ABOUT STABILITY OF THE EQUILIBRIUM STATE FOR A HYDRODYNAMICAL MODEL OF CHARGE TRANSPORT IN SEMICONDUCTORS*

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Доказана устойчивость и, при некоторых дополнительных предположениях на плотность легирования, асимптотическая устойчивость (по Ляпунову) состояния равновесия гидродинамической модели переноса заряда в полупроводниках в линейном приближении.

1. Preliminaries

Hydrodynamical models are widely used in mathematical modelling of physical phenomena in modern semiconductor devices. Derivation of such models is based on the study of the transport equation for the charge carriers density in an electric field. A conservation system with infinite number of equations (i. e., a system of conservation laws) is obtained from the transport equation with the help of a special technique of moments. Then, in a view of some physically justified assumptions, the system is exposed to simplification. As the result, the system of infinite number of moments equations reduces to one or another system of the hydrodynamical type.

In this paper we take, as a basis, a hydrodynamical model of charge transport in semiconductors suggested in the recently published paper [1]. In [2], this model is written as a quasilinear system of dimensionless conservation laws. Using notation from [2], below we give a nondivergent variant of this model

$$R_\tau + uR_s + Ru_s = 0,$$

$$u_\tau + uu_s + \frac{\vartheta}{R}R_s + \vartheta_s + \frac{1}{R}\Sigma_s = Q - \frac{u}{\tau_p},$$

$$\vartheta_\tau + u\vartheta_s + \frac{2}{3}\left(\vartheta + \frac{\Sigma}{R}\right)u_s + \frac{2}{3R}\theta_s = \frac{2}{3}\left\{u^2\left(\frac{1}{\tau_p} - \frac{1}{2\tau_w}\right) + \frac{3(1-\vartheta)}{2\tau_w}\right\},$$

*This work was supported by Russian Foundation for Basic Research, Grant No. 96–01–01560.

$$\Sigma_\tau + u\Sigma_s + \frac{4R\vartheta + 7\Sigma}{3}u_s + \frac{8}{15}\theta_s = \frac{4Ru^2}{3\tau_p} - \frac{\Sigma}{\tau_\sigma},$$

$$\theta_\tau + u\theta_s + \frac{5}{2}(R\vartheta + \Sigma)\vartheta_s - \frac{\Sigma\vartheta}{R}R_s + \left(\vartheta - \frac{\Sigma}{R}\right)\Sigma_s + \frac{16}{5}\theta u_s = -\frac{\theta}{\tau_q} +$$

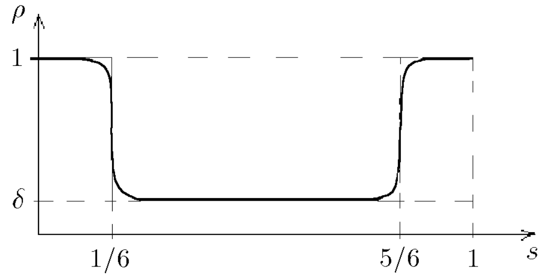
$$+\Sigma u\left(\frac{1}{\tau_\sigma} + \frac{1}{\tau_p} - \frac{1}{\tau_q}\right) + \frac{5}{2}Ru\vartheta\left(\frac{1}{\tau_p} - \frac{1}{\tau_q}\right) + \frac{Ru^3}{2}\left(\frac{5}{3\tau_w} - \frac{1}{\tau_q} - \frac{3}{\tau_p}\right) - \frac{5}{2}Ru\frac{1-\vartheta}{\tau_w}, \quad (1.1)$$

$$\varepsilon^2\varphi_{ss} = R - \rho. \quad (1.2)$$

Here $R,\ u,\ \vartheta,\ \Sigma,\ \theta,\ \varphi$ are the density, velocity, temperature, stress, heat flux and electric potential correspondingly; $\varepsilon^2 = 1/\beta,\ \beta$ is a positive constant (see [2]); $Q = \varphi_s$; $\tau_p = \tau_p(E),\ \tau_w = \tau_w(E),\ \tau_\sigma = \tau_\sigma(E),\ \tau_q = \tau_q(E)$ are the relaxation times; $E = \frac{u^2}{2} + \frac{3}{2}\vartheta$; the doping density $\rho = \rho(s)$ is a function given on $[0,\,1]$. In the sequel we will assume that the function $(\rho(s) - 1)$ is sufficiently smooth and finite, and

$$1 \geq \rho(s) \geq \delta > 0, \quad s \in [0,1].$$

A typical profile of the function taken in [2] is of the form:



Following [2], we will consider the well-known in physics of semiconductors problem on ballistic diode and formulate boundary conditions at $s = 0, 1$ (see. [3, 4]):

$$R(\tau,0) = R(\tau,1) = \vartheta(\tau,0) = \vartheta(\tau,1) = 1, \quad \Sigma(\tau,1) = 0, \quad (1.3)$$

$$\varphi(\tau,0) = A, \quad \varphi(\tau,1) = A + B, \quad (1.4)$$

where A, B are constants and the bias across the diode $B > 0$. Without the loss of generality, we assume that $A = 0$. Finally, as usual, at $\tau = 0$ we formulate initial data.

Following [2], we give an equivalent formulation of mixed problem (1.1)–(1.4). We will consider system (1.1) coupled with the relation

$$\varepsilon^2 Q_\tau = \int_0^1 R(\tau,\xi)u(\tau,\xi)\,d\xi - Ru \quad (1.2')$$

instead of the Poisson equation (1.2). Equation (1.2) rewritten in the form

$$\varepsilon^2 Q_s = R - \rho \quad (1.2'')$$

will be treated as an additional stationary law which the initial data, in particular, must meet. From boundary conditions (1.4) it follows that the relation

$$\int_0^1 Q(\tau,\xi)\,d\xi = B \tag{1.5}$$

is fulfilled and the initial data also satisfy this relation. The electric potential $\varphi = \varphi(\tau, s)$ is found from the evident equality

$$\varphi(\tau,s) = \int_0^s Q(\tau,\xi)\,d\xi. \tag{1.6}$$

Thus, instead of mixed problem (1.1) – (1.4) one can consider problem (1.1), (1.2′), (1.3) with additional requirements (1.2″), (1.5), which actually are requirements on the initial data. It is easily shown that these two formulations are equivalent, at least on smooth solutions.

Problem (1.1) – (1.4) has a stationary solution (the equilibrium state) at $B = 0$:

$$\begin{gathered} u(\tau,s) = \hat{u} = 0, \quad \vartheta(\tau,s) = \hat{\vartheta} = 1, \\ \Sigma(\tau,s) = \hat{\Sigma} = 0, \quad \theta(\tau,s) = \hat{\theta} = 0, \\ R(\tau,s) = \hat{R}(s) = e^{\hat{\varphi}(s)}, \quad \varphi(\tau,s) = \hat{\varphi}(s), \end{gathered} \tag{1.7}$$

where $\hat{\varphi}(s)$ satisfies the Poisson equation

$$\varepsilon^2 \hat{\varphi}'' = \hat{R} - \rho \tag{1.8}$$

with the boundary conditions

$$\hat{\varphi}(0) = \hat{\varphi}(1) = 0. \tag{1.9}$$

It is obvious that, at small ε, the solutions to boundary value problem (1.8), (1.9) can be presented as

$$\hat{\varphi}(s) = \ln \rho(s) + O(\varepsilon). \tag{1.10}$$

By this remark, we will assume in the sequel that the function $\hat{\varphi}(s)$ is sufficiently smooth and finite.

Remark 1.1. Let $B = 0$. From certain physical considerations (see [1]), the solution of (1.1) – (1.4) tends to the equilibrium state at $\tau \to \infty$ for any initial data, i. e.,

$$\begin{gathered} u(\tau,s) \to 0, \quad \vartheta(\tau,s) \to 1, \quad \Sigma(\tau,s) \to 0, \\ \theta(\tau,s) \to 0, \quad R(\tau,s) \to \hat{R}(s), \quad \varphi(\tau,s) \to \hat{\varphi}(s). \end{gathered}$$

Below we will prove this fact for a linearized mixed problem (though at a certain, very essential, restriction on the doping density $\rho(s)$).

Remark 1.2. Note that we can consider as the equilibrium state the following functions:

$$\begin{gathered} \hat{u} = 0, \quad \hat{\vartheta} = 1, \quad \hat{\Sigma} = 0, \\ \hat{\theta} = 0, \quad \hat{R}(s) = \rho(s), \quad \varphi(\tau,s) = \ln \rho(s), \end{gathered}$$

that are obviously solution of the system (1.1) and the Poisson equation

$$\varepsilon^2 \hat{\varphi}'' = \hat{R} - \tilde{\rho},$$

where

$$\tilde{\rho} = \rho - \varepsilon^2 (\ln \rho)''.$$

It is evident that functions ρ and $\tilde{\rho}$ have the same profile with only the small difference on some intervals $\frac{1}{6} - \Delta < s < \frac{1}{6} + \Delta$, $\frac{5}{6} - \Delta < s < \frac{5}{6} + \Delta$.

2. Linearization of mixed problem (1.1) – (1.4)

Let linearize original quasilinear problem (1.1) – (1.4) with respect to the equilibrium state (1.7). As the result, omitting intermediate, quite cumbersome calculations, we come to a linear system (small perturbations of the sought values are denoted with the same symbols):

$$r_\tau + u_s + \hat{\varphi}' u = 0,$$

$$\begin{gathered} u_\tau + r_s + \vartheta_s + \sigma_s + \hat{\varphi}'\vartheta + \hat{\varphi}'\sigma + \mu u = Q, \\ \frac{3}{2}\vartheta_\tau + u_s + q_s + \hat{\varphi}' q + \frac{3}{2}\nu\vartheta = 0, \\ \frac{3}{4}\sigma_\tau + u_s + \frac{2}{5}q_s + \frac{2}{5}\hat{\varphi}' q + \frac{3}{4}\chi\sigma = 0, \\ \frac{2}{5}q_\tau + \vartheta_s + \frac{2}{5}\sigma_s + \frac{2}{5}\gamma q = \bar{\mu} u, \end{gathered} \tag{2.1}$$

$$\varepsilon^2 Q_\tau = -\hat{R}u + \int\limits_0^1 \hat{R}(\xi) u(\tau,\xi)\, d\xi. \tag{2.2}$$

Here $r = \dfrac{R}{\hat{R}}$, $\sigma = \dfrac{\Sigma}{\hat{R}}$, $q = \dfrac{\theta}{\hat{R}}$, $\mu = \dfrac{1}{\hat{\tau}_p} > 0$, $\nu = \dfrac{1}{\hat{\tau}_w} > 0$, $\chi = \dfrac{1}{\hat{\tau}_\sigma} > 0$, $\gamma = \dfrac{1}{\hat{\tau}_q} > 0$, $\bar{\mu} = \mu - \gamma$, $\hat{\tau}_p = \tau_p\left(\dfrac{3}{2}\right)$, $\hat{\tau}_w = \tau_w\left(\dfrac{3}{2}\right)$, $\hat{\tau}_\sigma = \tau_\sigma\left(\dfrac{3}{2}\right)$, $\hat{\tau}_q = \tau_q\left(\dfrac{3}{2}\right)$. It follows from [2] that the constants μ, ν, χ, γ, β are large parameters. Boundary conditions to system (2.1) become of the form

$$r(\tau,0) = r(\tau,1) = \vartheta(\tau,0) = \vartheta(\tau,1) = \sigma(\tau,1) = 0. \tag{2.3}$$

Linearization of relations (1.2″), (1.5), (1.6) results in the following conditions

$$\begin{gathered} \varepsilon^2 Q_s = \hat{R} r, \\ \int\limits_0^1 Q(\tau,\xi)\, d\xi = 0, \\ \varphi(\tau,s) = \int\limits_0^s Q(\tau,\xi)\, d\xi, \quad \text{i.e.,} \quad \varphi_s = Q. \end{gathered} \tag{2.4}$$

The aim of this paper is the study of stability (by Lyapunov) of the trivial solution to linear problem (2.1) – (2.3).

In the next section we will derive an a priori estimate which implies stability of the trivial solution. Besides, in the last section we will prove asymptotic stability of this solution under an additional condition on the function $\rho(s)$.

We rewrite system (2.1) in the vector form

$$\boldsymbol{A}U_\tau + \boldsymbol{B}U_s + \boldsymbol{D}U = \boldsymbol{F}, \tag{2.1'}$$

where

$$U = \begin{pmatrix} r \\ u \\ \vartheta \\ \sigma \\ q \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & \frac{2}{5} & 0 \end{pmatrix},$$

$$\boldsymbol{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \frac{3\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{3\chi}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{2\gamma}{5} \end{pmatrix} + \hat{\varphi}' \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} 0 \\ Q \\ 0 \\ 0 \\ \bar{\mu} u \end{pmatrix}.$$

System (2.1′) is symmetric t-hyperbolic (by Friedrichs) (see [5]). The matrix $\boldsymbol{B}$ has two positive, two negative and one zero eigenvalues. This means (see [5]) that one boundary condition in (2.3) is redundant. In fact, condition $\sigma(\tau, 1) = 0$ is automatically fulfilled if the initial function $\sigma_0(s) = \sigma(0, s)$ possesses the property

$$\sigma_0(1) = 0.$$

Indeed, from boundary conditions

$$r(\tau, 0) = r(\tau, 1) = \vartheta(\tau, 0) = \vartheta(\tau, 1) = 0$$

and from the first and the third equations of system (2.1) it follows that

$$u_s(\tau, 0) = u_s(\tau, 1) = q_s(\tau, 0) = q_s(\tau, 1) = 0. \tag{2.5}$$

While deriving (2.5), we took into account that $\hat{\varphi}(s)$ is a sufficiently smooth and finite function. By (2.5), from the fourth equation of system (2.1) we obtain

$$\sigma_\tau(\tau, 0) + \chi\sigma(\tau, 0) = 0,$$
$$\sigma_\tau(\tau, 1) + \chi\sigma(\tau, 1) = 0$$

i. e.,

$$\sigma(\tau, 0) = e^{-\chi\tau}\sigma_0(0),$$
$$\sigma(\tau, 1) = e^{-\chi\tau}\sigma_0(1).$$

Thus, boundary condition $\sigma(\tau, 1) = 0$ is fulfilled if $\sigma_0(1) = 0$. Later we will suppose also that $\sigma_0(0) = 0$, i. e.,

$$\sigma(\tau, 0) = 0. \tag{2.6}$$

The energy integral identity for system (2.1′) in the differential form is (see [5]):

$$\{\hat{R}(\boldsymbol{A}U, U)\}_\tau + \{\hat{R}(\boldsymbol{B}U, U)\}_s + \hat{R}(2\mu u^2 - 2\bar{\mu} uq + \frac{4}{5}\gamma q^2 + 3\nu\vartheta^2 + \frac{3}{2}\chi\sigma^2) = 2\hat{R}uQ, \tag{2.7}$$

where $(\boldsymbol{B}U, U) = 2(ur + u\vartheta + u\sigma + \vartheta q + \frac{2}{5}\sigma q)$, and, in a view of boundary conditions (2.3),

$$(\boldsymbol{B}U, U)|_{s=0,1} = 0. \tag{2.8}$$

Integrating (2.7) by s from 0 to 1 (with account of (2.8)), we come to the energy integral identity in the integral form for system (2.1′):

$$\frac{d}{d\tau}||U(\tau)||_A^2 + 2\int_0^1 \hat{R}(\mu u^2 - \bar{\mu} uq + \frac{2\gamma}{5}q^2 + \frac{3\nu}{2}\vartheta^2 + \frac{3\chi}{4}\sigma^2)\, d\xi = 2\int_0^1 \hat{R}uQ\, d\xi. \tag{2.9}$$

Here $||U(\tau)||_A^2 = \int_0^1 \hat{R}(\boldsymbol{A}U, U)\, d\xi$.

Multiplying (2.2) by $2Q$, integrating the obtained expression by s from 0 to 1 (accounting the relation $\int_0^1 Q(\tau,\xi)\, d\xi = 0$, see (2.4)) and summing it up with (2.9), we finally have:

$$\frac{d}{d\tau}I^{(0)}(\tau) + 2\int_0^1 \hat{R}(\mu u^2 - \bar{\mu} uq + \frac{2\gamma}{5}q^2 + \frac{3\nu}{2}\vartheta^2 + \frac{3\chi}{4}\sigma^2)\, d\xi = 0, \tag{2.10}$$

where $I^{(0)}(\tau) = ||U(\tau)||_A^2 + \varepsilon^2 \int_0^1 Q^2(\tau,\xi)\, d\xi$.

Fulfillment of (2.10) means well-posedness of linear mixed problem (2.1) – (2.3) since (2.10) implies the a priori estimate:

$$\frac{d}{d\tau}I^{(0)}(\tau) \leq 0, \quad \text{i.e., } I^{(0)}(\tau) \leq I^{(0)}(0), \quad \tau > 0 \tag{2.11}$$

provided that

$$|\bar{\mu}| \leq \sqrt{\frac{8}{5}\mu\gamma}. \tag{2.12}$$

In following we will assume that inequality (2.12) is fulfilled.

A priori estimate (2.11) means that

$$U(\tau, s) \in L_2(0,1),$$
$$Q(\tau, s) \in L_2(0,1)$$

at every $\tau > 0$. From (2.4) it follows that

$$||\varphi(\tau)||_{L_2(0,1)} \leq ||Q(\tau)||_{L_2(0,1)},$$

i. e., $\varphi(\tau, s) \in \overset{\circ}{W}{}_2^1(0,1)$ at every $\tau > 0$, and

$$||\varphi(\tau)||_{\overset{\circ}{W}{}_2^1(0,1)} \leq ||Q(\tau)||_{L_2(0,1)}.$$

A priori estimate (2.11) also implies stability of the trivial solution to linear mixed problem (2.1) – (2.3). In the subsequent sections we will obtain more delicate a priori estimates which allow to prove existence of a smooth solution to linear mixed problem (2.1) – (2.3) for all $\tau > 0$ and stability and asymptotic stability (by Lyapunov) of its trivial solution.

Remark 2.1. Indeed, from (2.4), it follows that

$$Q(\tau, s) \in W_2^1(0,1),$$
$$\varphi(\tau, s) \in \overset{\circ}{W}{}_2^2(0,1)$$

for all $\tau > 0$.

Remark 2.2. Treating the Poisson equation (see (2.4))

$$\varphi_{ss} = \beta \hat{R} r,$$

as an ordinary differential equation for the unknown function φ with the boundary conditions

$$\varphi(\tau, 0) = \varphi(\tau, 1) = 0,$$

we arrive at

$$\varphi(\tau, s) = \beta \int_0^1 g(s, \xi) \hat{R}(\xi) r(\tau, \xi)\, d\xi \tag{2.13}$$

and

$$Q(\tau, s) = \varphi_s(\tau, s) = \beta \int_0^s \hat{R}(\xi) r(\tau, \xi)\, d\xi - \beta \int_0^1 (1 - \xi) \hat{R}(\xi) r(\tau, \xi)\, d\xi. \tag{2.14}$$

Here $g(s, \xi)$ is the Green function

$$g(s, \xi) = \begin{cases} \xi(s-1) & \text{for } 0 < \xi \le s, \\ s(\xi - 1) & \text{for } s < \xi < 1. \end{cases}$$

It is seen that the functions φ, Q, given by (2.13), (2.14), satisfy conditions (2.4).

Remark 2.3. Differentiate (2.1′) by τ. Then, for U_τ, we have the system

$$\boldsymbol{A}(U_\tau)_\tau + \boldsymbol{B}(U_\tau)_s + \boldsymbol{D} U_\tau = \boldsymbol{F}_\tau, \tag{2.15}$$

where

$$\boldsymbol{F}_\tau = (0, Q_\tau, 0, 0, \bar{\mu} u_\tau)^*$$

$*$ stands for transposition.

For system (2.15) it is easy to derive the energy integral identity in the integral form:

$$\frac{d}{d\tau} I^{(1)}(\tau) + 2 \int_0^1 \hat{R}(\mu u_\tau^2 - \bar{\mu} u_\tau q_\tau + \frac{2\gamma}{5} q_\tau^2 + \frac{3\nu}{2} \vartheta_\tau^2 + \frac{3\chi}{4} \sigma_\tau^2)\, d\xi = 0, \tag{2.16}$$

where $I^{(1)}(\tau) = ||U_\tau(\tau)||_A^2 + \varepsilon^2 \int\limits_0^1 Q_\tau^2(\tau, \xi)\, d\xi$.

The a priori estimate follows from (2.16)

$$I^{(1)}(\tau) \le I^{(1)}(0), \quad \tau > 0. \tag{2.17}$$

Differentiating system (2.15) once again by τ, we finally arrive at the estimate of the form:

$$I^{(2)}(\tau) \le I^{(2)}(0), \quad \tau > 0 \quad \text{etc.} \tag{2.18}$$

Here $I^{(2)}(\tau) = ||U_{\tau\tau}(\tau)||_A^2 + \varepsilon^2 \int\limits_0^1 Q_{\tau\tau}^2(\tau,\xi)\,d\xi$.

Remark 2.4. Let introduce a value

$$t^2 = ||U_0||^2_{W_2^2(0,1)} = \int\limits_0^1 \hat{R}[(U_0,U_0) + (U_0',U_0') + (U_0'',U_0'')]\,d\xi,$$

i. e., the squared norm of the vector of initial data $U_0(s) = U(0,s) = (r_0(s),\, u_0(s),\, \vartheta_0(s),\, \sigma_0(s),\, q_0(s))^*$ in the space $W_2^2(0,1)$.

We can prove the following estimates

$$\int\limits_0^1 Q^2(0,\xi)\,d\xi \le K_1 t^2, \quad \int\limits_0^1 Q_\tau^2(0,\xi)\,d\xi \le K_2 t^2, \quad \int\limits_0^1 Q_{\tau\tau}^2(0,\xi)\,d\xi \le K_3 t^2, \tag{2.19}$$

where $K_1,\ K_2,\ K_3$ are the positive constants.

Indeed, by the formula (2.14) we have

$$\begin{aligned}|Q(0,s)|^2 &= |\beta \int\limits_0^s \hat{R}(\xi) r(0,\xi)\,d\xi - \beta \int\limits_0^1 (1-\xi)\hat{R}(\xi) r(0,\xi)\,d\xi|^2 \le \\ &\le [2\beta \int\limits_0^1 \hat{R}(\xi)|r(0,\xi)|\,d\xi]^2 = 4\beta^2 (\int\limits_0^1 \hat{R}^{1/2}|r_0|\hat{R}^{1/2}\,d\xi)^2 \le \\ &\le 4\beta^2 \int\limits_0^1 \hat{R}\,d\xi \int\limits_0^1 \hat{R} r_0^2\,d\xi = 4\beta^2 K_0 \int\limits_0^1 \hat{R} r_0^2\,d\xi \le K_1 t^2,\end{aligned}$$

where $K_0 = \int\limits_0^1 \hat{R}\,d\xi > 0,\ K_1 > 0$. Thus, the first of estimates (2.19) is fulfilled. Later, since

$$\begin{aligned}|Q_\tau(0,s)| &\le \beta(|\int\limits_0^s \hat{R}(\xi) r_\tau(0,\xi)\,d\xi| + |\int\limits_0^1 (1-\xi)\hat{R}(\xi) r_\tau(0,\xi)\,d\xi| \le \\ &\le 2\beta \int\limits_0^1 \hat{R}(\xi)|r_\tau(0,\xi)|\,d\xi \le 2\beta(\int\limits_0^1 \hat{R}\,d\xi)^{1/2}(\int\limits_0^1 \hat{R} r_\tau^2\,d\xi)^{1/2},\end{aligned}$$

than, if the inequality

$$\int\limits_0^1 \hat{R}(\xi) r_\tau^2(0,\xi)\,d\xi \le K_4 t^2, \quad K_4 > 0 \tag{2.20}$$

is held, than the estimate

$$\int\limits_0^1 Q_\tau^2(0,\xi)\,d\xi \le K_2 t^2$$

took place.

By the analogy, under condition

$$\int\limits_0^1 \hat{R}(\xi) r_{\tau\tau}^2(0,\xi)\,d\xi \le K_5 t^2, \quad K_5 > 0 \tag{2.21}$$

we can show correctness of the third of estimates (2.19). The proof of inequalities (2.20), (2.21) is given in section 3.

Finally note that

$$\int\limits_0^1 \hat{R} Q_s^2(0,\xi)\,d\xi \le \int\limits_0^1 \hat{R}\beta^2 \hat{R}^2 r^2(0,\xi)\,d\xi \le \beta^2 \max_{s\in[0,1]} |\hat{R}^2(s)| \int\limits_0^1 \hat{R} r_0^2\,d\xi \le K_6 t^2. \tag{2.22}$$

Remark 2.5. Unfortunately, we did not succeed in obtaining the additional entropy conservation law for system (1.1). Such a law is easily derived for linearized system (2.1) by exclusion of the derivatives of u_s, q_s from the first, third, fourth equations of system (2.1):

$$(r + \vartheta - \frac{5}{4}\sigma)_\tau + \hat{\varphi}' u + \nu\vartheta - \frac{5}{4}\chi\sigma = 0. \tag{2.23}$$

Remark 2.6. We will say that the trivial solution of linear problem (2.1)–(2.3) is stable, if for any $\hat{\varepsilon} > 0$ there exists $\hat{\delta}(\hat{\varepsilon}) > 0$ such that from the inequality

$$||U_0|| \le \hat{\delta}$$

will follow the inequality

$$||U(\tau)|| \le \hat{\varepsilon}$$

for all $\tau > 0$.

Remark 2.7. We will say that the trivial solution to linear problem (2.1) – (2.3) is asymptotically stable, if, at an arbitrary initial data $U_0(s)$ from a Sobolev space, the solution $U(\tau, s)$ tends to zero at $\tau \to \infty$, also in the Sobolev space.

3. Stability of trivial solution to problem (2.1) – (2.3)

Since we assume that the function $\hat{\varphi}(s)$ is sufficiently smooth (see section 1), henceforth we can suggest that $|\hat{\varphi}'(s)|$, $|\hat{\varphi}''(s)| < C_0$, where C_0 is the positive constant.

From the first equation of system (2.1) it follows that

$$r_\tau^2 = u_s^2 + (\hat{\varphi}')^2 u^2 + 2u_s \hat{\varphi}' u \le 2(u_s^2 + C_0^2 u^2),$$

so

$$\int\limits_0^1 \hat{R}(\xi) r_\tau^2(0,\xi)\,d\xi \le K_4 t^2.$$

Remark 3.1. The constants K_1, K_2, K_3, K_4 (as well as other positive constants K_i, $i = 5, \ldots, 7$, N_j, $j = 1, \ldots, 3$, M_k, $k = 1, \ldots, 18$, which appear in the sequel in this section) are determined finally via the constants μ, ν, χ, γ, β and the function $\rho(s)$.

Let us differentiate the second equation of (2.1) by s:

$$u_{\tau s} = Q_s - (r_{ss} + \vartheta_{ss} + \sigma_{ss} + \hat{\varphi}'(\vartheta_s + \sigma_s) + \mu u_s + \hat{\varphi}''(\vartheta + \sigma)).$$

Then, with account to relation (2.22) it yields the inequality

$$\int_0^1 \hat{R}(\xi) u_{\tau s}^2(0, \xi)\, d\xi \le K_7 t^2.$$

Now, differentiating the first equation of (2.1) by τ:

$$r_{\tau\tau} + u_{\tau s} + \hat{\varphi}' u_\tau = 0,$$

we arrive at the estimate

$$\int_0^1 \hat{R}(\xi) r_{\tau\tau}^2(0, \xi)\, d\xi \le K_5 t^2.$$

Thus the inequalities (2.20), (2.21) from Remark 2.4 are valid, so the estimates (2.19) are proved.

Note that

$$(\boldsymbol{A} U_\tau, U_\tau) \le N_1 \{(U, U) + (U_s, U_s) + Q^2\} \tag{3.1}$$

since at the desired functions in (2.1) we have either the constants or the known bounded functions.

From the differentiated by s system (2.1) we obtain

$$(\boldsymbol{A} U_{\tau s}, U_{\tau s}) \le N_2 \{(U, U) + (U_s, U_s) + (U_{ss}, U_{ss})\}.$$

And differentiating (2.1) by τ, we finally have

$$(\boldsymbol{A} U_{\tau\tau}, U_{\tau\tau}) \le N_2 \{(U, U) + (U_s, U_s) + (U_{ss}, U_{ss}) + Q^2 + Q_\tau^2\}. \tag{3.2}$$

Gathering estimates (2.11), (2.17), (2.18), (2.19), (3.1) and (3.2), we can write that

$$\int_0^1 \hat{R}[r^2(\tau, \xi) + u^2(\tau, \xi) + \vartheta^2(\tau, \xi) + \sigma^2(\tau, \xi) + q^2(\tau, \xi) +$$
$$+ r_\tau^2(\tau, \xi) + u_\tau^2(\tau, \xi) + \vartheta_\tau^2(\tau, \xi) + \sigma_\tau^2(\tau, \xi) + q_\tau^2(\tau, \xi) +$$
$$+ r_{\tau\tau}^2(\tau, \xi) + u_{\tau\tau}^2(\tau, \xi) + \vartheta_{\tau\tau}^2(\tau, \xi) + \sigma_{\tau\tau}^2(\tau, \xi) + q_{\tau\tau}^2(\tau, \xi)]\, d\xi \le M_1 t^2, \quad \tau > 0. \tag{3.3}$$

Completing the derivation of an a priory estimate, we deduce some auxiliary inequalities. First, we will derive estimates for the first derivatives by s of the desired functions. From the first equation of system (2.1) it follows that

$$u_s^2 \le r_\tau^2 + C_0^2 u^2,$$

so

$$\int_0^1 \hat{R}(\xi)u_s^2(\tau,\xi)\,d\xi \le M_2 t^2, \quad \tau > 0. \tag{3.4}$$

Evaluate the derivative q_s with the help of the third equation of system (2.1):

$$q_s = -\left(\frac{3}{2}\vartheta_\tau + u_s + \hat{\varphi}' q + \frac{3\nu}{2}\vartheta\right),$$

i. e.,

$$\int_0^1 \hat{R} q_s^2\,d\xi \le M_3 t^2, \quad \tau > 0. \tag{3.5}$$

While deriving similar results for ϑ_s, σ_s, r_s it is necessary to have estimates for derivatives $u_{\tau s}$, $r_{\tau s}$, $\vartheta_{\tau s}$, $\sigma_{\tau s}$. Differentiating the first equation of (2.1) by τ, we will have

$$r_{\tau\tau} + u_{\tau s} + \hat{\varphi}' u_\tau = 0.$$

Whence we easily obtain

$$\int_0^1 \hat{R} u_{\tau s}^2\,d\xi \le M_4 t^2, \quad \tau > 0. \tag{3.6}$$

Now let us use the second and fifth equation of system (2.1) differentiated by τ:

$$r_{\tau s} + \vartheta_{\tau s} + \sigma_{\tau s} = \mathcal{F}_1 = Q_\tau - u_{\tau\tau} - \hat{\varphi}'\vartheta_\tau - \hat{\varphi}'\sigma_\tau - \mu u_\tau, \tag{3.7}$$

$$\vartheta_{\tau s} + \frac{2}{5}\sigma_{\tau s} = \mathcal{F}_2 = \bar{\mu} u_\tau - \frac{2}{5} q_{\tau\tau} - \frac{2\gamma}{5} q_\tau, \tag{3.8}$$

with the help of (3.7) and (3.8) we transform the additional relation (2.23) into:

$$\left(r_{\tau s} + \vartheta_{\tau s} - \frac{5}{4}\sigma_{\tau s}\right)_\tau + \bar{\nu}\left(r_{\tau s} + \vartheta_{\tau s} - \frac{5}{4}\sigma_{\tau s}\right) = \mathcal{F}_3 = \bar{\nu}\mathcal{F}_1 - \nu\mathcal{F}_2 - \hat{\varphi}' u_{\tau s} - \hat{\varphi}'' u_\tau, \tag{3.9}$$

where $\bar{\nu} = \frac{5}{9}\chi + \frac{8}{45}\nu$. Multiplying (3.9) by $2\hat{R}\left(r_{\tau s} + \vartheta_{\tau s} - \frac{5}{4}\sigma_{\tau s}\right)$ and integrating by s from 0 to 1, we get

$$\frac{\partial}{\partial\tau}\left\{\int_0^1 \hat{R}\left(r_{\tau s} + \vartheta_{\tau s} - \frac{5}{4}\sigma_{\tau s}\right)^2 d\xi\right\} + (2\bar{\nu} - 1)\int_0^1 \hat{R}\left(r_{\tau s} + \vartheta_{\tau s} - \frac{5}{4}\sigma_{\tau s}\right)^2 d\xi \le M_5 t^2,$$

then for $\bar{\nu} > \frac{1}{2}$ we have

$$\int_0^1 \hat{R}\left(r_{\tau s} + \vartheta_{\tau s} - \frac{5}{4}\sigma_{\tau s}\right)^2 d\xi \le M_6 t^2.$$

Since from (3.7) it follows that

$$\int\limits_0^1 \hat{R}\left(r_{\tau s}+\vartheta_{\tau s}+\sigma_{\tau s}\right)^2\, d\xi \le M_7 t^2,$$

than we receive

$$\int\limits_0^1 \hat{R}\sigma_{\tau s}^2\, d\xi \le M_8 t^2,$$

and after that from (3.8) and (3.7) we sequentially deduce analogous estimates for $\vartheta_{\tau s}^2$ and $r_{\tau s}^2$. Finally

$$\int\limits_0^1 \hat{R}[r_{\tau s}^2+\vartheta_{\tau s}^2+\sigma_{\tau s}^2]\, d\xi \le M_9 t^2, \quad \tau>0. \tag{3.10}$$

Later, from the first equation of system (2.1), we obtain

$$u_{ss}=-r_{\tau s}-(\hat{\varphi}' u)_s,$$

i. e.,

$$\int\limits_0^1 \hat{R}u_{ss}^2\, d\xi \le M_{10} t^2, \quad \tau>0. \tag{3.11}$$

In order to evaluate the derivative q_{ss}, we differentiate by s the third and the fourth equations of system (3.1). From these equations and from the last equation of system (3.1), we find

$$\left(\frac{2}{3\nu}+\frac{16}{75\chi}\right)q_{ss}=-\frac{1}{\nu}\vartheta_{\tau s}-\frac{2}{5\chi}\sigma_{ss}-\left(\frac{2}{3\nu}+\frac{16}{75\chi}\right)u_{ss}-\left(\frac{2}{3\nu}+\frac{16}{75\chi}\right)(\hat{\varphi}' u)_s+\frac{2}{5}q_\tau++\frac{2}{5}\gamma q-\bar{\mu}u.$$

So, if we assume that

$$0<\nu_0<\nu, \quad 0<\chi_0<\chi,$$

than we have

$$\int\limits_0^1 \hat{R}q_{ss}^2\, d\xi \le M_{11} t^2, \quad \tau>0. \tag{3.12}$$

To evaluate the derivatives ϑ_s, σ_s, we differentiate the third and forth equations of system (2.1) by s and multiply them by $2\hat{R}\vartheta_s$ and $2\hat{R}\sigma_s$, correspondingly. Finally, we have the inequalities:

$$\frac{3}{2}\frac{\partial}{\partial\tau}\left\{\int\limits_0^1 \hat{R}\vartheta_s^2\, d\xi\right\}+(3\nu-1)\int\limits_0^1 \hat{R}\vartheta_s^2\, d\xi \le M_{12} t^2,$$

$$\frac{3}{4}\frac{\partial}{\partial\tau}\left\{\int\limits_0^1 \hat{R}\sigma_s^2\,d\xi\right\}+\left(\frac{3\chi}{2}-1\right)\int\limits_0^1 \hat{R}\sigma_s^2\,d\xi\le M_{13}t^2,$$

i. e.,

$$\int\limits_0^1 \hat{R}[\vartheta_s^2+\sigma_s^2]\,d\xi\le M_{14}t^2,\quad \tau>0. \tag{3.13}$$

While deriving (3.13) we assumed that $\nu>\frac{1}{3}$ and $\chi>\frac{2}{3}$ that is true (see [2]).

From the second equation of system (2.1) it follows that

$$r_s=\mathcal{F}_4=Q-(u_\tau+\vartheta_s+\sigma_s+\hat{\varphi}'\vartheta+\hat{\varphi}'\sigma+\mu u),$$

so

$$\int\limits_0^1 \hat{R}r_s^2\,d\xi\le M_{15}t^2,\quad \tau>0. \tag{3.14}$$

From the third equation of (2.1) differentiated by τ it follows that

$$q_{\tau s}^2\le\frac{9}{3}\vartheta_{\tau\tau}^2+u_{\tau s}^2+C_0^2q_\tau^2+\frac{9\nu^2}{4}\vartheta_\tau^2,$$

i. e.,

$$\int\limits_0^1 \hat{R}q_{\tau s}^2\,d\xi\le M_{16}t^2,\quad \tau>0. \tag{3.15}$$

Finally to estimate the derivatives $\vartheta_{ss},\ r_{ss},\sigma_{ss}$ we will use again the method we have used while deriving the derivatives $\vartheta_{\tau s},\ r_{\tau s},\sigma_{\tau s}$. To this end we differentiate the second and fifth equations of system (2.1) by s:

$$r_{ss}+\vartheta_{ss}+\sigma_{ss}=G_1=\beta\hat{R}r-u_{\tau s}-(\hat{\varphi}'\vartheta)_s-(\hat{\varphi}'\sigma)_s-\mu u_s, \tag{3.16}$$

$$\vartheta_{ss}+\frac{2}{5}\sigma_{ss}=G_2=\bar{\mu}u_s-\frac{2}{5}q_{\tau s}-\frac{2\gamma}{5}q_s, \tag{3.17}$$

and (2.23) by s twice:

$$\left(r_{ss}+\vartheta_{ss}-\frac{5}{4}\sigma_{ss}\right)_\tau+\frac{11}{15}\nu\left(r_{ss}+\vartheta_{ss}-\frac{5}{4}\sigma_{ss}\right)=G_3=-\nu G_2+\frac{11}{15}\nu G_1-(\hat{\varphi}'u)_{ss}. \tag{3.18}$$

Then it follows from (3.16), (3.17), (3.18) that

$$\int\limits_0^1 \hat{R}[r_{ss}^2+\vartheta_{ss}^2+\sigma_{ss}^2]\,d\xi\le M_{11}t^2,\quad \tau>0. \tag{3.19}$$

Gathering estimates (3.3) – (3.6), (3.10) – (3.15), (3.19), we can write down the desired a priori estimate:

$$\int_0^1 \hat{R}\,[(U,U)+(U_\tau,U_\tau)+(U_s,U_s)+(U_{\tau\tau},U_{\tau\tau})+(U_{\tau s},U_{\tau s})+(U_{ss},U_{ss})]\,d\xi \le M_{12}t^2,\ \ \tau>0. \quad (3.20)$$

From (3.20) it follows that

$$U(\tau,s)\in W_2^2(0,1),\quad Q(\tau,s)\in W_2^3(0,1),\quad \varphi(\tau,s)\in \overset{\circ}{W}{}_2^4(0,1)\ \text{ for all } \tau>0.$$

It also implies stability (by Lyapunov) of the equilibrium state in the linear approximation.

Remark 3.2. More exactly,

$$r(\tau,s),\ \vartheta(\tau,s),\ \sigma(\tau,s)\in \overset{\circ}{W}{}_2^2(0,1).$$

Besides,

$$||U(\tau)||^2_{W_2^2(0,1)} \le M_{12}t^2,\quad \tau>0,$$

which means (see Remark 2.5) stability (by Lyapunov) of the trivial solution to mixed problem (2.1) – (2.3). Here

$$||U(\tau)||^2_{W_2^2(0,1)} = \int_0^1 \hat{R}[(U,U)+(U_s,U_s)+(U_{ss},U_{ss})]\,d\xi.$$

4. Asymptotic stability of trivial solution

Evidently, the first equation of system (2.1) can be rewritten as

$$(\hat{R}r)_\tau + (\hat{R}u)_s = 0.$$

It is expedient to introduce into consideration a potential $\Psi=\Psi(\tau,s)$ such that

$$\begin{aligned}\hat{R}u &= \Psi_\tau,\\ \hat{R}r &= -\Psi_s.\end{aligned} \quad (4.1)$$

First two boundary conditions in (2.3) imply

$$\Psi_s(\tau,0)=\Psi_s(\tau,1)=0. \quad (4.2)$$

The remained conditions in (2.3) and assumption (2.6) give

$$(\hat{R}\vartheta)(\tau,0)=(\hat{R}\vartheta)(\tau,1)=0, \quad (4.3)$$
$$(\hat{R}\sigma)(\tau,0)=(\hat{R}\sigma)(\tau,1)=0. \quad (4.4)$$

Note that equation (2.2) with regard to (4.1) transforms into

$$\left(\varepsilon^2 Q + \Psi - \int_0^1 \Psi\,d\xi\right)_\tau = 0,$$

i. e.,

$$\varepsilon^2 Q + \Psi - \int_0^1 \Psi \, d\xi = A_0(s),$$

where $A_0(s)$ is an arbitrary function. On the other hand, by (2.4), we have

$$\varepsilon^2 Q_s = \hat{R} r, \qquad \text{i. e., } A_0'(s) = 0 \text{ and } A_0 = \text{const};$$
$$\text{since } \int_0^1 Q \, d\xi = 0, \quad A_0 = 0.$$

So,

$$Q = \beta \left(\int_0^1 \Psi \, d\xi - \Psi \right) = \beta h(\tau, s). \tag{4.5}$$

Rewrite the second condition of system (2.1) as

$$(\hat{R} u)_\tau + (\hat{R} r)_s - \hat{\varphi}' \hat{R} r + (\hat{R} \vartheta)_s + (\hat{R} \sigma)_s + \mu \hat{R} u = \hat{R} Q,$$

and then, with the use of relations (4.1), in the form

$$\Psi_{\tau\tau} - \Psi_{ss} + (\hat{R} \vartheta)_s + (\hat{R} \sigma)_s + \mu \Psi_\tau + \hat{\varphi}' \Psi_s = \hat{R} \beta h. \tag{4.6}$$

By the analogy, the third, fourth, fifth equations from (2.1) can be rewritten as follows:

$$\frac{3}{2}(\hat{R}\vartheta)_\tau + \Psi_{\tau s} - \hat{\varphi}' \Psi_\tau + (\hat{R} q)_s + \frac{3\nu}{2}(\hat{R}\vartheta) = 0, \tag{4.7}$$

$$\frac{3}{4}(\hat{R}\sigma)_\tau + \Psi_{\tau s} - \hat{\varphi}' \Psi_\tau + \frac{2}{5}(\hat{R} q)_s + \frac{3\chi}{4}(\hat{R}\sigma) = 0, \tag{4.8}$$

$$\frac{2}{5}(\hat{R} q)_\tau + (\hat{R}\vartheta)_s - \hat{\varphi}'(\hat{R}\vartheta) + \frac{2}{5}(\hat{R}\sigma)_s - \frac{2}{5}\hat{\varphi}'(\hat{R}\sigma) + \frac{2\gamma}{5}(\hat{R} q) - \bar{\mu} \Psi_\tau = 0. \tag{4.9}$$

Later we will use equations obtained from (4.6) – (4.9) by differentiating by s:

$$H_{\tau\tau} - H_{ss} + (\hat{R}\vartheta)_{ss} + (\hat{R}\sigma)_{ss} + \mu H_\tau + \hat{\varphi}' H_s + \beta(2\hat{R} - \rho) H = \hat{R}' \beta h, \tag{4.10}$$

$$\frac{3}{2}(\hat{R}\vartheta)_{\tau s} + H_{\tau s} - (\hat{\varphi}' \Psi_\tau)_s + (\hat{R} q)_{ss} + \frac{3\nu}{2}(\hat{R}\vartheta)_s = 0, \tag{4.11}$$

$$\frac{3}{4}(\hat{R}\sigma)_{\tau s} + H_{\tau s} - (\hat{\varphi}' \Psi_\tau)_s + \frac{2}{5}(\hat{R} q)_{ss} + \frac{3\chi}{4}(\hat{R}\sigma)_s = 0, \tag{4.12}$$

$$\frac{2}{5}(\hat{R} q)_{\tau s} + (\hat{R}\vartheta)_{ss} - (\hat{\varphi}' \hat{R}\vartheta)_s + \frac{2}{5}(\hat{R}\sigma)_{ss} - \frac{2}{5}(\hat{\varphi}' \hat{R}\sigma)_s + \frac{2\gamma}{5}(\hat{R} q)_s - \bar{\mu} H_\tau = 0, \tag{4.13}$$

where $H = \Psi_s$.

Remark 4.1. The aggregate h (see (4.5)) can be rewritten

$$h(\tau, s) = \int_0^1 \Psi(\tau, \xi)\, d\xi - \Psi(\tau, s) = \int_0^1 [\Psi(\tau, \xi) - \Psi(\tau, s)]\, d\xi =$$

$$= \int_0^1 \left[\int_s^\xi \Psi_z(\tau, z)\, dz \right] d\xi = \int_0^1 \int_s^\xi H(\tau, z)\, dz\, d\xi. \tag{4.14}$$

Now we proceed to derivation of the desired estimate. With this purpose, we multiply equation (4.6) by $2\Psi_\tau$, equation (4.7) by $2(\hat{R}\vartheta)$, equation (4.8) by $2(\hat{R}\sigma)$, equation (4.9) by $2(\hat{R}q)$, sum them up and integrate it by s from 0 to 1 (accounting boundary conditions (4.2)–(4.4)):

$$\frac{d}{d\tau}\left\{ \int_0^1 \left[\Psi_\tau^2 + \Psi_s^2 + \frac{3}{2}(\hat{R}\vartheta)^2 + \frac{3}{4}(\hat{R}\sigma)^2 + \frac{2}{5}(\hat{R}q)^2 \right] d\xi \right\} +$$

$$+2 \int_0^1 \left[\mu\Psi_\tau^2 + \frac{3\nu}{2}(\hat{R}\vartheta)^2 + \frac{3\chi}{4}(\hat{R}\sigma)^2 + \frac{2\gamma}{5}(\hat{R}q)^2 + \hat{\varphi}'\Psi_\tau\Psi_s - \hat{\varphi}'\Psi_\tau(\hat{R}\vartheta) - \hat{\varphi}'\Psi_\tau(\hat{R}\sigma) - \right.$$

$$\left. -\bar{\mu}\Psi_\tau(\hat{R}q) - \hat{\varphi}'(\hat{R}\vartheta)(\hat{R}q) - \frac{2}{5}\hat{\varphi}'(\hat{R}\sigma)(\hat{R}q) \right] d\xi = 2\beta \int_0^1 \hat{R}\Psi_\tau h\, d\xi. \tag{4.15}$$

By analogy, multiplying (4.10) by $2H_\tau$, (4.11) by $2(\hat{R}\vartheta)_s$, (4.12) by $2(\hat{R}\sigma)_s$, (4.13) by $2(\hat{R}q)_s$, summing up, integrating by s (accounting (4.2)–(4.4) and also (2.5)), we have:

$$\frac{d}{d\tau}\left\{ \int_0^1 \left[H_\tau^2 + H_s^2 + \frac{3}{2}(\hat{R}\vartheta)_s^2 + \frac{3}{4}(\hat{R}\sigma)_s^2 + \frac{2}{5}(\hat{R}q)_s^2 + \beta(2\hat{R} - \rho)H^2 \right] d\xi \right\} +$$

$$+2 \int_0^1 \left[\mu H_\tau^2 + \frac{3\nu}{2}(\hat{R}\vartheta)_s^2 + \frac{3\chi}{4}(\hat{R}\sigma)_s^2 + \frac{2\gamma}{5}(\hat{R}q)_s^2 + \hat{\varphi}' H_\tau H_s - (\hat{\varphi}'\Psi_\tau)_s(\hat{R}\vartheta)_s - (\hat{\varphi}'\Psi_\tau)_s(\hat{R}\sigma)_s - \right.$$

$$\left. -\bar{\mu}H_\tau(\hat{R}q)_s - (\hat{\varphi}'\hat{R}\vartheta)_s(\hat{R}q)_s - \frac{2}{5}(\hat{\varphi}'\hat{R}\sigma)_s(\hat{R}q)_s \right] d\xi = 2\beta \int_0^1 \hat{R}' H_\tau h\, d\xi. \tag{4.16}$$

We will use the result of integration by s from 0 to 1 (with account to boundary conditions) of equation (4.10) multiplied by $2H$:

$$\frac{d}{d\tau}\left\{ \int_0^1 \left[2HH_\tau + \mu H^2 \right] d\xi \right\} + 2 \int_0^1 \left[-H_\tau^2 + H_s^2 - H_s(\hat{R}\vartheta)_s - \right.$$

$$-H_s(\hat{R}\sigma)_s + \frac{\beta}{2}(3\hat{R}-\rho)H^2\Big]\,d\xi = 2\beta\int\limits_0^1 \hat{R}'Hh\,d\xi. \tag{4.17}$$

Summing up (4.15)–(4.17), we obtain the expression:

$$\frac{d}{d\tau}J^{(0)} + J^{(1)} = 2\beta\int\limits_0^1 \hat{R}\Psi_\tau h\,d\xi + 2\beta\int\limits_0^1 \hat{R}'H_\tau h\,d\xi + 2\beta\int\limits_0^1 \hat{R}'Hh\,d\xi \le$$

$$\le 2\beta\left(\int\limits_0^1 \hat{R}^2\,d\xi\right)^{\frac{1}{2}}\left(\int\limits_0^1 (\hat{R}u)^2\,d\xi\right)^{\frac{1}{2}}\left(\int\limits_0^1 (\hat{R}r)^2\,d\xi\right)^{\frac{1}{2}} +$$

$$+2\beta\left(\int\limits_0^1 (\hat{R}')^2\,d\xi\right)^{\frac{1}{2}}\left(\int\limits_0^1 (\hat{R}r)^2_\tau\,d\xi\right)^{\frac{1}{2}}\left(\int\limits_0^1 (\hat{R}r)^2\,d\xi\right)^{\frac{1}{2}} + 2\beta\left(\int\limits_0^1 (\hat{R}')^2\,d\xi\right)^{\frac{1}{2}}\int\limits_0^1 (\hat{R}r)^2\,d\xi. \tag{4.18}$$

Note that the estimate of the right hand part in (4.18) is derived with the help of the inequalities of Hölder and Cauchy, and formula (4.14). Besides, in (4.18)

$$J^{(0)} = \int\limits_0^1 \Big[(\hat{R}u)^2 + \frac{3}{2}(\hat{R}\vartheta)^2 + \frac{3}{4}(\hat{R}\sigma)^2 + \frac{2}{5}(\hat{R}q)^2 + (\hat{R}r_\tau)^2 +$$

$$+ (\hat{R}r)^2_s + \frac{3}{2}(\hat{R}\vartheta)^2_s + \frac{3}{4}(\hat{R}\sigma)^2_s + \frac{2}{5}(\hat{R}q)^2_s +$$

$$+ (\beta(2\hat{R}-\rho) + \mu + 1)(\hat{R}r)^2 + 2(\hat{R}r)(\hat{R}r_\tau)\Big]\,d\xi,$$

$$J^{(1)} = 2\int\limits_0^1 \Big[\mu(\hat{R}u)^2 + \frac{3\nu}{2}(\hat{R}\vartheta)^2 + \frac{3\chi}{4}(\hat{R}\sigma)^2 + \frac{2\gamma}{5}(\hat{R}q)^2 +$$

$$+ (\mu-1)(\hat{R}r)^2_\tau + (\hat{R}r)^2_s + \frac{3\nu}{2}(\hat{R}\vartheta)^2_s + \frac{3\chi}{4}(\hat{R}\sigma)^2_s + \frac{2\gamma}{5}(\hat{R}q)^2_s +$$

$$+ \frac{\beta}{2}(3\hat{R}-\rho)(\hat{R}r)^2 - \hat{\varphi}'(\hat{R}u)(\hat{R}r) - \hat{\varphi}'(\hat{R}u)(\hat{R}\vartheta) - \hat{\varphi}'(\hat{R}u)(\hat{R}\sigma) -$$

$$- \bar{\mu}(\hat{R}u)(\hat{R}q) - \hat{\varphi}'(\hat{R}\vartheta)(\hat{R}q) - \frac{2}{5}\hat{\varphi}'(\hat{R}\sigma)(\hat{R}q) + \hat{\varphi}'(\hat{R}r)_\tau(\hat{R}r)_s -$$

$$- (\hat{\varphi}'\hat{R}u)_s(\hat{R}\vartheta)_s - (\hat{\varphi}'\hat{R}u)_s(\hat{R}\sigma)_s + \bar{\mu}(\hat{R}r)_\tau(\hat{R}q)_s - (\hat{\varphi}'\hat{R}\vartheta)_s(\hat{R}q)_s -$$

$$- \frac{2}{5}(\hat{\varphi}'\hat{R}\sigma)_s(\hat{R}q)_s + (\hat{R}r)_s(\hat{R}\vartheta)_s + (\hat{R}r)_s(\hat{R}\sigma)_s\Big]\,d\xi.$$

Rewrite (4.18) with account to the estimate of the right hand part as the following inequality

$$\frac{d}{d\tau}J^{(0)} + J^{(2)} \le 0, \tag{4.19}$$

where

$$J^{(2)} = J^{(1)} - \beta \left(\int_0^1 \hat{R}^2 \, d\xi \right)^{\frac{1}{2}} \varepsilon_1 \int_0^1 (\hat{R}u)^2 \, d\xi -$$

$$-\beta \left(\int_0^1 \hat{R}^2 \, d\xi \right)^{\frac{1}{2}} \frac{1}{\varepsilon_1} \int_0^1 (\hat{R}r)^2 \, d\xi - \beta \left(\int_0^1 (\hat{R}')^2 \, d\xi \right)^{\frac{1}{2}} \varepsilon_2 \int_0^1 (\hat{R}r)_\tau^2 \, d\xi -$$

$$-\beta \left(\int_0^1 (\hat{R}')^2 \, d\xi \right)^{\frac{1}{2}} \frac{1}{\varepsilon_2} \int_0^1 (\hat{R}r)^2 \, d\xi - 2\beta \left(\int_0^1 (\hat{R}')^2 \, d\xi \right)^{\frac{1}{2}} \int_0^1 (\hat{R}r)^2 \, d\xi.$$

While deriving $J^{(2)}$ we used the Cauchy inequality with some positive constants $\varepsilon_1,\ \varepsilon_2$.

In $J^{(0)}$ and $J^{(2)}$ the expressions under the integral sign are positive definite squared forms of the variables $\hat{R}r,\ \hat{R}u,\ \hat{R}\vartheta,\ \hat{R}\sigma,\ \hat{R}q,\ (\hat{R}r)_s,\ (\hat{R}\vartheta)_s,\ (\hat{R}\sigma)_s,\ (\hat{R}q)_s,\ \hat{R}r_\tau$, since the parameters $\mu,\ \nu,\ \chi,\ \gamma,\ \beta$ are sufficiently large, with $\mu > \dfrac{1}{5}\gamma$ and if we choose constants $\varepsilon_1,\ \varepsilon_2$ such that

$$\varepsilon_i < \frac{3\mu - \gamma}{\beta}, \quad i = 1, 2,$$

$$\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} < 2l.$$

Here positive constant l is founded from the inequality

$$\delta - \left(\int_0^1 (\rho'(\xi))^2 \, d\xi \right)^{\frac{1}{2}} > l, \tag{4.20}$$

which exactly is the essential restriction on the function $\rho(s)$ first mentioned in Remark 1.1. Note, that $\bar{\mu} < 0$ (see [1]) and with high accuracy, by (1.10),

$$3\hat{R} - \rho = 2\rho.$$

Under fulfillment of (4.20) there exists a constant $M_0 > 0$ such that

$$J^{(2)} \geq M_0 J^{(0)}. \tag{4.21}$$

By (4.21) inequality (4.19) transforms into

$$\frac{d}{d\tau} J^{(0)} + M_0 J^{(0)} \leq 0,$$

i. e.,

$$J^{(0)}(\tau) \leq e^{-M_0 \tau} J^{(0)}(0). \tag{4.22}$$

Remind relations (4.1) and rewrite (4.22) as

$$\int_0^1 \left[r^2(\tau, s) + u^2(\tau, s) + \vartheta^2(\tau, s) + \sigma^2(\tau, s) + q^2(\tau, s) + r_\tau^2(\tau, s) + \right.$$

$$+ r_s^2(\tau,s) + u_s^2(\tau,s) + \vartheta_s^2(\tau,s) + \sigma_s^2(\tau,s) + q_s^2(\tau,s)\big]\, d\xi \le M_1 e^{-M_0\tau} t^2. \tag{4.23}$$

Here $M_1 > 0$ is a constant, $t^2 = ||U_0||^2_{W_2^1(0,1)} = \int\limits_0^1 [(U_0,U_0) + (U_0', U_0')]\, d\xi$ is the squared norm of the vector of initial data $U_0(s) = U(0,s) = (r_0(s),\ u_0(s),\ \vartheta_0(s),\ \sigma_0(s),\ q_0(s))^*$ in the Sobolev's space $W_2^1(0,1)$.

Remark 4.2. Constants $M_0,\ M_1$, as well as positive constants $M_2,\ M_3$, are finally determined via the constants $\mu,\ \nu,\ \chi,\ \gamma,\ \beta$ and the function $\rho(s)$.

Derivatives u_τ, ϑ_τ, σ_τ and q_τ are estimated with the help of system (2.1):

$$\begin{aligned}
u_\tau &= Q - (r_s + \vartheta_s + \sigma_s + \varphi'\vartheta + \hat{\varphi}'\sigma + \mu u),\\
\vartheta_\tau &= -\frac{2}{3}(u_s + q_s + \hat{\varphi}' q + \frac{3}{2}\nu\vartheta),\\
\sigma_\tau &= -\frac{4}{3}(u_s + \frac{2}{5}q_s + \frac{2}{5}\hat{\varphi}' q + \frac{3}{4}\nu\vartheta),\\
q_\tau &= -\frac{5}{2}(\vartheta_s + \frac{2}{5}\sigma_s + \frac{2}{5}\gamma q - \bar{\mu} u).
\end{aligned}$$

Consequently,

$$\int_0^1 \left[u_\tau^2(\tau,s) + \vartheta_\tau^2(\tau,s) + \sigma_\tau^2(\tau,s) + q_\tau^2(\tau,s)\right]\, d\xi \le M_2 e^{-M_0\tau} t^2. \tag{4.24}$$

Combining estimates (4.23) and (4.24), we come to the desired a priori estimate:

$$\int\limits_0^1 [(U,U) + (U_\tau, U_\tau) + (U_s, U_s)]\, d\xi \le M_3 e^{-M_0\tau} t^2, \quad \tau > 0. \tag{4.25}$$

From (4.25) it follows that

$$\begin{aligned}
&U(\tau,s) \in W_2^1(0,1),\\
&Q(\tau,s) \in W_2^2(0,1),\\
&\varphi(\tau,s) \in \overset{\circ}{W}{}_2^3(0,1), \quad \text{for all } \tau > 0,
\end{aligned}$$

and the equilibrium state in the linear approximation is asymptotically stable (by Lyapunov).

Remark 4.3. Precisely,

$$r(\tau,s),\ \vartheta(\tau,s), \sigma(\tau,s) \in \overset{\circ}{W}{}_2^1(0,1).$$

Besides,

$$||U(\tau)||^2_{W_2^1(0,1)} \le M_3 e^{-M_0\tau} t^2, \quad \tau > 0,$$

just this means (see Remark 2.6) asymptotic stability (by Lyapunov) of the trivial solution to mixed problem (2.1)–(2.3). Remind that

$$||U(\tau)||^2_{W_2^1(0,1)} = \int_0^1 [(U,U) + (U_s,U_s)]\, d\xi.$$

5. Conclusions

The analysis, carried out in the paper, states a very important (from the applications point of view) fact on asymptotic stability of the equilibrium state for the antidemocratic hydrodynamical model (see [1]) of charge transport in semiconductors. Indeed, in absence of *the bias* across the real semiconductor devices, transport of charge carriers (i. e., electric flow) must be absent. Consequently, applying hydrodynamical models in description of physical phenomena of charge transport in semiconductors, we must require of them the adequate description of these phenomena (including correct description of the transition process in semiconductor devices in absence of the bias across the diode).

Unfortunately, the fact of asymptotical stability of the equilibrium state is proved under essential restriction (4.20) on the doping density $\rho(s)$ and in the linear approximation as yet. It should be noted at the same time that proof of stability of the equilibrium state does not contain any restrictions on the doping density.

We gratefully thank Prof. A. M. Anile for many helpful discussions. We also appreciate A. A. Iohrdanidy for efficient cooperation.

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Received for publication November 24, 1998,
in revised form January 6, 1999