G^{ij} = \frac{5}{3} \frac{W}{m^*} \delta^{ij}$, $i = 1, 2, 3$, $\mathbb{C}_W^i$, $i = 1, 2, 3$ are the components of the production of the energy flux balance equations.

Since the electric field is related to the electric potential Φ as

$$E^i = -\frac{\partial \Phi}{\partial x^i}, i = 1, 2, 3,$$

the system (1.1) – (1.4) is coupled with the Poisson equation

$$\epsilon \triangle \Phi = e(n - N), \tag{1.5}$$

where $N = N_D - N_A$, N_D and N_A being the donor and accepter density respectively, ϵ is the dielectric constant.

Let us introduce the adimensional variables

$$R = \frac{n}{N^+}, \quad u^i = \frac{V^i}{\mathbb{C}_0}, \quad E = \frac{W}{m^*\mathbb{C}_0^2},$$
$$q^i = \frac{S^i}{m^*\mathbb{C}_0^3}, \quad \varphi = \frac{\Phi e}{m^*\mathbb{C}_0^2},$$
$$\tau = \frac{\mathbb{C}_0 t}{L}, \quad \tilde{x}^i = \frac{x^i}{L}.$$

Here R, E, φ, u^i, q^i, $i = 1,2,3$ are the new dependent variables, τ, $\tilde{x}^i$, $i = 1,2,3$ are the new independent variables (further we will write again x^i instead of $\tilde{x}^i$), N^+ is the doping density N in the n^+ region (see [1]), $\mathbb{C}_0 = \sqrt{\dfrac{K_{\mathrm{B}}T_0}{m*}}$ is a sort of sound speed, T_0 is the lattice temperature, K_{B} is the Boltzmann constant, L is the width of the $n^+ - n - n^+$ channel.

The evolution equations in the adimensional divergent form read

$$\left.\begin{aligned}
&\frac{\partial R}{\partial \tau} + \frac{\partial (Ru^i)}{\partial x^i} = 0,\\
&\frac{\partial (Ru^i)}{\partial \tau} + \frac{\partial\left(\frac{2}{3}RE\right)}{\partial x^i} = R\frac{\partial \varphi}{\partial x^i} + R\widetilde{\mathbb{C}}_P^i, \ i = 1,2,3,\\
&\frac{\partial (RE)}{\partial \tau} + \frac{\partial (Rq^i)}{\partial x^i} = Ru^i\frac{\partial \varphi}{\partial x^i} + R\widetilde{\mathbb{C}}_W,\\
&\frac{\partial (Rq^i)}{\partial \tau} + \frac{\partial\left(\frac{10}{9}RE^2\right)}{\partial x^i} = \frac{5}{3}RE\frac{\partial \varphi}{\partial x^i} + R\widetilde{\mathbb{C}}_W^i, \ i = 1,2,3,
\end{aligned}\right\} \tag{1.6}$$

$$\varepsilon^2 \triangle \varphi = R - \rho, \tag{1.7}$$

where

$$\rho = \frac{N}{N^+}, \quad \varepsilon^2 = \frac{1}{\beta}, \quad \beta = \frac{e^2L^2N^+}{\epsilon m^*\mathbb{C}_0^2},$$
$$\widetilde{\mathbb{C}}_P^i = \mathbb{C}_P^i\frac{L}{m^*\mathbb{C}_0^2}, \quad \widetilde{\mathbb{C}}_W = \mathbb{C}_W\frac{L}{m^*\mathbb{C}_0^3}, \quad \widetilde{\mathbb{C}}_W^i = \mathbb{C}_W^i\frac{L}{m^*\mathbb{C}_0^4}.$$

For one dimensional problems the system of balance equations (1.6), (1.7) reads

$$U_\tau + \mathcal{B}U_x = F(Q,U), \tag{1.6$'$}$$

$$\varepsilon^2 \varphi_{xx} = R - \rho. \tag{1.7$'$}$$

Here

$$U = \begin{pmatrix} R \\ J \\ RE \\ Rq \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{10}{9}E^2 & 0 & \frac{20}{9}E & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ RQ + R\widetilde{\mathbb{C}}_P^1 \\ JQ + R\widetilde{\mathbb{C}}_W \\ \frac{5}{3}REQ + R\widetilde{\mathbb{C}}_W^1 \end{pmatrix},$$

$$J = Ru, \quad Q = \varphi_x,$$

$$\begin{pmatrix} R\widetilde{\mathbb{C}}_P^1 \\ R\widetilde{\mathbb{C}}_W^1 \end{pmatrix} = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} \\ \tilde{c}_{21} & \tilde{c}_{22} \end{pmatrix} \begin{pmatrix} J \\ Rq \end{pmatrix} \quad \text{(see Appendix A from [5])},$$

$$R\widetilde{\mathbb{C}}_W = \hat{c}P \quad \text{(see Appendix A from [5])}, \quad P = R\left(\frac{2}{3}E - 1\right).$$

It is easy to obtain the hyperbolicity condition for the balance equations of charge transport (1.6′). In point of fact the eigenvalues of the matrix $\mathcal{B}$ are the next

$$\left.\begin{aligned} \lambda_{1,2} &= \pm\left(\frac{10+2\sqrt{10}}{9}E\right)^{1/2}, \\ \lambda_{3,4} &= \pm\left(\frac{10-2\sqrt{10}}{9}E\right)^{1/2}, \end{aligned}\right\} \tag{1.8}$$

i.e., the hyperbolicity condition is $E > 0$.

The system (1.6′) can be rewritten in the next form

$$\left.\begin{aligned} & R_\tau + J_x = 0, \\ & J_\tau + R_x + P_x = RQ + \hat{c}_{11}J + \tilde{c}_{12}\Theta, \\ & \frac{3}{2}P_\tau + J_x + \Theta_x = JQ + \hat{c}P, \\ & \frac{2}{5}\Theta_\tau + (P + \frac{P^2}{R})_x = PQ + \hat{c}_{21}J + \hat{c}_{22}\Theta, \end{aligned}\right\} \tag{1.6″}$$

where $\Theta = Rq - \frac{5}{2}J$, $\hat{c}_{11} = \tilde{c}_{11} + \frac{5}{2}\tilde{c}_{12}$, $\hat{c}_{21} = \frac{2}{5}\tilde{c}_{21} - \tilde{c}_{11} + \frac{5}{2}\hat{c}_{22}$, $\hat{c}_{22} = \frac{2}{5}\tilde{c}_{22} - \tilde{c}_{12}$.

The scaled doping density $\rho = \rho(x)$ must be considered as a known function defined on $[0,1]$. The coefficients $\hat{c}$, $\tilde{c}_{11}$, $\tilde{c}_{12}$, $\tilde{c}_{21}$, $\tilde{c}_{22}$ must be considered as functions of ζ, where $\zeta = \dfrac{3\xi}{4E}$, $\xi = \dfrac{\hbar\omega_{np}}{K_{\mathrm{B}}T_0}$, $\hbar$ is the Planck constant, $\hbar\omega_{np}$ is the optical phonon energy (about of finding of the coefficients $\hat{c}$, $\tilde{c}_{11}$, $\tilde{c}_{12}$, $\tilde{c}_{21}$, $\tilde{c}_{22}$ the interested reader can see [6]; see too Appendix A from [5]).

We will consider the test problem of the ballistic diode, well-known in physics of semiconductors (see for example [1–3]). It is a one dimensional problem representing a semiconductor devices which is divided into several regions. The first and the third regions present a high doping concentration and for this reason are called n^+ regions while in the intermediate part we have a zone of low doping concentration, named n region. In this connection we will assume further that the function $(\rho(x) - 1)$ is sufficiently smooth and finite and $1 \geq \rho(x) \geq \delta > 0$, $x \in [0,1]$.

For the test problem of the ballistic diode the boundary conditions at $x = 0, 1$ for equations (1.6″), (1.7′) are given by (see [2–4]; see too (1.8))

$$\left.\begin{aligned} R(\tau,0) = R(\tau,1) = 1, \\ P(\tau,0) = P(\tau,1) = 0, \end{aligned}\right\} \tag{1.9}$$

$$\varphi(\tau,0) = A, \quad \varphi(\tau,1) = A + B, \tag{1.10}$$

where A, B are constants, and $B > 0$ representing by the bias across the diode. Without the loss of generality, we assume that $A = 0$. Of course we must also assign at $\tau = 0$ the initial data.

Following [4], we give an equivalent formulation of mixed problem (1.6″), (1.7′), (1.9), (1.10). We will consider system (1.6″) coupled with the relation

$$\varepsilon^2 Q_\tau = \int_0^1 J(\tau, s)\, ds - J(\tau, x) = l(J) \tag{1.11}$$

instead of the Poisson equation (1.7′). Equation (1.7′) rewritten in the form

$$\varepsilon^2 Q_x = R - \rho \tag{1.12}$$

will be treated as an additional stationary law which the initial data, in particular, must meet. From boundary conditions (1.10) it follows that the relation

$$\int_0^1 Q(\tau, s)\, ds = B \tag{1.13}$$

is fulfilled and the initial data also should satisfy this relation. The electric potential $\varphi = \varphi(\tau, x)$ is found from the evident equality

$$\varphi(\tau, x) = \int_0^x Q(\tau, s)\, ds. \tag{1.14}$$

Thus, instead of mixed problem (1.6″), (1.7′), (1.9), (1.10) one can consider problem (1.6″), (1.11), (1.9), with additional requirements (1.12), (1.13), which actually are requirements on the initial data. It is easily to show that these two formulations are equivalent, at least on smooth solutions.

Problem (1.6″), (1.7′), (1.9), (1.10) has for $B = 0$ the stationary solution of global thermodynamical equilibrium:

$$\left.\begin{aligned}
&J(\tau, x) = \widehat{J} = 0, \quad \text{i. e.} \quad u(\tau, x) = \hat{u} = 0,\\
&P(\tau, x) = \widehat{P} = 0, \quad \text{i. e.} \quad E(\tau, x) = \widehat{E} = \frac{3}{2},\\
&\Theta(\tau, x) = \widehat{\Theta} = 0, \quad \text{i. e.} \quad q(\tau, x) = \hat{q} = 0,\\
&R(\tau, x) = \widehat{R}(x) = e^{\hat{\varphi}(x)},\\
&\varphi(\tau, x) = \hat{\varphi}(x),
\end{aligned}\right\} \tag{1.15}$$

where $\hat{\varphi}(x)$ satisfies the Poisson equation

$$\varepsilon^2 \hat{\varphi}'' = \widehat{R} - \rho \tag{1.16}$$

with the boundary conditions

$$\hat{\varphi}(0) = \hat{\varphi}(1) = 0. \tag{1.17}$$

It is obvious that if ε can be considered as a small parameter, the solution to boundary value problem (1.16), (1.17) can be approximated as

$$\hat{\varphi}(x) = \ln \rho(x) + O(\varepsilon). \tag{1.18}$$

We will assume in the sequel that the function $\hat{\varphi}(x)$ is sufficiently smooth and finite.

From the physical considerations we have the following

Remark 1.1. *Let $B = 0$. One expects that the solution to $(1.6'')$, $(1.7')$, (1.9), (1.10) tends to the equilibrium state as $\tau \to \infty$ for any initial data, i. e.,*

$$\begin{aligned} &J(\tau, x) \to 0, \\ &P(\tau, x) \to 0, \\ &\Theta(\tau, x) \to 0, \\ &R(\tau, x) \to \widehat{R}(x), \\ &\varphi(\tau, x) \to \hat{\varphi}(x). \end{aligned}$$

Below we will prove this fact. At the same time our proof of asymptotic behavior of global solution will contain some restrictions (for example, on the doping density, the initial data and coefficients $\hat{c}$, $\tilde{c}_{11}$, $\tilde{c}_{12}$, $\tilde{c}_{21}$, $\tilde{c}_{22}$, β).

Remark 1.2. *A similar fact was proven in [5] for a simpler mixed problem than $(1.6'')$, $(1.7')$, (1.9), (1.10). The problem $(1.6'')$, $(1.7')$, (1.9), and (1.10) was studied in [7].*

2. Formulation of the auxiliary problems

Now we proceed to proving the existence of global solution to mixed problem $(1.6'')$, (1.11), and (1.9) with additional requirements (1.12), (1.13) ($B = 0!$) and construct global a priori estimate. First we formulate some auxiliary problems. We consider the potential $H = H(\tau, x)$ (do not mix up with the electric potential φ) such that

$$\left. \begin{aligned} J &= H_\tau, \\ R &= -H_x. \end{aligned} \right\} \tag{2.1}$$

Then the first condition in $(1.6'')$ is fulfilled automatically. In terms of H, (1.9) can be rewritten as follows:

$$\left. \begin{aligned} &H_x(\tau, 0) = H_x(\tau, 1) = -1, \\ &P(\tau, 0) = P(\tau, 1) = 0. \end{aligned} \right\} \tag{1.9'}$$

In a view of (2.1), (1.11) comes into

$$\left\{ \varepsilon^2 Q(\tau, x) - l(H) \right\}_\tau = 0$$

or

$$\varepsilon^2 Q = A_0(x) + l(H),$$

where

$$l(H) = \int_0^1 H(\tau, s)\, ds - H(\tau, x).$$

From (1.12) it follows that

$$A_0'(x) = -\rho(x),$$

i.e.

$$A_0(x) = C - \int_0^x \rho(s)\, ds.$$

Here C is an arbitrary as yet constant. Accounting (1.13), we arrive at

$$C = \int_0^1 (1-s)\rho(s)\, ds$$

and

$$A_0(x) = -\int_0^x \rho(s)\, ds + \int_0^1 (1-s)\rho(s)\, ds.$$

Consequently,

$$Q(\tau, x) = \beta\left[l(H) + A_0(x)\right]. \tag{2.2}$$

For the sake of convenience we introduce a new independent variable $U(\tau, x)$ instead of H:

$$H(\tau, x) = U(\tau, x) + \hat{u}(x) - x, \tag{2.3}$$

where $\hat{u}(x)$ is the solution to the boundary value problem:

$$\left.\begin{aligned} &\hat{u}'' = \beta(\hat{u}' - 1)\left[\hat{v}(x) + \widehat{A}_0(x)\right], \quad 0 < x < 1, \\ &\hat{u}'(0) = \hat{u}'(1) = 0, \end{aligned}\right\} \tag{2.4}$$

moreover,

$$\hat{v}(x) = l(\hat{u}) = \int_0^1 \hat{u}(s)\, ds - \hat{u}(x),$$
$$\widehat{A}_0(x) = x - \int_0^1 (1-s)\, ds + A_0(x).$$

Remark 2.1. *Changing variables in (2.4)*

$$1 - \hat{u}'(x) = e^{\hat{\varphi}(x)} = \widehat{R}(x),$$

we obtain that $\hat{\varphi}(x)$ is the solution to the problem (1.16), (1.17). Besides, (see (2.2))

$$Q(\tau, x) = \beta\left[l(H) + A_0(x)\right] = \beta l(U) + \hat{\varphi}'(x). \tag{2.2$'$}$$

In a view of (2.3), from (2.1) we derive:

$$\left.\begin{aligned} J &= U_\tau, \\ \mathfrak{L} &= \widehat{R} - R = U_x, \end{aligned}\right\} \tag{2.1$'$}$$

besides the second equation in (1.6$''$) can be rewritten as follows:

$$J_\tau - \mathfrak{L}_x - \hat{c}_{11}J + P_x - \widehat{R}\beta l(U) - \tilde{c}_{12}\Theta + \left[\beta l(U) + \underline{\hat{\varphi}'}\right]\mathfrak{L} = 0, \tag{2.5}$$

and (1.9$'$) takes the form:

$$\mathfrak{L}(\tau,0) = \mathfrak{L}(\tau,1) = P(\tau,0) = P(\tau,1) = 0. \tag{1.9$''$}$$

Differentiating (2.5) by x, we have:

$$\mathfrak{L}_{\tau\tau} - \mathfrak{L}_{xx} + P_{xx} + \mathfrak{F} = 0. \tag{2.6}$$

Here

$$\begin{aligned}
&\mathfrak{F} = t_1\mathfrak{L}_\tau + t_2P_\tau + \chi_1\mathfrak{L} + \chi_2P + \mathfrak{F}_0,\\
&\mathfrak{F}_0 = \left(f_0 - \frac{\sigma}{R}f_1\right)\mathfrak{L}_x - \beta\mathfrak{L}^2 - \beta l(U)\underline{\widehat{R}'} - f_1\frac{P_x}{R} - \tilde{c}_{12}f_0J + \frac{\sigma}{R}\underline{\widehat{R}'}\,f_1,\\
&t_1 = \tilde{c}_{12} - \hat{c}_{11}, \quad t_2 = \frac{3}{2}\tilde{c}_{12},\\
&\chi_1 = \beta\left(2\widehat{R} - \rho\right), \quad \chi_2 = -\hat{c}\tilde{c}_{12},\\
&f_0 = \beta l(U) + \underline{\hat{\varphi}'}, \quad f_1 = \hat{c}_{11}'J + \tilde{c}_{12}'\Theta,\\
&\sigma = \frac{P}{R}, \quad \hat{c}_{11}' = \frac{d}{d\sigma}\hat{c}_{11}(\zeta) \quad \text{and so on. Note that (see section 1):}\\
&\zeta = \frac{3\xi}{4E}, \quad \frac{2}{3}E = \sigma + 1.
\end{aligned}$$

Differentiating cross-wise the two last equations in (1.6$''$), we exclude Θ from the left parts and come to the relation:

$$\tilde{a}P_{\tau\tau} - \tilde{b}P_{xx} + \mathfrak{L}_{xx} + G = 0, \tag{2.7}$$

where

$$\begin{aligned}
&\tilde{a} = \frac{3}{2\hat{a}}, \quad \tilde{b} = \frac{1 + \frac{5}{2}(1+2\sigma)}{\hat{a}}, \quad \hat{a} = 1 - \frac{5}{2}\sigma^2,\\
&G = t_3\mathfrak{L}_\tau + t_4P_\tau + \chi_3\mathfrak{L} + \chi_4P + G_0,\\
&G_0 = \frac{1}{\hat{a}}\left\{-5Rf_2^2 + \frac{5}{2}\underline{\widehat{R}''}\sigma^2 - g_0 - \mathfrak{F}_0\right\},\\
&t_3 = \frac{1}{\hat{a}}\left[\frac{5}{2}\left(\hat{c}_{21} - \hat{c}_{22}\right) - \left(\tilde{c}_{12} - \hat{c}_{11}\right)\right],
\end{aligned}$$

$$
\begin{aligned}
&t_4 = -\frac{1}{\hat{a}}\left(\hat{c} + \frac{15}{4}\hat{c}_{22} + \frac{3}{2}\tilde{c}_{12}\right),\\
&\chi_3 = -\frac{1}{\hat{a}}\chi_1, \quad \chi_4 = \frac{1}{\hat{a}}\hat{c}\left(\frac{5}{2}\hat{c}_{22} + \tilde{c}_{12}\right),\\
&f_2 = \frac{1}{R}\left[P_x - \sigma\left(\underline{\widehat{R}'} - \mathfrak{L}_x\right)\right],\\
&g_0 = f_0\left[\widehat{R}\beta l(U) - \mathfrak{L}f_0 + \hat{c}_{11}J + \tilde{c}_{12}\Theta - P_x + \mathfrak{L}_x\right] + \beta J l(J) +\\
&\quad +\hat{c}'\frac{P}{R}(P_\tau + \sigma\mathfrak{L}_\tau) - \frac{5}{2}\left[f_0 P_x + P\left(\underline{\hat{\varphi}''} - \beta\mathfrak{L}\right) + \hat{c}_{22}Jf_0 + (\hat{c}'_{21}J + \hat{c}'_{22}\Theta)\, f_2\right].
\end{aligned}
$$

Remark 2.2. *While deriving (2.6), (2.7), we express Θ_x using the third equation in (1.6″).*

Remark 2.3. *Underlined aggregates turn into zero in the case of uniform doping, i.e. when $\rho(x) \equiv 1, \quad 0 \le x \le 1$ (in this case $\widehat{R}(x) \equiv 1$, $\hat{\varphi}(x) \equiv 0$, $0 \le x \le 1$).*

Unifying (2.6), (2.7), we obtain the system:

$$
AL_{\tau\tau} - BL_{xx} + TL_\tau + XL + \Lambda = 0. \tag{2.8}
$$

Here

$$
\begin{aligned}
&L = \begin{pmatrix} \mathfrak{L} \\ P \end{pmatrix}, \quad A = \operatorname{diag}(1, \tilde{a}), \quad B = \begin{pmatrix} 1 & -1 \\ -1 & \tilde{b} \end{pmatrix},\\
&T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} = \begin{pmatrix} t_1 & t_0 \\ t_0 & t_4 \end{pmatrix} + \begin{pmatrix} 0 & \tilde{t} \\ -\tilde{t} & 0 \end{pmatrix} = ST + CT,\\
&X = \begin{pmatrix} \chi_1 & \chi_2 \\ \chi_3 & \chi_4 \end{pmatrix} = \begin{pmatrix} \chi_1 & \chi_0 \\ \chi_0 & \chi_4 \end{pmatrix} + \begin{pmatrix} 0 & \tilde{\chi} \\ -\tilde{\chi} & 0 \end{pmatrix} = SX + CX,\\
&t_0 = \frac{t_2 + t_3}{2}, \quad \tilde{t} = \frac{t_2 - t_3}{2}, \quad \chi_0 = \frac{\chi_2 + \chi_3}{2}, \quad \tilde{\chi} = \frac{\chi_2 - \chi_3}{2},\\
&\Lambda = \begin{pmatrix} \mathfrak{F}_0 \\ G_0 \end{pmatrix}.
\end{aligned}
$$

The boundary conditions (1.9″) take the form:

$$
L(\tau, 0) = L(\tau, 1) = 0. \tag{1.9‴}
$$

Finally, differentiating (2.8) by τ, we come to the system

$$
AD_{\tau\tau} - BD_{xx} + TD_\tau + XD + \mathfrak{K} = 0, \tag{2.9}
$$

where $D = L_\tau$,

$$
\mathfrak{K} = \Lambda_\tau + A_\tau D_\tau - B_\tau L_{xx} + T_\tau D + X_\tau L,
$$

moreover, (see (2.5), (2.8) and the last equation in (1.6″)):

$$J_\tau = \mathcal{L}_x + \hat{c}_{11} J - P_x + \widehat{R}\beta l(U) + \tilde{c}_{12}\Theta - [\beta l(U) + \hat{\varphi}'] \mathcal{L},$$
$$L_{xx} = B^{-1} [AD_\tau + TD + XL + \Lambda],$$
$$\Theta_\tau = \frac{5}{2} \left\{PQ + \hat{c}_{21} J + \hat{c}_{22}\Theta - P_x - (R\sigma^2)_x\right\}.$$

The boundary conditions for (2.9) follow from (1.9″).

3. Asymptotic stability of the equilibrium state

Now we start to construct global a priori estimate. In the sequel we will use the following almost obvious relations (A, B are symmetric matrices):

$$2\,(D_\tau, AD_{\tau\tau}) = (D_\tau, AD_\tau)_\tau - (D_\tau, A_\tau D_\tau),$$
$$2\,(D_\tau, BD_{xx}) = 2\,(D_\tau, BD_x)_x - (D_x, BD_x)_\tau - 2\,(D_\tau, B_x D_x) + (D_x, B_\tau D_x),$$
$$(D, AD_{\tau\tau}) = (D, AD_\tau)_\tau - (D_\tau, AD_\tau) - (D, A_\tau D_\tau),$$
$$(D, BD_{xx}) = (D, BD_x)_x - (D_x, BD_x) - (D, B_x D_x)$$

and so on.

We multiply (2.9) by $2D_\tau$ and obtain that in a view of the relations from above

$$\begin{gathered}\{(D_\tau, AD_\tau) + (D_x, BD_x) + (D, SXD)\}_\tau - 2(D_\tau, BD_x)_x + \\ +2\{(D_\tau, STD_\tau) + (D_\tau, CXD)\} + 2\,(D_\tau, \mathcal{K}) - (D_\tau, A_\tau D_\tau) - \\ -(D, SX_\tau D) + 2\,(D_\tau, B_x D_x) - (D_x, B_\tau D_x) = 0.\end{gathered} \tag{3.1}$$

Now we multiply the same system by $2D$ and come to the expression

$$\begin{gathered}\{2\,(D, AD_\tau) + (D, STD)\}_\tau - 2(D, BD_x)_x + \\ +2\{-(D_\tau, AD_\tau) + (D_x, BD_x) + (D, SXD) + (D, CTD_\tau)\} + \\ +2\,(D, \mathcal{K}) - (D, ST_\tau D) - 2\,(D, A_\tau D_\tau) + \\ +2\,(D_x, BD_x) + 2\,(D, B_x D_x) = 0.\end{gathered} \tag{3.2}$$

Arguing in the same way, we multiply (2.8) first by $2L_\tau$, then by $2L$ and come to

$$\begin{gathered}\{(D, AD) + (L_x, BL_x) + (L, SXL)\}_\tau - 2(D, BL_x)_x + \\ +2\{(D, STD) + (D, CXL)\} + 2\,(D, \Lambda) - (D, A_\tau D) - \\ -(L, SX_\tau L) + 2\,(D, B_x L_x) - (L_x, B_\tau L_x) = 0;\end{gathered} \tag{3.3}$$

$$\begin{gathered}\{2\,(L, AD) + (L, STL)\}_\tau - 2(L, BL_x)_x + \\ +2\{-(D, AD) + (L_x, BL_x) + (L, SXL) + (L, CTD)\} + \\ +2\,(L, \Lambda) - (L, ST_\tau L) - 2\,(L, A_\tau D) + \\ +2\,(L_x, BL_x) + 2\,(L, B_x L_x) = 0.\end{gathered} \tag{3.4}$$

The equation (2.5) and two last equations in (1.6″) together yield the evident identity:

$$\begin{gathered}\left\{J^2+\mathfrak{L}^2+\frac{2}{5}\Theta^2+\frac{3}{2}P^2\right\}_\tau-2(J\mathfrak{L})_x+2(JP)_x+2(P\Theta)_x+\\+2\left\{-\hat{c}_{11}J^2-\tilde{c}_{12}J\Theta-\hat{c}_{22}\Theta^2-\hat{c}_{21}J\Theta-\hat{c}P^2-\widehat{R}\beta l(U)J\right\}+\\+2\left(J\mathfrak{L}-PJ-\Theta P\right)f_0+2\Theta\left(P_x\sigma+Pf_2\right)=0.\end{gathered}\tag{3.5}$$

We integrate (3.1) – (3.5) by x from 0 to 1 with account to the boundary conditions (1.9″), multiply them by positive arbitrary constants α_1, α_2, α_3, α_4, α_3, summ up the result, and finally come to

$$\frac{d}{d\tau}J^{(0)}+J^{(1)}=\Pi.\tag{3.6}$$

Here

$$J^{(0)}=\int_0^1\left\{(Y,\mathcal{A}_0Y)+\alpha_1(D_x,BD_x)+\alpha_3(L_x,BL_x)+\alpha_3\left(\frac{2}{5}\Theta^2+J^2\right)\right\}dx,$$

$$Y=\begin{pmatrix}D_\tau\\D\\L\end{pmatrix},\quad \mathcal{A}_0=\begin{pmatrix}\alpha_1A&\alpha_2A&0\\\alpha_2A&\mathfrak{R}&\alpha_4A\\0&\alpha_4A&\widehat{\mathfrak{R}}\end{pmatrix},$$

$$\mathfrak{R}=\alpha_3A+\alpha_2ST+\alpha_1SX,$$

$$\widehat{\mathfrak{R}}=\alpha_3\widehat{A}+\alpha_4ST+\alpha_3SX,\quad \widehat{A}=\operatorname{diag}\left(1,\frac{3}{2}\right),$$

$$\begin{gathered}J^{(1)}=2\int_0^1\left\{(Y,\mathcal{A}_1Y)+\alpha_2(D_x,BD_x)+\alpha_4(L_x,BL_x)+\right.\\\left.+\alpha_3\left(-\hat{c}_{11}J^2-\mu_{12}J\Theta-\hat{c}_{22}\Theta^2-\widehat{R}\beta l(U)J\right)\right\}dx,\end{gathered}$$

$$\mu_{12}=\tilde{c}_{12}+\hat{c}_{21},$$

$$\mathcal{A}_1=\begin{pmatrix}\alpha_1ST-\alpha_2A&\alpha_1CX&0\\\alpha_2CT&\alpha_2SX+\alpha_3ST-\alpha_4A&\alpha_3CX\\0&\alpha_4CT&\alpha_4SX+\alpha_3\operatorname{diag}(0,-\hat{c})\end{pmatrix};$$

the aggregate Π, although is not given here, can be easily written down.

Remark 3.1. *From*

$$l(U)=\int_0^1U(\tau,s)\,ds-U(\tau,x)=$$

$$=\int_0^1\left[\int_x^sU_z(\tau,z)\,dz\right]ds=\int_0^1\left[\int_x^s\mathfrak{L}(\tau,z)\,dz\right]ds,$$

it follows that

$$|l(U)| \le \int_0^1 |\mathfrak{L}|\, ds \le \left(\int_0^1 \mathfrak{L}^2\, ds\right)^{1/2}.$$

Then

$$\left|2\int_0^1 \widehat{R}\beta l(U)J\, dx\right| \le 2\beta\int_0^1 |l||J|\, dx \le \beta\left[\hat{\varepsilon}\int_0^1 J^2\, dx + \frac{1}{2\hat{\varepsilon}}\int_0^1 \mathfrak{L}_x^2\, dx\right].$$

While deriving the last inequality, we have used the Cauchy inequality with $\hat{\varepsilon} > 0$ and the Poincare inequality (see [8]):

$$\int_0^1 \mathfrak{L}^2\, ds \le \frac{1}{2}\int_0^1 \mathfrak{L}_x^2\, dx.$$

We also assume that $\rho(x) \equiv 1$ (see section 2, Remark 2.3), i.e. $\widehat{R}(x) \equiv 1$.

Further reasoning are in certain degree standard for the problems of such type (see, for example, [9, 10]). We will assume that the problem $(1.6'')$, (1.11), and (1.9) have the smooth (classical) local solution on an interval $[0, \tau_*]$. We define the constant

$$M_* = \max\left\{\max_{\tau\in[0,\tau_*]}\|L(\tau)\|_{C[0,1]}, \max_{\tau\in[0,\tau_*]}\|L_x(\tau)\|_{C[0,1]}, \max_{\tau\in[0,\tau_*]}\|L_\tau(\tau)\|_{C[0,1]}\right\}.$$

The value of M_* is sufficiently small. We then consider the quadratic form under the integral sign in $J^{(0)}$ to be positive definite. Positive definiteness is first stated at the equilibrium state (see Appendix); since the constant M_* is small, the statement on positive definiteness stays valid in a small neighborhood of the equilibrium state.

The equality (3.6) can be rewritten as follows:

$$\frac{d}{d\tau}J^{(0)} + J^{(2)} \le |\Pi|, \tag{3.6$'$}$$

where

$$J^{(2)} = 2\int_0^1\Big\{(Y, \mathcal{A}_1 Y) + \alpha_2(D_x, BD_x) + \alpha_4(L_x, BL_x) + \\ +\alpha_3\left[(-\hat{c}_{11}J^2 - \mu_{12}J\Theta - \hat{c}_{22}\Theta^2) - \frac{\beta\hat{\varepsilon}}{2}J^2 - \frac{\beta}{4\hat{\varepsilon}}\mathfrak{L}_x^2\right]\Big\}\, dx.$$

We again assume that the quadratic form under the integral sign in $J^{(2)}$ is positive definite (see Appendix); there exists a constant $M_1 > 0$ (which is finally determined through the constant M_*) such that

$$J^{(2)} \ge M_1 J^{(0)}. \tag{3.7}$$

In a view of (3.7), the inequality $(3.6')$ can be rewritten as follows:

$$\frac{d}{d\tau}J^{(0)} + M_1 J^{(0)} \le M_2\big(J^{(0)}\big)^{3/2}. \tag{3.8}$$

Here $M_2 > 0$ is a constant which is determined by M_*. The right-hand side in (3.8) is obtained while estimating $|\Pi|$ (see [5, 10]) with the use of simplest embedding theorems (see, for example, [11 – 13]).

Remark 3.2. *Here we place examples of such simplest embedding theorems (see inequality (3.6′)):*

$$\|L(\tau)\|_{C[0,1]}, \|L_x(\tau)\|_{C[0,1]} \le M_b \|L(\tau)\|_{W_2^2(0,1)} \le \widetilde{M}_b \|L_{xx}(\tau)\|_{L_2(0,1)} \le \widetilde{M}_b M_3 \big(J^{(0)}(\tau)\big)^{1/2},$$

$$\|J(\tau)\|_{C[0,1]}, \|J_x(\tau)\|_{C[0,1]} \le M_b \|J(\tau)\|_{W_2^2(0,1)} \le M_b M_4 \big(J^{(0)}(\tau)\big)^{1/2}, \quad \textit{and so on}$$

Here M_b, $\widetilde{M}_b$ are constants of embedding; $M_3, M_4 > 0$ are constants determined by M_*.

If $z \ge 0$, then $F(z) = -M_1 z + M_2 z^{3/2}$ is negative at $0 < z < \left(\dfrac{M_1}{M_2}\right)^2$. Consequently, if the initial data are sufficiently small and

$$J^{(0)}(0) < \left(\frac{M_1}{M_2}\right)^2,$$

then it follows from (3.8) that (see [10]):

$$J^{(0)}(\tau) \le e^{-\nu\tau} J^{(0)}(0), \quad \tau > 0. \tag{3.9}$$

Here ν is a constant such that $0 < \nu < M_1$.

Remark 3.3. *Global estimate (3.9) has been derived provided that $\rho(x) \equiv 1$, $0 \le x \le 1$, i.e. $\widehat{R}(x) \equiv 1$, $\hat{\varphi}(x) \equiv 0$, $0 \le x \le 1$. Clearly, this estimation stays valid if the doping density $\rho(x)$ slightly differs from the uniform density $\rho(x) \equiv 1$, $0 \le x \le 1$.*

Existence of the estimate (3.9) means that

$$\begin{aligned} &J(\tau,x), \Theta(\tau,x) \in W_2^2(0,1),\\ &L(\tau,x) \in W_2^2(0,1) \cap \mathring{W}_2^1(0,1),\\ &\varphi(\tau,x) \in W_2^4(0,1) \cap \mathring{W}_2^1(0,1) \quad \text{for all} \quad \tau \ge 0. \end{aligned}$$

Clearly, that

$$\begin{aligned} &J(\tau,x), P(\tau,x), \Theta(\tau,x) \to 0,\\ &R(\tau,x) \to \widehat{R}(x) \quad \text{in } C^1[0,1], \text{ if } \quad \tau \to \infty;\\ &\varphi(\tau,x) \to \hat{\varphi}(x) \quad \text{in } C^3[0,1], \text{ if } \quad \tau \to \infty. \end{aligned}$$

Conclusions

The analysis carried out in this paper constitutes a very important step in the mathematical analysis of the hydrodynamical models of charge transport in semiconductors. In fact the stability of the equilibrium state in an essential conditions to be satisfied. Usually for physical model described by means of hyperbolic systems the previous property is investigated by constructing an appropriate entropy function. However in our case is not easy to get an explicit form of the entropy and a special analysis is required.

The stability analysis has been performed for the non-linear macroscopic balance equations of charge transport (see (1.6″)). The analysis of the asymptotic stability in the presence of an arbitrary doping profile is under current investigations by the authors.

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Appendix: about $J^{(0)}$, $J^{(2)}$

Here we describe the conditions which make the quadratic forms under the integral signs in $J^{(0)}$, $J^{(2)}$ positive definite (entries of the matrices A, B, T, and X are taken at the equilibrium state, see (1.15)).

First we note that the matrices A, B at the equilibrium state are positive definite. We begin with $J^{(0)}$. Using the Poincare inequality (see [8]), we obtain:

$$\begin{aligned} J^{(0)} = \int\limits_0^1 \Big\{ \Big(Y, \widehat{\mathcal{A}}_0 Y\Big) + \alpha_1 (D_x, BD_x) + \alpha_3 (L_x, BL_x) + \\ + \alpha_3 \Big(\frac{2}{5}\Theta^2 + J^2\Big)\Big\}\, dx \geq \int\limits_0^1 \Big\{ \Big(Y, \widehat{\mathcal{A}}_0 Y\Big) + \frac{\alpha_1}{2}(D_x, BD_x) + \\ + \tfrac{\alpha_3}{2}(L_x, BL_x) + \alpha_1 (D, BD) + \alpha_3 (L, BL) + \alpha_3\Big(\frac{2}{5}\Theta^2 + J^2\Big)\Big\}\, dx \geq \\ \geq \int\limits_0^1 \Big\{ \Big(Y, \widehat{\mathcal{A}}_2 Y\Big) + \alpha_1 \Big(D, (SX + B)D\Big) + \alpha_3 \Big(L, (SX + B)L\Big) + \\ + \frac{\alpha_1}{2}(D_x, BD_x) + \frac{\alpha_3}{2}(L_x, BL_x) + \alpha_3 \Big(\frac{2}{5}\Theta^2 + J^2\Big)\Big\}\, dx, \end{aligned} \tag{0.1}$$

where

$$\widehat{\mathcal{A}}_2 = \begin{pmatrix} \alpha_1 \widehat{A} & \alpha_2 \widehat{A} & 0 \\ \alpha_2 \widehat{A} & (\alpha_3 + \alpha_2 k)\widehat{A} & \alpha_4 \widehat{A} \\ 0 & \alpha_4 \widehat{A} & (\alpha_3 + \alpha_4 k)\widehat{A} \end{pmatrix}, \quad k = \frac{\alpha_2}{\alpha_1}.$$

While deriving (0.1), we have also assumed that

$$ST > kA. \tag{0.2}$$

By $\varkappa$, y, z we denote the values:

$$\varkappa = \frac{\alpha_1}{\alpha_2} = k^{-1}, \quad y = \frac{\alpha_3}{\alpha_2}, \quad z = \frac{\alpha_4}{\alpha_2}$$

and write down

$$\varkappa = k^{-1} = \frac{\tilde{t}}{\tilde{\chi}}, \quad z = yk = y\frac{\tilde{\chi}}{\tilde{t}}. \tag{0.3}$$

Let $\tilde{c}_{12} = 0$ at the equilibrium state. Then

$$k = \frac{\beta}{-\hat{c}_{11} - \dfrac{5}{2}(\hat{c}_{21} - \hat{c}_{22})} > 0$$

(we assume that at the equilibrium state $\hat{c}_{11} < 0$, $\hat{c}_{11} + \dfrac{5}{2}(\hat{c}_{21} - \hat{c}_{22}) < 0$). If

$$SX + B > 0, \tag{0.4}$$

then the quadratic form under the integral sign in the right-hand side of (0.1) is positive definite. Indeed, the matrix $\widehat{A}_2$ is positive definite if the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_2 & (\alpha_3+\alpha_2 k) & \alpha_4 \\ 0 & \alpha_4 & (\alpha_3+\alpha_4 k) \end{pmatrix} > 0.$$

It is easy to check that this matrix is positive definite.

For small k, (0.2) is valid if

$$ST > 0,$$

i.e.

$$\hat{c}_{11}\left(\hat{c}+\frac{15}{4}\hat{c}_{22}\right)-\frac{\left[\hat{c}_{11}+\frac{5}{2}(\hat{c}_{21}-\hat{c}_{22})\right]^2}{4} > 0.$$

The last inequality is true if $\hat{c} < 0$ is sufficiently large by moduli. The inequality (0.4) is reduced to the following:

$$\hat{c}\hat{c}_{22}+\frac{13}{10}-\frac{\beta}{10} > 0,$$

which is a priori valid if $\hat{c}_{22} < 0$ and β is sufficiently small.

It is easy to show that, provided that (0.2), (0.4), and (0.3) are fulfilled, the quadratic form under the integral sign in $J^{(2)}$ is positive definite too (if the constant β is sufficiently small). The constant $\hat{\varepsilon}$ is taken to be equal to $\dfrac{2}{\beta}$. Besides, we must have

$$(\hat{c}_{11}+1)\hat{c}_{22}-\frac{\hat{c}_{21}^2}{4} > 0.$$

So, if at the equilibrium state

$$\begin{aligned}
&\hat{c}_{11}, \hat{c}_{22}, \hat{c} < 0, \quad \tilde{c}_{12} = 0,\\
&\hat{c}_{11}+\frac{5}{2}(\hat{c}_{21}-\hat{c}_{22}) < 0,\\
&\hat{c}_{11}\left(\hat{c}+\frac{15}{4}\hat{c}_{22}\right)-\frac{\left(\hat{c}_{11}+\frac{5}{2}(\hat{c}_{21}-\hat{c}_{22})\right)^2}{4} > 0,\\
&\hat{c}\hat{c}_{22}+\frac{13}{10}-\frac{\beta}{10} > 0, \ (\hat{c}_{11}+1)\hat{c}_{22}-\frac{\hat{c}_{21}^2}{4} > 0;
\end{aligned}$$

the constant $\alpha_2 > 0$ is arbitrary, $\alpha_1 = \varkappa\alpha_2$, $\alpha_3 = y\alpha_2$ ($y > 0$ is arbitrary constant), $\alpha_4 = \alpha_2 z = \alpha_2 y k$, $\hat{\varepsilon} = \dfrac{2}{\beta}$, $\varkappa = k^{-1}$, $k = -\dfrac{\beta}{\left(\hat{c}_{11}+\frac{5}{2}(\hat{c}_{21}-\hat{c}_{22})\right)} > 0$; then the quadratic forms under the integral signs in $J^{(0)}$, $J^{(2)}$ are positive definite (if the constant M_* is small, this statement is valid in a small neighborhood of the equilibrium state).

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