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OPTIMAL CONTROL THEORY: ON THE GLOBAL STRUCTURE AND CONNECTIONS WITH OPTIMIZATION PART 1

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Вводятся и исследуются различные понятия *глобальных структур нелинейных задач теории оптимального управления*

$$\mathcal{P}(\ell, L, F, E, G) \quad \begin{cases} \text{Min} \quad \mathcal{I}(x,u) := \ell(x(a), x(b)) + \int_a^b L(t, x(t), u(t))\, dt, \quad \text{где} \\ \dot{x}(t) = F(t, x(t), u(t)) \quad (t \in [a,b]), \\ E(x(a), x(b)) = 0, \quad G(t, x(t), u(t)) \geq 0 \quad (t \in [a,b]), \end{cases}$$

таких, что *нелинейные задачи оптимизации* вида

$$\mathcal{P}(\mathbf{f}, \mathbf{E}, \mathbf{G}) \quad \begin{cases} \text{Min } \mathbf{f}(\mathbf{x}), \quad \text{где} \\ \mathbf{E}(\mathbf{x}) = 0, \quad \mathbf{G}(\mathbf{y}, \mathbf{x}) \geq 0 \quad (\mathbf{y} \in Y) \end{cases}$$

становятся содержательными при линейных ограничениях на F. Если в *частичных структурах* управляющая переменная u трактуется как параметр, то в *композитных структурах* рассматриваются полные зависимости от x и u. Поэтому т. н. неявная *функция Куна — Таккера* становится также содержательной. Композитная концепция основывается на *принципе минимума* и оказывается гарантированно *полной* в смысле условия регулярности на допустимых множествах (Мангасарян — Фромовиц), точек Куна — Таккера (Коима) в рамках структуры кусочной дифференцируемости (трансверсальности). А именно, при некоторых условиях регулярности для описания структуры не нужны никакие типы неустойчивости (вырождение, бифуркация). Эти два понятия включают в себя соответствующие понятия глобальной (топологической) устойчивости. При общих предположениях компактности структурная устойчивость может быть описана в терминах наших условий регулярности. В части 1 описываются частичные структуры и основы композитных структур. Последние более полно описаны в части 2 [43], где также определяются и кусочно декомпозитные структуры.

1. Introduction

In nonlinear optimization, the *global* structure of an *ordinary* **s***emi*-**i***nfinite minimization problem* [3, 12, 18]

$$\mathcal{P}_{\mathcal{SI}}(\mathbf{f},\mathbf{E},\mathbf{G}) \quad \begin{cases} \text{Min } \mathbf{f}(\mathbf{x}) \text{ on } M_{\mathcal{SI}}[\mathbf{E},\mathbf{G}], \quad \text{where} & (1.1) \\ \left. \begin{array}{rl} M_{\mathcal{SI}}[\mathbf{E},\mathbf{G}] & := \{\mathbf{x} \in I\!R^n \mid \mathbf{e}_i(\mathbf{x}) = 0 \ \ (i \in I), \\ & \qquad \mathbf{G}(\mathbf{y},\mathbf{x}) \geq 0 \ \ (\mathbf{y} \in Y)\} \end{array} \right\} & (1.2) \end{cases}$$

can be defined by means of the whole entity of lower level sets

$$\mathcal{L}^{\theta}_{\mathcal{SI}}(\mathbf{f},\mathbf{E},\mathbf{G}) \ := \ \{\mathbf{x} \in M_{\mathcal{SI}}[\mathbf{E},\mathbf{G}] \mid \ \mathbf{f}(\mathbf{x}) \leq \ \theta\} \quad (\theta \in I\!R). \tag{1.3}$$

For the special case where Y consists of only finitely many elements this structure was analyzed by Jongen, Jonker, Twilt [19], Guddat, Jongen [9] and Jongen, Weber [22]. We abbreviate *finitely constrained* optimization by $\mathcal{F}$; hereby, we also write $\mathbf{G}_{\mathbf{y}}(\mathbf{x}) := \mathbf{G}(\mathbf{y},\mathbf{x})$. For the general semi-infinite case we refer to Jongen, Rückmann [19] and Weber [40]. All these investigations deal with the structural *(topological) stability* of the considered problems.

In this paper, we widen our scope to a large class of **optimal control problems**. First of all, we are concerned with an unfolding of the *structural* topological aspect. Later on, we also present and incorporate reasonable notions of *structural stability* into the structure. We shall also *characterize* this global stabilitity. Indeed, hereby certain *connections* to nonlinear minimization problems of the form (1.1), (1.2) always play a basic part. However, in order suitably to reflect the aspect of *infinity* which is characteristic for optimal control, we have to enrich the modelling and its notions. This will be started in the present *part 1* and continued in the following *part 2* [43].

Now, the *nonlinear optimal control problems* that we study, namely look as follows (for related problems see, e. g., [5, 11, 14, 25, 33, 35]):

$$\mathcal{P}(\ell,L,F,E,G) \quad \begin{cases} \text{Min } \mathcal{I}(x,u) \ := \ \ell(x(a),x(b)) + \int_a^b L(t,x(t),u(t))\,dt, \ \text{where} & (1.4) \\ x \in (C^0_{pw\,2}([a,b],I\!R))^n, \quad u \in (F_{pw\,2}([a,b],I\!R))^q, & (1.5) \\ \text{such that} & \\ \dot{x}(t) = F(t,x(t),u(t)) \quad (\text{for almost every } \ t \in [a,b]), & (1.6) \\ (x(a),x(b)) \in M[E], & (1.7) \\ x(t) \in M[G(t,\cdot,u(t))] \quad (\text{for almost every } \ t \in [a,b]). & (1.8) \end{cases}$$

There should not be misunderstandings caused by reserving the (fat) bold face style of writing functions for optimization problems, but not for our optimal control problem(s). With the study of $\mathcal{P}(\ell,L,F,E,G)$ we continue the *one parametric, generical* study [42] under the *structural* aspect, where the parameters may now be taken from the full functions' space.

Here, (L,F,G), (ℓ,E) will be of the class C^3, C^2, respectively. The notations

$$F_{pw\,k}([a,b],I\!R), \quad C^r_{pw\,k}([a,b],I\!R) \quad (r,k \in \mathbb{Z}_+ \cup \{+\infty\}, \ \ r<k)$$

stand for the class of functions or C^r-functions, which are piecewise C^k. They are important subclasses of the Lebesgue- or Sobolev-spaces of L^∞- or $W^{1,\infty}$-functions. Moreover, the sets $M[E]$, $M[G(t,\cdot,u(t))]$ are understood with the following meaning:

$$\left.\begin{array}{rcl} M[E] & = & \{(\mathbf{x}^1,\mathbf{x}^2) \in I\!R^n \times I\!R^n \mid e_i(\mathbf{x}^1,\mathbf{x}^2) = 0 \ (i \in I)\}, \\ M[G(t,\cdot,u(t))] & = & \{\mathbf{x} \in I\!R^n \mid g_j(t,\mathbf{x},u(t)) \geq 0 \ (j \in J)\}. \end{array}\right\} \tag{1.9}$$

Namely, with n, q denoting the number of coordinates of the space and control variable x, u, respectively, we fix the two index sets $I := \{1,\ldots,m\}$, where these numbers are chosen high enough: $m < n < q$, and $J := \{1,\ldots,s\}$. In the sequel, the functions ℓ, e_i $(i \in I)$ belong to $C^2(I\!R^{2n}, I\!R)$, while the functions L, $f_k\,(k \in \{1,\ldots,n\})$ and $g_j\,(j \in J)$ are elements of $C^3(I\!R^{n+q+1}, I\!R)$. Finally, we let F,E,G stand for $(f_1,\ldots,f_n)^T$, $(e_1,\ldots,e_m)^T$, and $(g_1,\ldots,g_s)^T$, respectively.

In order to guarantee the existence of a *flow* ([1]) induced by differential constraints which are related with (1.6), we assume *linear boundedness* of the right hand side:

Assumption (LB): *There exist positive functions $\alpha_0, \beta_0 \in C(I\!R^{q+1}, I\!R)$ such that the following estimation with respect to the Euclidean norm $||\cdot||$ of $I\!R^n$ holds:*

$$||F(t,\mathbf{x},\mathbf{u})|| \leq \alpha_0(t,\mathbf{u})||\mathbf{x}|| + \beta_0(t,\mathbf{u}) \quad ((t,\mathbf{x},\mathbf{u}) \in I\!R^{n+q+1}).$$

Because of a small argumentation from differential topology ([13]), in the case of compactness of the following set, formally being a feasible set in the sense of *finitely constrained* optimization,

$$M[G] = \{(t,\mathbf{x},\mathbf{u}) \in I\!R^{n+q+1} \mid t \in [a,b],\ g_j(t,\mathbf{x},\mathbf{u}) \geq 0 \ (j \in J)\}, \tag{1.10}$$

the assumption (LB) can be made even without loss of generality. Let us consider the following short explanation.

In our research, *differential topology* is useful in order to distinguish between the *local* and the *global* aspect. This means concentration on (relative) neighbourhoods and consideration of a whole given (feasible) set (or Euclidean space), respectively. A further use of differential topology consists in realizing some desired *perturbation(s)* of such functions which define a given optimization or optimal control problem. In some situations, these problems or parts of it may (by means of a small perturbation) be made stable in a certain sense. In the present context of Assumption (LB), it is enough to consider F in some neighbourhood $\mathcal{U}$ of $M[G]$. Later on, we shall make the assumption of compactness of the set $M[G]$ (hence, maybe, also of $\mathcal{U}$) within a suitably wide **compactness assumption**, being called **(COMP)**. Now, we may perturb F (with the help of differential topology) outside of $\mathcal{U}$ such that F fulfills the condition (LB); i.e., it really remains unchanged.

We are concerned with finding and giving positive answers to the following two questions:

($\mathcal{Q}_1$). *Is there a reasonable concept of the* global structure of $\mathcal{P}(\ell,L,F,E,G)$ *?*

($\mathcal{Q}_2$). *Can the topological* stability *of such a structure be* characterized *by means of qualitative (regular) properties of its parts?*

As we want to introduce different concepts of the structure of $\mathcal{P}(\ell,L,F,E,G)$, there may be more so-called *regularity conditions* (or assumptions) used in order to guarantee all parts of these structures to be *well-defined*. Then we call the structure *complete*. Otherwise some irregular or, say, bifurcation (or vanishing) behaviour of sets or points become characteristic for the composite structure. Therefore, as we shall explain in the sequel, the **perturbational** aspect of global *structural stability* will be considered as an *intrinsic* item (element) of the problem's structure. This topological item can be expressed both in the language of the problem's quality, say (ir)regularity, and the language of (no) bifurcation phenomena.

For a better understanding of that perturbational aspect, regarded as an intrinsic item, we give the following explanation and motivation.

In the three concepts of structure which we work out in this *part 1* and in *part* (paper) *2*, the different parts of the structure (optimization problems, implicit functions) are stated (or concluded under some regularity condition). First properties may immediately be realized. This is like a "photo" of the problem at *one* single moment, and we call the underlying viewpoint the **static aspect**. If, however, the problem is considered at, say, *different* moments (lying close to each other), then the **dynamical aspect** of perturbations enters the structural concept. The answer of the question whether the given optimal control problem qualitatively remains the same (or changes) under such perturbations, is already implied in the (ir)regularity of its different parts. Hence, by means of the behaviour under small perturbations (dynamical features) important qualitative properties of the given problem's structure can be studied.

The parts of this structure shall be optimization problems with their lower levels (studied under perturbation) and implicit functions with their underlying structure of piecewise differentiability (studied under perturbation, too). Some topology will turn out to be useful in order to quantify the behaviour under perturbation.

Let us indicate the importance of the dynamical aspect by means of two simplified examples and their comparison.

The function $u \mapsto L_1(p,u) := p+u^4$ $(u \in I\!R)$ has its (local or global) minimum at $u^1_\vee(p) \equiv 0$ (for all $p \in I\!R$). This critical point is *(locally) isolated* and, in this sense, $u^1_\vee(\cdot) \equiv 0$ is an *implicit* function. However, if we suitably and slightly perturb L_1 in the full space (C^2, C^2_S) (C^2_S: Whitney-topology, being introduced below), then we get several critical points, in particular two local minima, such that the (local) isolatedness gets lost (unstability). However, for $u \mapsto L_2(p,u) := p + u^2$ $(u \in I\!R)$ the implicit function $u^2_\vee(\cdot) \equiv 0$ is (locally) stable under small perturbations. In fact, here the implicit function theorem in Banach spaces applies. Hence, besides the static aspect of (local) isolatedness holding in both examples, there is some main difference which only appears due to the dynamical aspect.

However, the static and the dynamical aspect may also be equivalent. For instance, Kojima has shown for finitely constrained optimization that under the Mangasarian-Fromovitz constrained qualification (MFCQ) [27] on the feasible set there is an algebraic ("static") condition which *equivalently characterizes* the local uniqueness together with the ("dynamical") stability under perturbations of a Kuhn—Tucker point (here: of 0) (see [23]). (In Section 3, we shall give the definition of this *strong stability* and of (*MFCQ*).) Finally, at the end of *part 1* and in *part 2* [43], we shall look at (strongly stable implicit) Kuhn—Tucker functions $u_\vee$.

A similar *characterization theorem* can be stated due to (MFCQ) (using Farkas' lemma, [12, 18]).

Finally, Jongen, Rückmann, and the author showed that such small perturbations can in fact be realized ((re-)constructed) with the help of *dynamics* (vector fields, generated flows; see [22, 40]). Here, we conclude our motivation of the perturbational aspect.

In order to refer to those perturbations in a topological sense that in a constructive *and* analytical way takes account of local and global (asymptotic) characteristics, we need some more notation. By Df, $D^2 f$ ($D_{\mathbf{x}} f$, or $D_x f$, and $D^2_{\mathbf{x}} f$, or $D^2_x f$) we denote the row-vector of first or the matrix of second order derivatives of $f \in C^2(\mathcal{M}, I\!R)$, $\mathcal{M} \subseteq I\!R^p$ open, respectively (or, partially, due to *all* the components $\mathbf{x}_i$, or x_i, of the vector variable $\mathbf{x}$, or x). In the case $f \in (C^1(\mathcal{M}, I\!R))^v$, Df (and $D_x f$) denotes the functional matrix. Note, that for $v = 1$

we have $D^T f = \dot{f}$. A C^k-function ($k \in I\!N$) on the closure $\overline{\mathcal{M}}$, instead of the open set $\mathcal{M}$, is always understood in the sense that it can be continued to a C^k-function on a suitable open set $\mathcal{N}$, $\overline{\mathcal{M}} \subseteq \mathcal{N}$.

Now, the topology for the product $(\prod_{i=1}^{r_1} C^3(\mathcal{M}_i, I\!R)) \times (\prod_{j=r_1+1}^{r_2} C^2(\mathcal{M}_j, I\!R))$, with $\mathcal{M}_k$ being some $I\!R^r$, will be the product-topology generated by the *Whitney-* (or *strong*) *C^k-topology* C^k_S on each factor $C^k(\mathcal{M}_d, I\!R)$ $(k \in \{2, 3\}$, respectively; cf. [13, 18]). As, e. g., for $k = 2$ we need only to refer to the derivatives up to order 2, let us consider $k = 3$. A typical *base-neighbourhood* of a function $\chi^0 \in C^3(\mathcal{M}_{i_0}, I\!R)$ $(i_0 \in \{1, \ldots, r_1\})$ is the set $\chi^0 + \mathcal{W}_\epsilon$, where $\mathcal{W}_\epsilon$ is defined with the aid of a *controlling* continuous positive $\epsilon : \mathcal{M}_{i_0} \to I\!R$:

$$\begin{aligned} \mathcal{W}_\epsilon \;=\; & \{ \chi \in C^3(\mathcal{M}_{i_0}, I\!R) \mid |\chi(\omega)| + \textstyle\sum_i \left| \dfrac{\partial \chi}{\partial \omega_i}(\omega) \right| + \\ & + \textstyle\sum_{i,j} \left| \dfrac{\partial^2 \chi}{\partial \omega_i \partial \omega_j}(\omega) \right| + \textstyle\sum_{i,j,k} \left| \dfrac{\partial^3 \chi}{\partial \omega_i \partial \omega_j \partial \omega_k}(\omega) \right| < \epsilon(\omega) \\ & \text{for all } \omega \in \mathcal{M}_{i_0} \}. \end{aligned} \tag{1.11}$$

Each Whitney-topology has the following two advantages (see [40]). Namely, on the one hand, with this topology we can take account or force the asymptotic behaviour of functions χ. On the other hand, in some constructive sense it allows to distinuish between the *local* and the *global* aspect.

For our analysis on the qualitative problem structure we also mention the related investigations in general, maybe also *infinite* dimensional settings, which was presented, e. g., by Palais, Smale [29–31, 39], Schwartz [37] and Gromoll, Meyer [8]. The author once started his investigations on the global structure and on its stability in nonlinear optimal control with the *particular structure*. The essence of this structure, which we firstly work out in this *part 1*, is closely related with the approach that underlies the parametric and generical study [41]. Here, the variable u is regarded as a C^2-parameter such that our given problem $\mathcal{P}(\ell, L, F, E, G)$ becomes treated as the family $(\mathcal{P}^u(\ell, L, F, E, G))_{u \in (C^2([a,b], I\!R))^q}$ of problems from the *calculus of variations*. Each of the latter problems can be represented by means of a *semi-infinite* optimization problem. In these optimization problems the particular structure is founded by means of its lower level sets.

The main advantage of the particular structure *is its simplicity. Disadvantages of it are overcome in two more complex alternatives, called the* composite structure *and the* decomposite structure.

However, $\mathcal{P}(\ell, L, F, E, G)$ reveals a feasibility condition which more generally implies *piecewise twice continuous differentiability* and minimization jointly in x *and* u. Both these aspects are with the help of *first order necessary optimality conditions* incorporated in the following two concepts which have to be very refined, in comparison with the particular structure. Moreover, for our *global* study, *topological methods in nonlinear optimization* turn out to be very appropriate, together with the *local* study of Malanowski and Maurer [26].

The *main difference between the composite and the decomposite structure* is the following "dynamical" one. For the *composite* structure the causal relations between the different structural parts (optimization problems, implicit functions) are taken into account. By means of these relations, those parts are considered under perturbation of the given (initial) data (ℓ, L, F, E, G). For the *decomposite* structure the different structural parts are regarded as separate, de-coupled objects with their own defining functions under perturbation.

The composite and the decomposite structure are given at the end of this *part 1* and, in particular, in *part 2* [43]. Let us in some more detail state several main features of these two further structures.

On the one hand, the *composite structure* exhibits the different analytical stages, with their different finitely constrained or semi-infinite optimization problems and implicit so-called *Kuhn – Tucker functions*, respectively. Here, these "$\mathcal{SI}$" optimization problems will even be *generalized* ("$\mathcal{GSI}$") in the sense that $Y = Y(x)$ depends on x.

Those steps are more intimately related with each other, we say *causal*. On the other hand, it is initially based on the stability assumption of the feasible solutions (x^0, u^0) of those necessary optimality conditions, which come from the **minimum principle**. That latter assumption is due to slight perturbations of the defining functional data, and for it we use the notion of *continuity* (CONT). This *part 1* gives an introduction (including a motivation) into the composite concept (see Sections 1, 3).

By contrast, in the *decomposite structure* given in *part 2* [43] the problem reduction leads again to optimization problems, feasible sets and Kuhn—Tucker functions, which all are no longer considered from the viewpoint of their relations, but *stepwise* as our objects of interest. In so far, it is a stronger concept. However, it does not need that strong condition (CONT).

Remark. *In technological and in social sciences, control and optimization are studied and performed either from a more* local *or from a more* global *point of view. For example, in robotics we may distinguish between local and global motions [44], in social ethics or political economics we find local and global strategies of energy supply for the future [2]. Moreover, in cooperative game theory we find both local and global phenomena [32]. There is always a structural aspect being accomplished by dynamical features. In this article, we study the structure of optimal control problems from the* global *viewpoint whereby local "landscapes" come together in one global "landscape" being represented by all the lower level sets of objective function(al)s.*

Our study is of a more theoretical nature which, however, can be formalized and also illustrated by examples. Moreover, together with part 2 *it implies a critical comparison of three structural approaches and a discussion of their frontiers.*

2. The particular structure and its stability

Underlying the **particular** concept we have the following four *claims* that we try to fulfill:

PC$_1$. *The control variable u contributes to the problem's given data (parameters;* partial optimality*).*

PC$_2$. *The control variable u is of class C^2* (aspect of differentiability).

PC$_3$. *The structure is global* (aspect of globality).

PC$_4$. *The structure is based on a reduction* (aspect of reduction).

In fact, as it is claimed in $\mathbf{PC}_1$ (and $\mathbf{PC}_2$) for the particular structure, we look at *all* the following problems from the **calculus of variations**. For each C^2 control variable u we study the problem $\mathcal{P}^u(\ell, L, F, E, G)$ of minimization with respect to x, which comes from keeping u fixed in $\mathcal{P}(\ell, L, F, E, G)$. In this way, u becomes a *parameter*, i. e., it *contributes to the problem's given data* (ℓ, L, F, E, G).

Hence, the first two claims $\mathbf{PC}_{1,2}$ treat u as being dispensed from the minimization, i. e., from the originally joint optimization in (x, u). In comparison with the (joint) minimization $\mathrm{Min}_{x,u}\, \mathcal{I}(x, u)$ *(under the constraints given in (1.6) – (1.8))* due to the variable (x, u) in $\mathcal{P}(\ell, L, F, E, G)$, the minimization $\mathrm{Min}_x\, \mathcal{I}(x, u^*)$ *(under (1.6) – (1.8), $u = u^*$)* in $\mathcal{P}^{u^*}(\ell, L, F, E, G)$ is called *partial minimization* (with respect to x). The corresponding notion on (local and global) optimality is meant when we use the notion *partial optimality* with respect to x.

Now, we may also delete "almost every" in (1.6), (1.8). Moreover, in comparison with the piecewise concept of continuity and differentiability, now u consists of only *one* piece, namely the global one on $[a,b]$. In this way we arrive at the family

$$(\mathcal{P}^u(\ell, L, F, E, G))_{u\in(C^2([a,b],I\!R))^q}$$

of problems from the *calculus of variations*, where for each member the optimization is only partial in x. This basic simplifying parametrization (with the help of u) can be enriched by a parametric study where we walk from one control variable u^{*1} to another one, u^{*2}, e.g., along a path $s \mapsto u(s,\cdot)\,(s \in [\alpha,\beta])$. Therefore, we refer to [42], where this special aspect of connectedness is worked out.

Behind the particular structure, the *idea* of **partitioning** of the constrained problem. $\mathrm{Min}_{x,u}\,\mathcal{I}(x,u)$ into the problems $\mathrm{Min}_x\,\mathcal{I}(x,u)$ (being due to constrained C^2 control variables u) finally stands. Hereby, u becomes a parameter in the *inner* problem of the partitioned problem $\mathrm{Min}_u\,\{\,\mathrm{Min}_x\,\mathcal{I}(x,u)\,\}$, which may represent our given problem $\mathrm{Min}_{x,u}\,\mathcal{I}(x,u)$. Therefore, we are not concerned with questions of well-definedness and structure of the appearing inner function $u \mapsto \mathrm{Min}_x\,\mathcal{I}(x,u)$, but *we shall study these inner problems in detail.* However, in addition to this constructive idea of partitioning or *bilevel approach*, there is also the more *topological-qualitative viewpoint* of comparing some problems $\mathrm{Min}_x\,\mathcal{I}(x,u^{*1})$ and $\mathrm{Min}_x\,\mathcal{I}(x,u^{*2})$. Here, we refer again to the parametric approach with $u^{*1}(t) = u(\alpha,t)$, $u^{*2}(t) = u(\beta,t)$ from [42].

There is another bilevel approach which we shall work out, where nondifferentiabilities coming from the appearance of the minimum functions (as given above) are (generically) governed by means of establishing a piecewise C^2 structure. Namely, first the minimization is due to u (x being the parameter), and then with respect to x. This approach is based on the *minimum principle* and implied in the (de-)composite structure.

Now, we may concentrate on one particular control variable $u = u^*$ which may, but need not, jointly with some state variable $x = x^0$ to be optimal.

Let us shortly consider the meaning of the claim $\mathbf{PC}_4$. *Reduction* is understood in the sense that complexity becomes reduced by means of tracing back a given problem to one (or several) less complex problem(s).

In the case of the particular structure the complexity of an optimal control problem (in the variables x and u) becomes reduced to the complexity of problems from the calculus of variations (in the variable x) and, finally, to the complexity of semi-infinite optimization problems (in the variable $\mathbf{x} \in I\!R^n$). In particular, the "infinite dimensions" of the initial state (and control) variables' space are reduced to the finite dimensions of $I\!R^n$.

The assumption (LB) guarantees for the control variable u^*, which has in a C^2 way been continued outside of $[a,b]$, the complete integrability of the system of differential equations (1.6) (with $u = u^*$). Hence, it yields the existence of a global C^2 flow Φ^{u^*} generated by (1.6).

The following somewhat technical remark states several details on this flow $\Phi^{u^*}(\mathbf{x},t)$, which we (for the ease of notation) abbreviate by $\mathcal{Q}[\mathbf{x}](t) := \Phi^{u^*}(\mathbf{x},t)$, such that $\mathcal{Q}[\mathbf{x}](t)$ indicates an orbit which ends and emanates at $\mathbf{x}$, referring to negative and positive t, respectively. Hereby, the parameter u (now, $= u^*$) becomes suppressed for a while. If the reader is not very interested in these details, then he may skip the remark.

Remark. *The function $\mathcal{Q}[\cdot](\cdot)$ comes from the guiding partition, given by the first n components of the flow $(\check{\Phi}^{u^*} =)\ \check{\mathcal{Q}}[\check{\mathbf{x}}](t)$, which is due to the* dynamical system *in $I\!R^{n+1}$*

$$\dot{\check{x}} \;=\; \check{F}(t,\check{x}), \quad \check{x}(t_0) \;=\; \check{\mathbf{x}}_0 \qquad (\check{\mathbf{x}}_0 \in I\!R^{n+1},\; t_0 \in I\!R), \tag{2.1}$$

where $\check{\mathbf{x}} := (\mathbf{x}^T, \mathbf{x}_{n+1})^T$, $\check{F}(t,\check{\mathbf{x}}) := (F^T(t,\mathbf{x},u^*(t)),\, 1)^T$. *Here we restrict the last initial value component by fixing* $(\check{\mathbf{x}}_0)_{n+1} = t_0$. *We get all the solutions of the differential constraints (1.6), namely* $x(\cdot) = \mathcal{Q}[\mathbf{x}](\cdot - t_0)$, *where* $\mathbf{x}_0 = x(t_0)$. *In particular, it holds*

$$\left.\begin{aligned} \mathcal{Q}[\mathbf{x}](0) \;&=\; \mathbf{x}, \\ (\mathcal{Q}[\mathbf{x}](s))(t) \;&=\; \mathcal{Q}[\mathbf{x}](s+t). \end{aligned}\right\} \quad (\mathbf{x} \in I\!R^n,\; s,t \in I\!R). \tag{2.2}$$

The solutions of (1.6) do not depend on the choice of that continuation of u^*. *For more information on flows we refer to [1,18]. The notation* $\mathcal{Q}[\cdot](\cdot)$ *was suggested by* Craven [6].

With the help of $\mathcal{Q}[\cdot](\cdot)$ the whole problem $\mathcal{P}^{u^*}(\ell, L, F, E, G)$ can be traced back (collapsed) to the *initial time* $t_0 := a$. Hereby, we wander with the trajectories of the flow in reverse direction. Then, for $\mathbf{x} = x(a)$ we arrive at the following globally defined functions:

$$\mathbf{f}^*(\mathbf{x}) \;:=\; \mathcal{I}(\mathcal{Q}[\mathbf{x}](\cdot - a), u^*(\cdot - a)), \tag{2.3}$$

$$\mathbf{e}_i^*(\mathbf{x}) \;:=\; e_i(\mathcal{Q}[\mathbf{x}](b-a)) \quad (i \in I), \tag{2.4}$$

$$\mathbf{g}_j^*(t,\mathbf{x}) \;:=\; g_j(t, \mathcal{Q}[\mathbf{x}](t-a), u^*(t)) \quad (j \in J). \tag{2.5}$$

In this way we have generated the following (ordinary) **semi-infinite optimization problem** in the initial value variable $\mathbf{x}$ which *equivalently* reflects $\mathcal{P}^{u^*}(\ell, L, F, E, G)$ by the initial value dependence of the flow:

$$\mathcal{P}^{u^*}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*) \quad \left\{ \begin{array}{ll} \text{Min } \mathbf{f}^* \text{ on } M^{u^*}_{\mathcal{SI}}[\mathbf{E}^*, \mathbf{G}^*], \text{ where} & (2.6) \\[2ex] \left.\begin{array}{l} M^{u^*}_{\mathcal{SI}}[\mathbf{E}^*, \mathbf{G}^*] := \{\mathbf{x} \in I\!R^n \mid \mathbf{e}_i^*(\mathbf{x}) = 0 \;\; (i \in I), \\[1ex] \qquad\qquad \mathbf{g}_j^*(t,\mathbf{x}) \geq 0 \;\; (j \in J,\, t \in [a,b])\}. \end{array}\right\} & (2.7) \end{array} \right.$$

Here, the functions $\mathbf{f}^*$, and $\mathbf{E}^*$, $\mathbf{G}^*$, comprising $\mathbf{e}_i^*$ $(i \in I)$, $\mathbf{g}_j^*$ $(j \in J)$, are of class C^2. We note that we have **reduced** the infinite dimensional problem $\mathcal{P}^{u^*}(\ell, L, F, E, G)$ to a *finite dimensional* one; see also Figure 1. Hence, our claim $\mathbf{PC}_4$ is fulfilled, too.

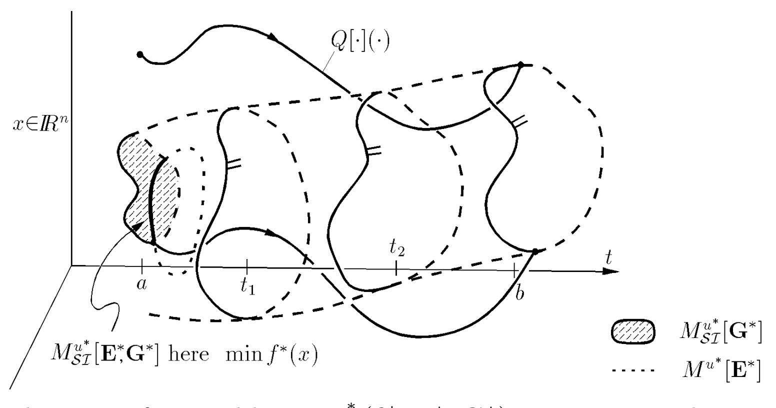

Fig. 1. On the introduction of a problem $\mathcal{P}^{u^*}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$, at time a, due to $\mathcal{P}(\ell, L, F, E, G)$. In this example, there is the periodic boundary constraint $\mathbf{E}(x(a), x(b)) := x(a) - x(b) = 0_n$, and $M[G(t,\cdot,u^*(t))]$ $(t \in [a,b])$ is represented by means of three times (t_1, t_2, b).

Remark. *Let us look at a* technical alternative. *By means of suitable translations of the set* $Y := [a,b]$*, which give us pairwise disjoint new index sets* Y^j $(j \in J)$*, corresponding transformations of the inequality constraints, and with the help of a suitable* C^∞*-partition of unity, we could finally glue together the inequality constraints, being restricted to* Y^j $(j \in J)$*, such that the following holds. We only had* one *real-valued constraint* C^2*-function* $\mathbf{G}^0$ *and only* one *corresponding index set* Y *being again representable as a feasible set in the sense of finitely constrained optimization, which together precisely reflect all the inequality constraints of* $\mathcal{P}^{u^*}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$*. With this change in the representation we would immediately have a model of the semi-infinite form (1.1), (1.2). This form is often assumed in the literature (e. g., in [20, 21, 36]).*

Such a C^∞*-partition of unity (subordinate to some locally finite open covering) is a family of globally defined* C^∞*-functions* $\gamma(\omega)$ *(with their support always lying in a member of the covering) constituting* ω*-dependent convex combinations. The values of* γ *are elements of the interval* $[0,1]$ *(see [13, 18]).*

Before we continue founding the particular structure of $\mathcal{P}(\ell, L, F, E, G)$, let us shortly come back to the idea of *partitioning*. By definition, the variable $\mathbf{x}$ of each problem $\mathcal{P}^{u}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$ (here $u = u^*$) represents an initial value condition. If a *(locally) minimal* initial value $\mathbf{x}$ is supplied, then the system (1.6) determines the (locally) minimal variable $x(\cdot)$ as a function of $u(\cdot)$. This dependence might symbolically be represented as $x(t) = \mathsf{Q}(u)(t)$. But then we may consider the *outer problem* in the form $Min_u\, \mathsf{J}(\mathsf{Q}(u), u)$, under the constraints on $u(\cdot)$ given by means of (1.6), (1.8), where $x = \mathsf{Q}(u)$. Now, let us come back to the structure of the inner problem(s).

The structure of our problem $\mathcal{P}^{u^*}(\ell, L, F, E, G)$ is defined by all the lower level sets $\mathcal{L}^{\theta, u^*}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$ $(\theta \in I\!R)$ of $\mathcal{P}^{u^*}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$. At last, looking at the *u-parametrized* entity of all such lower level sets, we arrive at the **particular structure** of our optimal control problem $\mathcal{P}(\ell, L, F, E, G)$ in a *global* way; see $\mathbf{PC_3}$. Until now, this structure is *completely*, but in a more static way realized.

As we regard the dynamical aspect of the topological behaviour under perturbations to be an *intrinsic* item within the structure, we also have to give the definition of particular structural stability of $\mathcal{P}(\ell, L, F, E, G)$. Hereby, for each semi-infinite problem of the form (2.6), (2.7) for which there is no "history" based on some control variable and flow, we delete u^* in the problem notation. So, we write, e. g., $\mathcal{P}_{\mathcal{SI}}(\mathbf{f}, \mathbf{E}, \mathbf{G})$.

Our following definition is also illustrated in Figure 2.

Definition 2.1. Let the following subdefinitions refer to defining functions of class C^2.

a) Two semi-infinite optimization problems $\mathcal{P}_{\mathcal{SI}}(\mathbf{f}^1, \mathbf{E}^1, \mathbf{G}^1)$, $\mathcal{P}_{\mathcal{SI}}(\mathbf{f}^2, \mathbf{E}^2, \mathbf{G}^2)$ are called *equivalent* if there are continuous functions $\phi_{\mathcal{SI}} : I\!R \times I\!R^n \to I\!R^n$ and $\psi_{\mathcal{SI}} : I\!R \to I\!R$ with the following three properties $(\mathcal{E}_{\mathcal{SI}\ 1,2,3})$:

   $(\mathbf{E}_{\mathcal{SI}\,1})$. For every $\theta \in I\!R$ the mapping $\phi_{\mathcal{SI},\theta} : I\!R^n \to I\!R^n$ is a homeomorphism, where $\phi_{\mathcal{SI},\theta}(\mathbf{x}) := \phi_{\mathcal{SI}}(\theta, \mathbf{x})$.

   $(\mathbf{E}_{\mathcal{SI}\,2})$. The mapping $\psi_{\mathcal{SI}} : I\!R \to I\!R$ is a monotonically increasing homeomorphism.

   $(\mathbf{E}_{\mathcal{SI}\,3})$. $\phi_{\mathcal{SI},\theta}(\mathcal{L}^{\theta}_{\mathcal{SI}}(\mathbf{f}^1, \mathbf{E}^1, \mathbf{G}^1)) = \mathcal{L}^{\psi_{\mathcal{SI}}(\theta)}_{\mathcal{SI}}(\mathbf{f}^2, \mathbf{E}^2, \mathbf{G}^2)$ for all $\theta \in I\!R$.

b) Let a variable $u^* \in (C^2([a,b], I\!R))^q$ be given. We say that both the semi-infinite optimization problem $\mathcal{P}^{u^*}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$ and its underlying (generating) problem from the calculus of variations $\mathcal{P}^{u^*}(\ell, L, F, E, G)$ are *structurally stable* if there exist a C^2_S-neighbourhood

$\mathcal{O}^*$ of $(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$ such that for each element $(\tilde{\mathbf{f}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{G}}) \in \mathcal{O}^*$ the problem $\mathcal{P}_{\mathcal{SI}}(\tilde{\mathbf{f}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{G}})$ is equivalent with $\mathcal{P}_{\mathcal{SI}}^{u^*}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$.

c) The optimal control problem $\mathcal{P}(\ell, L, F, E, G)$ is called *particularly structurally stable* if for all $u \in (C^2([a,b], I\!R))^q$ the problem $\mathcal{P}^u(\ell, L, F, E, G)$ is structurally stable.

This definition is founded in the introduction of *structural stability* of a nonlinear optimization problem which was at first given in [9] for the (special) case of finitely constrained optimization. Later, in [19, 40], it became extended to the semi-infinite case. The condition of *equivalence* which topologically identifies two optimization problems, constitutes an *equivalence relation. Structural stability* of an optimization problem also implies that *descent is preserved* under small perturbations. For illustrations and more basic information on equivalence and structural stability we refer to [19, 22, 40]; for the related concept in the theory of dynamical systems we mention [4, 38].

For finitely constrained or semi-infinite optimization [19, 22, 40] show that, in essence, two regularity conditions are **sufficient** for this problem to be structurally stable. Namely, there is the (in the semi-infinite case *extended*) *Mangasarian — Fromovitz constraint qualification* ((E)MFCQ) for the feasible set and the *strong stability* in the sense of *Kojima (and Rückmann)* for the Kuhn — Tucker points. These conditions will be noted in the next Section (and in *part 2*, respectively). If, moreover, the feasible set is compact, then [9, 19] (and [40]) also show that the regularity conditions are **necessary** for structural stability, which then has become (equivalently) **characterized**.

Now, we shall realize that structural stability with its *dynamically constructable* homeomorphisms being explained below, equivalently reflects the *static qualitative* properties of the *structure*. Hereby, for generating the dynamics, *flows* are very important, while in the *necessity part* some algebraic topology [16] is used.

Reversely, a violation of one of these regularity conditions can be interpreted with a dynamical meaning, namely as a *bifurcation or vanishing* of a feasible set or of a Kuhn — Tucker point. With such a "pathological" (irregular) behaviour we are again in the dynamical context, and we remember our explanations on the perturbational-dynamical and on the static aspect given in Section 1.

Let us for a moment pay attention to those dynamically constructable homeomorphisms. We already introduced and illustrated the conditions of structural stability of $\mathcal{P}(\ell, L, F, E, G)$.

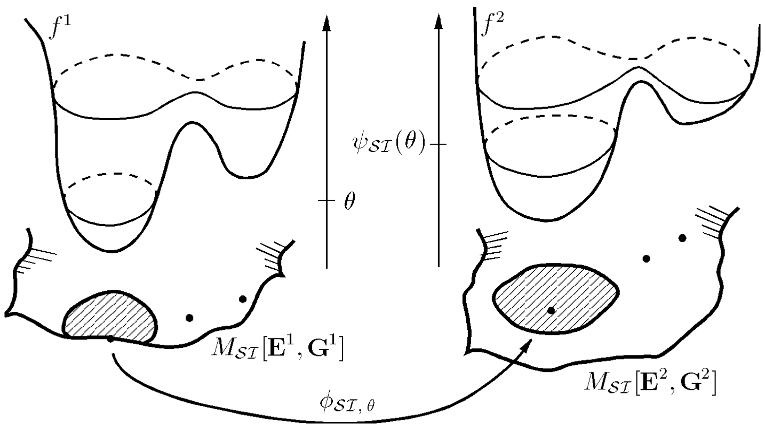

Fig. 2. Equivalence of $\mathcal{P}_{\mathcal{SI}}(\mathbf{f}^1, \mathbf{E}^1, \mathbf{G}^1)$ with $\mathcal{P}_{\mathcal{SI}}(\mathbf{f}^2, \mathbf{E}^2, \mathbf{G}^2)$; structural stability of $\mathcal{P}_{\mathcal{SI}}^{u^*}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$ and $\mathcal{P}^{u^*}(\ell, L, F, E, G)$, whereby $(\mathbf{f}^1, \mathbf{E}^1, \mathbf{G}^1) = (\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$ and $(\mathbf{f}^2, \mathbf{E}^2, \mathbf{G}^2) = (\tilde{\mathbf{f}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{G}}) \in \mathcal{O}^*$.

Hereby, for each of the problems $\mathcal{P}^{u^*}(\ell, L, F, E, G)$ we referred to *homeomorphisms* $\psi_{\mathcal{SI}}$ and $\phi_{\mathcal{SI},\theta}$, which take over the levels θ and the lower level sets $\mathcal{L}^{\theta}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$ of the unperturbed problem (on)to corresponding levels $\psi_{\mathcal{SI}}(\theta)$ and lower level sets $\mathcal{L}^{\psi_{\mathcal{SI}}(\theta)}_{\mathcal{SI}}(\tilde{\mathbf{f}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{G}})$ of the *perturbed* problem, respectively. In [22, 40], for the case of finitely constrained optimization $(\mathcal{F})$ it was proved that such homeomorphisms can be constructed by means of vectorfields (hence, of flows; "dynamically") if some regularity conditions are fulfilled. The latter ones are the *Mangasarian — Fromovitz constraint qualification (MFCQ)* on the feasible sets, the *strong stability* of all Kuhn — Tucker points, and some technical separateness condition on the critical values.

In principle, the same "dynamical construction" can be made in our present context where there is semi-infinite optimization ($\mathcal{SI}$ instead of $\mathcal{F}$). This way was basically observed in [40], where a (large) number of illustrations are given for the lower level sets and homeomorphisms in finitely constrained and in semi-infinite optimization. Later, Rückmann presented a satisfying (algebraically characterizable) condition of *strong stability* for the semi-infinite case (cf. [36]; a short presentation is also given in *part 2*, [43]). With the help of that condition, Jongen and Rückmann could take over the characterization of structural stability from finitely constrained to semi-infinite optimization, namely by means of the *extended Mangasarian — Fromovitz constraint qualification (EMFCQ)* (the extended (MFCQ)), *strong stability*, and the separateness condition.

Those compactness conditions are guaranteed if one of the sets $M[G]$, $M[E]$ is compact or, without loss of generality, if it holds

Assumption (COMP): *The sets* $M[E]$, $M[G]$ *are* **compact**.

Namely, if only one of the two feasible sets should be compact then the other one could outside of a sufficiently large ball be made compact with the help of differential topology (cf. [13,18], and Section 1) on E or G, respectively. The whole virtue of (COMP) will be used for the composite and for the decomposite structure.

Now, the investigation [19], being based on [9, 22, 36], gives us the opportunity to *characterize* the particular structural stability of $\mathcal{P}(\ell, L, F, E, G)$ in the following way.

Corollary 2.2 (**Characterization Theorem** $(\mathcal{A})$). *Let the optimal control problem* $\mathcal{P}(\ell, L, F, E, G)$ *with defining* C^2-, C^3-*functions, respectively, be given and the Assumptions (LB), (COMP) hold.*

Then, $\mathcal{P}(\ell, L, F, E, G)$ *is* particularly structurally stable if and only if *with respect to* all *the problems* $\mathcal{P}^{u^*}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$, *being equivalent with* $\mathcal{P}^{u^*}(\ell, L, F, E, G)$, $(u^* \in (C^2([a,b], I\!R))^q)$ *the following three conditions* $(\mathcal{CA}_{1,2,3})$ are *fulfilled:*

($\mathbf{CA}_1$). (EMFCQ) *holds for* $M^{u^*}_{\mathcal{SI}}[\mathbf{E}^*, \mathbf{G}^*]$ $(u^* \in (C^2([a,b], I\!R))^q)$.

($\mathbf{CA}_2$). *All the* Kuhn — Tucker points $\overline{\mathbf{x}}$ *of* $\mathcal{P}^{u^*}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$ *are* strongly stable $(u^* \in (C^2([a,b], I\!R))^q)$.

($\mathbf{CA}_3$). *For each two different Kuhn — Tucker points* $\overline{\mathbf{x}}^1 \neq \overline{\mathbf{x}}^2$ *of* $\mathcal{P}^{u^*}_{\mathcal{SI}}(\mathbf{f}^*, \mathbf{E}^*, \mathbf{G}^*)$ *the corresponding critical values are different* (separate), *too:* $\mathbf{f}^*(\overline{\mathbf{x}}^1) \neq \mathbf{f}^*(\overline{\mathbf{x}}^2)$ $(u^* \in (C^2([a,b], I\!R))^q)$.

Let us introduce *(EMFCQ)* and *strong stability* in the unfolding of the composite structure structure being subsequently founded in *Section 3*. These two analytical conditions can also shortly be interpreted as follows. While (EMFCQ) assures the considered feasible set to be a topological manifold with Lipschitzian coordinates [20], strong stability of a Kuhn — Tucker point [23, 34] means its local uniqueness and continuous dependence on the functional data. Under our compactness assumption (COMP), (EMFCQ) is also equivalent with the global topological stability of the feasible set [20]. Hence, both our static-qualitative regularity conditions have a dynamical-quantitative meaning which is also the nature of our structural stability.

For more illustrations on these regularity conditions, on structural stability and on their violation we refer to [40] and mention [42]. In the latter article, generic types of even more general control problems are presented, together with examples and normal forms of such problems.

Remark 2.3. With its parts (b), (c), Definition 2.1 takes into account slight perturbations of optimization problems' data. In a sense given in part 1, this is a detail with decomposite character, namely by taking into account problems which are raised by means of a reduction, as the problems of our topological interest. An *alternative* approach is due to the composite point of view, presented in Section 3 and part 2, where we causally recur to the original data, i. e., to the defining functions of our optimal control problem $\mathcal{P}(\ell, L, F, E, G)$. Then, we would for each (C^2-) control variable u^* refer to a neighbourhood $C^3_S \times C^2_S$-neighbourhood $\mathcal{R}^*$ of $((L, F, G), (\ell, E))$ such that for each element $((\widetilde{L}, \widetilde{F}, \widetilde{G}), (\widetilde{\ell}, \widetilde{E})) \in \mathcal{R}^*$ the problems $\mathcal{P}^{u^*}_{\mathcal{SI}}(\tilde{\mathbf{f}}, \widetilde{\mathbf{E}}, \widetilde{\mathbf{G}})$, $\mathcal{P}^{u^*}_{\mathcal{SI}}(\mathbf{f}, \mathbf{E}, \mathbf{G})$, being generated by $\mathcal{P}^{u^*}(\tilde{\ell}, \widetilde{L}, \widetilde{F}, \widetilde{E}, \widetilde{G})$, $\mathcal{P}^{u^*}(\ell, L, F, E, G)$, are equivalent.

If assumption (COMP) holds, then that *alternative* — more composite — particular structural stability is equivalent with our more decomposite condition given in Definition 2.1. This fact results from the two corresponding characterization theorems which reveal the same regularity conditions. Finally, we note for each alternative that, with the help of the continuous dependence of $\mathcal{Q}[\cdot](\cdot)\,(= \Phi^{u^*})$ on the initial values $x(a)$, the particular structural stability can be reformulated with the lower level sets of $\mathcal{I}(\cdot, u^*)$, i. e., in the space $(C^2([a,b], I\!R))^n$ of the state variables x.

Our considerations on the particular structure and its stability can be generalized for the case where the objective functional is of the nondifferentiable maximum type [15]. For that purpose, we equivalently turn to the height function over the epigraph of a max-type objective function. This technique can be found in [40, 41] (cf. also [42]).

Finally, we conclude the following result:

Result 2.4. (**Modelling Theorem** $(\mathcal{A}_{\mathcal{M}})$). *Under the Assumptions (LB) and (COMP), there is a first positive answer (given above) to our questions* $(\mathcal{Q}_{1,2})$. *Hereby, the claims* $PC_{1,2,3,4}$ *are fulfilled.*

Let us remember, that we have already made the claims $\mathrm{PC}_{1,2,3,4}$ become both definite and by means of the particular structure being satisfied. Therefore, we make a cross-reference to the beginning of Section 2 and to the succeeding explantions and definitions. Now, let us also realize the fulfillment of our two questions $(\mathcal{Q}_{1,2})$, being given in Section 1.

$(\mathcal{Q}_1)$: Our particular structure and its condition on particular structural stability are **global** ones. In fact, the (dimensional) reduction is made *globally*, and, in particular, it finally refers to *global* lower level sets (cf. also Figure 2). The concept of referring to the lower level sets in optimization, originally being due to Jongen, is convincing. Namely, in the context of Definition 2.1 and Figure 2 we indicated why this concept is from the (constructive-) topological and from the numerical viewpoint a very adequate one.

However, the composite and the decomposite structure do also fulfill $(\mathcal{Q}_1)$. For these two further structures the global concept of referring to lower level sets *and* implicit functions is worked out in a more enriched bilevel structure.

$(\mathcal{Q}_2)$: Above, we tried to make more transparent why the particular structure really fulfills the question $(\mathcal{Q}_2)$. We remember all our reflections on ((E)MFCQ) and strong stability, and we refer to our Characterization Theorem $(\mathcal{A})$ (Corollary 2.2).

Moreover, there is the following *didactic* aspect. The composite and the decomposite structure do also fulfill $(\mathcal{Q}_2)$ (*part 2*). In view of the introduction given in Section 1, and having already presented the particular structure, we are better prepared to understand these more complicated composite or decomposite structures with both their advantages and disadvantages.

In the Modelling Theorem $(\mathcal{A}_{\mathcal{M}})$, the word *first* indicates that there are some objections against the particular structure. Namely, that *partial optimality* does not fully satisfy further claims. In particular, there is some *lack of causal dependence* of our perturbed semi-infinite optimization problems with respect to the given optimal control problems. Finally, the control variable u should be admitted to reveal a finite number of *jumps*. With the subsequent *composite structure*, being motivated by some different claims, we shall overcome these disadvantages. However, for that purpose the complexity of our model will have to increase.

Before we in a detailed way turn to the composite structure and, in *part 2*, to the decomposite structure, we anticipate reflecting their essential concepts in Figure 3. For this figure, we may also remember our explanations given in Section 1.

Based on these concepts with necessarily higher complexity, we subsequently satisfy more demanding claims better than with the help of the particular structure. Within our qualitative-topological context these claims typically rise from optimal control theory. Hereby, our *qualitative conditions* will in detail be presented, and our *bilevel* idea of *partitioning* will become strongly incorporated.

3. The composite structure and its stability; some foundations

The **composite** concept means a strong refinement of the *particular* concept from Section 2, and it is devoted to the following four more sophisticated *claims*. Namely,

CC$_1$. *The minimization is jointly in* (x, u) (optimality).

CC$_2$. *The structure reveals* compatibility with the minimum principle.

CC$_3$. *The structure is based on a* global and analytical approach.

CC$_4$. *The structure is unfolded in a* causal *(connected) way.*

In order to "address" the right hand sides of our differential equations' system (1.6) we introduce an auxiliary parameter $\mathbf{w} \in I\!R^n$. We introduce parametric families of minimization problems; these families will be subdivided into two classes. In this *part 1* we start with minimizing due to the control vector $\mathbf{u} \in I\!R^q$ and *in part 2* ([43]) we end with minimizing due to the state vector $\mathbf{x} \in I\!R^n$.

Let us make the overall **compactness assumption (COMP)**, given in Section 1. *Part 2* will show, why (COMP) is in the suitable "wide" way formulated (incorporating $M[E]$) for our composite structure. We write t as a further parameter $\mathbf{t}$, and we refer to the following projectively defined *index set*

$$M_{pr}^{\eta_0}[F,G] \;=\; \left.\begin{array}{l} \{(\mathbf{t},\mathbf{x},\mathbf{w}) \in I\!R^{2n+1}\,|, \\ (\mathbf{t},\mathbf{x},\mathbf{u}) \in M[G],\;\; ||\,F(\mathbf{t},\mathbf{x},\mathbf{u}) - \mathbf{w}\,|| \le \eta_0 \;\text{ for some }\; \mathbf{u} \in I\!R^q \end{array}\right\} \tag{3.1}$$

where $\eta_0 > 0$, maybe chosen sufficiently small, stands for some relaxation and admits a study on the limit $\eta_0 \to 0$. By means of a continuity argumentation it follows from the assumption (COMP), that the set $M_{pr}^{\eta_0}[F,G]$ is *compact*, too. Now, let us first of all for each $(\mathbf{t},\mathbf{x},\mathbf{w}) \in$

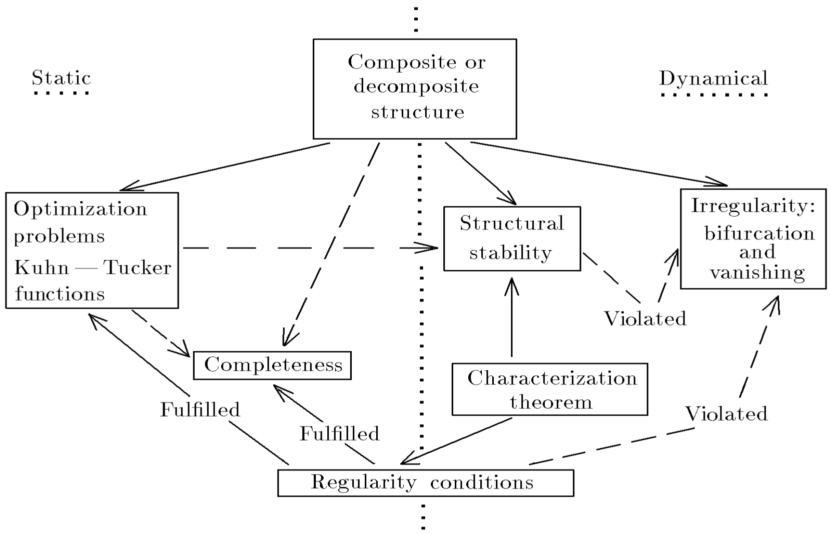

Fig. 3. The concepts of the composite structure and of the decomposite structure.

$M_{pr}^{\eta_0}[F,G]$ look at the corresponding **finitely constrained optimization problem**

$$(*) \qquad \mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L,F,G) \begin{cases} \mathrm{Min}_{\mathbf{u}\in I\!R^q} \;\; L(\mathbf{t},\mathbf{x},\mathbf{u}), \quad \text{where} & (3.2) \\ \mathbf{u} \,\in\, M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F-\mathbf{w},G]. & (3.3) \end{cases}$$

Hereby, the feasible set is given by

$$\left.\begin{array}{l} M_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}[F,G] \;:=\; M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F-\mathbf{w},G], \;\; \text{i.e.} \\ \quad = \{\mathbf{u}\in I\!R^q \mid \;\; f_k(\mathbf{t},\mathbf{x},\mathbf{u}) - \mathbf{w}_k = 0 \;(k\in\{1,\dots,n\}), \\ \quad\; g_j(\mathbf{t},\mathbf{x},\mathbf{u}) \geq 0 \;(j\in J)\} \end{array}\right\} ((\mathbf{t},\mathbf{x},\mathbf{w}) \in M_{pr}^{\eta_0}[F,G]). \qquad (3.4)$$

Of course, such a set may be empty. However, if the condition (MFCQ) (being introduced below), holds for $M_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}[F,G]$, then it is a nonempty topological manifold [10, 20]. At the end of this paper we shall study examples for both the case of empty feasible sets and the case of nonempty ones.

The *global structure* of $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L,F,G)$ (see $\mathbf{CC}_3$) is given by means of the level called ζ, and the set of active inequality constraints at a point $\mathbf{u_0} \in M_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}[F,G]$ is defined as follows:

$$J_0(\mathbf{t},\mathbf{x},\mathbf{u_0}) \;:=\; \{j\in J \mid g_j(\mathbf{t},\mathbf{x},\mathbf{u_0}) \;=\; 0\}. \qquad (3.5)$$

For the purpose, in *part 2* to guarantee the well-definedness of the *structure*(s) of Kuhn—Tucker functions (later on being introduced) which will (as some items) contribute to the composite structure, we introduce some *regularity conditions.*

On the one hand, there is the subsequent definition which firstly introduces a stronger and, then, a weaker constraint qualification on the feasible sets in the present *finitely constrained* optimization. For *semi-infinite* optimization, these two constraint qualifications can straightforwardly be *extended.* We remark that ((E)LICQ) implies ((E)MFCQ) [18, 20, 40].

Definition 3.1. Let a parameter point $(\mathbf{t},\mathbf{x},\mathbf{w}) \in M_{pr}^{\eta_0}[F,G]$ and a point $\mathbf{u_0} \in M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F{-}\mathbf{w},G]$ be given.

$(\mathcal{ELI})$.

a) The *linear independence constraint qualification* (LICQ) is said to hold at $\mathbf{u_0}$ if the gradients $D_{\mathbf{u}}(f_k - \mathbf{w_k})(\mathbf{t}, \mathbf{x}, \mathbf{u_0})$ $(= D_{\mathbf{u}} f_k(\mathbf{t}, \mathbf{x}, \mathbf{u_0}))$, $k \in \{1, \ldots, n\}$, $D_{\mathbf{u}} g_j(\mathbf{t}, \mathbf{x}, \mathbf{u_0})$, $j \in J_0(\mathbf{t}, \mathbf{x}, \mathbf{u_0})$, of the active constraints are linearly independent.

b) *(LICQ)* is said to hold for $M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}, G]$ if (LICQ) is fulfilled at each element of this feasible set.

c) For a semi-infinite optimization problem the straightforward extension of (LICQ), in which the set of active inequality constraints has to be finite, is called the *extended linear independence constraint qualification (ELICQ).*

$(\mathcal{EMF})$.

a) The *Mangasarian – Fromovitz constraint qualification (MFCQ)* is said to hold at $\mathbf{u_0}$ if the following conditions $(\mathcal{MF}_{\mathcal{F}\ 1,2}^{\mathbf{t},\mathbf{x},\mathbf{w}})$ are satisfied:

$(\mathbf{MF}_{\mathcal{F}1}^{\mathbf{t},\mathbf{x},\mathbf{w}})$. The vectors $D_{\mathbf{u}}(f_k - \mathbf{w}_k)(\mathbf{t}, \mathbf{x}, \mathbf{u_0})$, $k \in \{1, \ldots, n\}$, are linearly independent.

$(\mathbf{MF}_{\mathcal{F}2}^{\mathbf{t},\mathbf{x},\mathbf{w}})$. There exists a vector $\xi_{\mathbf{0}} \in I\!R^q$, such that it holds

$$\left\{ \begin{array}{l} D_{\mathbf{u}}(f_k - \mathbf{w}_k)(\mathbf{t}, \mathbf{x}, \mathbf{u}_0)\, \xi_{\mathbf{0}} \; = \; 0 \quad (k \in \{1, \ldots, n\}), \\ \\ D_{\mathbf{u}} g_j(\mathbf{t}, \mathbf{x}, \mathbf{u}_0)\, \xi_{\mathbf{0}} \; > \; 0 \quad (j \in J_0(\mathbf{t}, \mathbf{x}, \mathbf{u_0})). \end{array} \right\} \tag{3.6}$$

We call $\xi_{\mathbf{0}}$ it an *MF-vector* at $\mathbf{u_0}$.

b) *(MFCQ)* is said to hold for $M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}, G]$ if (MFCQ) is fulfilled at each element of $M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}, G]$.

c) For a semi-infinite optimization problem the straightforward extension of (MFCQ) is called the *extended Mangasarian – Fromovitz constraint qualification (EMFCQ)*, and each of the corresponding vectors $\xi_{\mathbf{0}}$ is called an *EMF-vector*.

Under the condition (COMP), the *static* qualitative condition ((E)MFCQ) on the existence of (relatively) inwardly pointing directions $\xi_{\mathbf{0}}$ has also a *dynamical* meaning. Namely, it means the (global) topological stability of the feasible set $M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}, G]$ under small perturbations of the defining data (see [10] and, for the semi-infinite case, [20]).

For more information on ((E)MFCQ) we mention [12, 20, 27]. On the other hand, we define the *strong stability* of a *Kuhn – Tucker point.*

Definition 3.2. Let a parameter point $(\mathbf{t}, \mathbf{x}, \mathbf{w}) \in M_{pr}^{\eta_0}[F, G]$ *and a point* $\overline{\mathbf{u}} \in M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}, G]$ be given. Then, $\overline{\mathbf{u}}$ is called a *Kuhn – Tucker point* for $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L, F, G)$ if there exist numbers λ_k, $\mu_j \in I\!R$ $(k \in \{1, \ldots, n\}$, $j \in J_0(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}))$ such that the subsequent two conditions $(\mathcal{KT}_{\mathcal{F}\ 1,2}^{\mathbf{t},\mathbf{x},\mathbf{w}})$ hold:

$(\mathbf{KT}_{\mathcal{F}1}^{\mathbf{t},\mathbf{x},\mathbf{w}})$. $\quad D_{\mathbf{u}} L(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}) \; = \; \sum_{k \in \{1,\ldots,n\}} \lambda_k D_{\mathbf{u}}(f_k - \mathbf{w}_k)(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}) + \sum_{j \in J_0(\mathbf{t},\mathbf{x},\overline{\mathbf{u}})} \mu_j D_{\mathbf{u}} g_j(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}})$.

$(\mathbf{KT}_{\mathcal{F}2}^{\mathbf{t},\mathbf{x},\mathbf{w}})$. $\quad \mu_j \geq 0 \quad (j \in J_0(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}))$.

The numbers λ_k, μ_j $(k \in \{1, \ldots, n\}$, $j \in J_0(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}))$ are called *Lagrange multipliers.*

For the following introduction of *strong stability* of a Kuhn — Tucker point for $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L,F,G)$, $(\mathbf{t},\mathbf{x},\mathbf{w}) \in M_{pr}^{\eta_0}[F,G]$ being given, we could either refer to (C_S^3-)perturbations of (L,F,G) or, alternatively, to perturbations of $(L,F,G)(\mathbf{t},\mathbf{x},\cdot)$. These two approaches lead to conditions which are *equivalent*. Let us choose the second alternative. For the necessity of the *by 1 increased order of differentiability* (here: 3) we refer to a calculation given in [42].

In order to measure the local "distance" between the given and the perturbed functional data we put for each C^3-triplet (L^1,F^1,G^1), defining a problem $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L^1,F^1,G^1)$ of the same form as $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L,F,G)$, for some $(\mathbf{t},\mathbf{x},\mathbf{w}) \in M_{pr}^{\eta_0}[F,G]$ and for each set $\Omega \subseteq I\!R^q$:

$$\left.\begin{array}{l}
\mathrm{norm}_{\mathcal{F}}[(L^1(\mathbf{t},\mathbf{x},\cdot),F^1(\mathbf{t},\mathbf{x},\cdot),G^1(\mathbf{t},\mathbf{x},\cdot)),\Omega] \;:= \\
\qquad \sup_{\omega\in\Omega}\, \max\Big\{ |\chi(\omega)| + \sum_{i=1}^q \left|\dfrac{\partial\chi}{\partial\omega_i}(\omega)\right| + \\
\qquad\quad + \sum_{i_1,i_2=1}^q \left|\dfrac{\partial^2\chi}{\partial\omega_{i_1}\partial\omega_{i_2}}(\omega)\right| + \sum_{i_1,i_2,i_3=1}^q \left|\dfrac{\partial^3\chi}{\partial\omega_{i_1}\partial\omega_{i_2}\partial\omega_{i_3}}(\omega)\right|, \;\Big| \\
\quad \chi \in \{L^1(\mathbf{t},\mathbf{x},\cdot)\} \cup \{(f_k^1-\mathbf{w}_k)(\mathbf{t},\mathbf{x},\cdot) \mid k\in\{1,\ldots,n\}\} \cup \{g_j^1(\mathbf{t},\mathbf{x},\cdot) \mid j\in J\}\Big\}.
\end{array}\right\} \quad (3.7)$$

Let $(\Omega =)\,\mathcal{B}(\omega,\delta)$ stand for the open ball of radius $\delta > 0$ around a point ω.

Definition 3.3. Let for a parameter point $(\mathbf{t},\mathbf{x},\mathbf{w}) \in M_{pr}^{\eta_0}[F,G]$ it the point $\overline{\mathbf{u}}$, be a Kuhn — Tucker point for $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L,F,G)$.

Then $\overline{\mathbf{u}}$ is called *strongly stable* if for some $\overline{\delta} > 0$ and for each $\delta \in (0,\overline{\delta}]$ there exists an $\alpha > 0$ such that for each C^3-function $(\widetilde{L},\widetilde{F},\widetilde{G})$ with

$$\mathrm{norm}_{\mathcal{F}}[(\widetilde{L}(\mathbf{t},\mathbf{x},\cdot)-L(\mathbf{t},\mathbf{x},\cdot),(\widetilde{F}-\mathbf{w})(\mathbf{t},\mathbf{x},\cdot)-(F-\mathbf{w})(\mathbf{t},\mathbf{x},\cdot),\widetilde{G}(\mathbf{t},\mathbf{x},\cdot)-G(\mathbf{t},\mathbf{x},\cdot)),\mathcal{B}(\overline{\mathbf{u}},\delta)] \le \alpha$$

the ball $\mathcal{B}(\overline{\mathbf{u}},\delta)$ contains a Kuhn — Tucker point $\tilde{\mathbf{u}}$ for $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(\tilde{L},\widetilde{F},\widetilde{G})$ which is unique in $\mathcal{B}(\overline{\mathbf{u}},\overline{\delta})$.

Strong stability will be a basic concept of conditions used for various *finitely constrained* and *semi-infinite* optimization problems. The last definition was originally given by Kojima [23] (see also [24], and, for a related research, [34]). It is more *dynamically* formulated. Kojima also gave an *algebraical characterization* for the strong stability of a Kuhn — Tucker point $\overline{\mathbf{u}}$ in the case where the condition (MFCQ) holds at $\overline{\mathbf{u}}$. This algebraic characterization reflects strong stability in a qualitative (*static*) way.

Here, we need some preparations in notation.

For a given parameter vector $(\mathbf{t},\mathbf{x},\mathbf{w}) \in M_{pr}^{\eta_0}[F,G]$ and a given Kuhn — Tucker point $\overline{\mathbf{u}}$ for $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L,F,G)$ we write $\mu_{\mathbf{0}}$ for the vector $(\ldots,\mu_j,\ldots)$ of Lagrange multipliers μ_j ($j \in J_0(\mathbf{t},\mathbf{x},\overline{\mathbf{u}})$) being ordered in some way. Such a vector $\mu_{\mathbf{0}}$ can be complemented to a multiplier vector μ being defined with the Lagrange multipliers μ_j of Definition 3.2 and with $\mu_j := 0$ for $j \in J \setminus J_0(\mathbf{t},\mathbf{x},\overline{\mathbf{u}})$. We put:

$$\Lambda^{\mathbf{t},\mathbf{x},\mathbf{w}}(\overline{\mathbf{u}}) \;=\; \big\{(\lambda,\mu_{\mathbf{0}}) \mid \; \lambda \in I\!R^n,\ \mu_{\mathbf{0}} \in I\!R^{|J_0(\mathbf{t},\mathbf{x},\overline{\mathbf{u}})|} \ \text{ fulfill } \ \mathrm{KT}_{1,2}^{\mathbf{t},\mathbf{x},\mathbf{w}}\big\}, \qquad (3.8)$$

being a compact polyhedron if (MFCQ) holds at $\overline{\mathbf{u}}$ (cf. [6]), and having the cardinality $|\Lambda^{\mathbf{t},\mathbf{x},\mathbf{w}}(\overline{\mathbf{u}})|$. For each element $(\lambda,\mu_{\mathbf{0}}) \in \Lambda^{\mathbf{t},\mathbf{x},\mathbf{w}}(\overline{\mathbf{u}})$ we define the *Lagrange function* $\mathrm{L}_{[\lambda,\mu_{\mathbf{0}}]}^{\mathbf{t},\mathbf{x},\mathbf{w}} : I\!R^q \to I\!R$:

$$\mathrm{L}_{[\underline{\lambda},\underline{\mu}_0]}^{\mathbf{t},\mathbf{x},\mathbf{w}}(\mathbf{u}) \;:=\; L(\mathbf{t},\mathbf{x},\mathbf{u}) \,-\, \sum_{k=1}^n \lambda_k\,(f_k-\mathbf{w}_k)(\mathbf{t},\mathbf{x},\mathbf{u}) \,-\, \sum_{j\in J_0(\mathbf{t},\mathbf{x},\mathbf{u})} \mu_j\, g_j(\mathbf{t},\mathbf{x},\mathbf{u}). \qquad (3.9)$$

Finally, we put for each $\omega \in I\!R^q$ and each $\widetilde{J} \subseteq J$:

$$W^{\mathbf{t},\mathbf{x}}(\omega, \widetilde{J}) = \{\xi \in I\!R^q |,$$

$$D_{\mathbf{u}}(f_k - \mathbf{w}_k)(\mathbf{t}, \mathbf{x}, \omega)\xi = 0 \ (k \in \{1, \ldots, n\}), D_{\mathbf{u}} g_j(\mathbf{t}, \mathbf{x}, \omega)\xi = 0 \ (j \in \widetilde{J})\}. \tag{3.10}$$

Lemma 3.4. ([23], Corollary 4.3 and Theorem 7.2). *Let a parameter point* $(\mathbf{t}, \mathbf{x}, \mathbf{w}) \in M_{pr}^{\eta_0}[F, G]$ *and a Kuhn — Tucker point* $\overline{\mathbf{u}}$ *be a Kuhn — Tucker point for* $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L, F, G)$ *be given.*

a) *If (LICQ) holds at* $\overline{\mathbf{u}}$, *then* $\overline{\mathbf{u}}$ *is* strongly stable *if and only if the matrix* $D^2_{\mathbf{u}} \mathrm{L}^{\mathbf{t},\mathbf{x},\mathbf{w}}_{[\lambda,\mu_{\mathbf{0}}]}(\overline{\mathbf{u}})$ $(|\Lambda^{\mathbf{t},\mathbf{x},\mathbf{w}}(\overline{\mathbf{u}})| = 1)$ *has nonvanishing determinants with a common sign on the subspaces* $W^{\mathbf{t},\mathbf{x}}(\overline{\mathbf{u}}, \widetilde{J})$ *for all* $\widetilde{J}$ *with* $J_+(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}) \subseteq \widetilde{J} \subseteq J_0(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}})$, *where*

$$J_+(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}) := \{j \in J_0(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}) \mid \mu_j > 0\}. \tag{3.11}$$

b) *If (MFCQ) holds at* $\overline{\mathbf{u}}$ *where, however, (MFCQ) is violated, then* $\overline{\mathbf{u}}$ *is* strongly stable *if and only if for each* $(\lambda, \mu_{\mathbf{0}}) \in \Lambda^{\mathbf{t},\mathbf{x},\mathbf{w}}(\overline{\mathbf{u}})$ *the matrix* $D^2_{\mathbf{u}} \mathrm{L}^{\mathbf{t},\mathbf{x},\mathbf{w}}_{[\lambda,\mu_{\mathbf{0}}]}(\overline{\mathbf{u}})$ *is positive definite on the subspace* $W^{\mathbf{t},\mathbf{x}}(\overline{\mathbf{u}}, J_+(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}))$ *(with the latter index set in the sense of (3.11)).*

Hereby, a quadratic matrix A of type $r_1 \times r_1$ over an $r_2 (\leq r_1)$-dimensional subspace W is understood as the matrix $A|W := B^T A B$ where the columns of B are the members of some basis of W. The algebraic conditions in Lemma 3.4 do not depend on the special choice of this basis. In the notation of the lemma we call the Kuhn — Tucker point $\overline{\mathbf{u}}$ **nondegenerate** [16] if (LICQ) holds at $\mathbf{u}$, the equation $J_+(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}}) = J_0(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}})$ is valid and the Hessian matrix $D^2_{\mathbf{u}} \mathrm{L}^{\mathbf{t},\mathbf{x},\mathbf{w}}_{[\lambda,\mu_{\mathbf{0}}]}(\overline{\mathbf{u}})$ is nonsingular on the *tangent space* $T_{\overline{\mathbf{u}}} = T_{\overline{\mathbf{u}}} M^{\mathbf{t},\mathbf{x}}_{\mathcal{F}}[F - \mathbf{w}, G]$, defined by

$$T_{\overline{\mathbf{u}}} = W^{\mathbf{t},\mathbf{x}}(\overline{\mathbf{u}}, J_0(\mathbf{t}, \mathbf{x}, \overline{\mathbf{u}})). \tag{3.12}$$

Obviously, it follows that *strong stability* is a more *general* concept than nondegeneracy. For example, while a strongly stable Kuhn — Tucker point, at which (MFCQ) holds and which lies on the boundary (relative in the zero-set $M^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}]$ of $F - \mathbf{w}$), may due to a small perturbation shift into the interior, this is impossible in the case of its nondegeneracy.

For more information on strong stability in finitely constrained optimization see [24]. In the *semi-infinite* case [36, 40] where strong stability has the same analytical meaning as in the finite case (Definition 3.3), an algebraic characterization was given by Rückmann ([36], see also *part 2*).

Let us assume that for *each* parameter $(\mathbf{t}, \mathbf{x}, \mathbf{w}) \in M_{pr}^{\eta_0}[F, G]$ of our parametric optimization problem $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L, F, G)$ both *(MFCQ)* and the *strong stability* holds for the feasible set and for all Kuhn — Tucker points, respectively. Moreover, let us think (COMP) to be fulfilled. Then, the set $M_{pr}^{\eta_0}[F, G]$ has also to be compact, and looking at $(\mathbf{t}, \mathbf{x}, \mathbf{w})$ as another perturbational parameter, we may choose the *same* numbers δ, α for all $(\mathbf{t}, \mathbf{x}, \mathbf{w}) \in M_{pr}^{\eta_0}[F, G]$.

Moreover, under the compactness assumption (COMP) and even **generically**, there exist continuous *implicitly* defined so-called **Kuhn — Tucker functions** $u_\vee : M_{pr}^{\eta_0}[F, G] \to I\!R^q$. Their values are Kuhn — Tucker points $\mathbf{u} = u_\vee(\mathbf{t}, \mathbf{x}, \mathbf{w}) \in M^{\mathbf{t},\mathbf{x}}_{\mathcal{F}}[F - \mathbf{w}, G]$ being due to the parameters $(\mathbf{t}, \mathbf{x}, \mathbf{w}) \in M_{pr}^{\eta_0}[F, G]$, such that the following holds. Namely, these functions represent *all* Kuhn — Tucker points of our parametric problems.

That *genericity*, being a condition stronger than density (cf. [13, 18]), may be interpreted as a basic representativity of those problem data $((L,F,G),(\ell,E))$, for which *all* the Kuhn—Tucker functions, which may locally in $M_{pr}^{\eta_0}[F,G]$ be given, are always *globally* (on $M_{pr}^{\eta_0}[F,G]$) extendable, locally isolated (implicit) functions, and, herewith, *well-defined*. By definition, a generic set contains a subset which is the intersection of some sequence of open and dense sets (here: in the space $(C^3 \times C^2, C_S^3 \times C_S^2)$).

Our genericity comes from the openess and density (hence, genericity) of the qualitative conditions (MFCQ) and strong stability (cf., e. g., [20, 24, 40]), and from the genericity of the globality (of the domain) property.

The latter fact can be realized by means of exhausting the path components $\mathcal{K}$ of the compact set $M_{pr}^{\eta_0}[F,G]$ in a *one parametric* way with the help of paths, together with generic *one parametric* nonlinear programming [17, 21], by means of continuity and gluing of functions (cf. [13, 18]). Let us only mention that for these programming problems the appearance of turning points and stopping of some Kuhn—Tucker trajectory (in $I\!R^{q+1}$) is well understood [17, 21], such that we prove genericity with the help of suitable perturbations.

We remember that, hereby, the *implicit definedness* of these functions $u_\vee$ reflects the local uniqueness (isolatedness) and continuous (data) dependence (stability) of all the Kuhn—Tucker points, while the *(global) existence* is (generically) guaranteed with the help of both (COMP) and the qualitative conditions. Therefore, we note that a local minimum $\overline{\mathbf{u}} \in M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F-\mathbf{w},G]$ fulfilling (MFCQ) is a Kuhn—Tucker point (see [13, 21]). More such (perturbational) argumentations on implicitness are given in [23, 24, 40, 42].

In the special case, where $M_{pr}^{\eta_0}[F,G]$ has only *one* path component, and in view of our Assumption (COMP), the number of these Kuhn—Tucker functions can be chosen to be of a *finite* number. We remark that, in the general case of one or more path components, the number of those components itself is finite if $M_{pr}^{\eta_0}[F,G]$ can be represented as a feasible set (in the sense on finitely constrained optimization) fulfilling (LICQ) [9, 16, 40].

Whenever the number of path components is greater than 1, then the number of Kuhn—Tucker functions increases because of combinatorical reasons. This fact will *not* cause problems.

Within each path component $\mathcal{K} \subseteq M_{pr}^{\eta_0}[F,G]$ the *types* of the values $u_\vee(\mathbf{t},\mathbf{x},\mathbf{w})$ of a given *Kuhn — Tucker function* $u_\vee$, namely, *local maximum*, *saddlepoint*, or *local minimum* for the underlying finite problem, does not depend on the special choice of the argument $(\mathbf{t},\mathbf{x},\mathbf{w}) \in \mathcal{K}$ (cf. also [24]). Hence, we could talk about the *type of* $u_\vee|\mathcal{K}$.

The globality condition of the Kuhn—Tucker functions is very strong, but comfortable for the ease of exposition. However, in *part 2* this globality condition will be deleted by means of admitting jumps between locally defined "pieces" of Kuhn—Tucker functions. These jumps will happen at n-dimensional Lipschitzian manifolds in the $(n+1)$-dimensional $(\mathbf{t},\mathbf{x})$-space. Then *genericity* is guaranteed in a basic sense.

Example and concluding remarks 3.5. The dimensional restrictions $m<n<q$ are made in order, under certain constraint qualifications, to guarantee the feasible sets $M_{\mathcal{F}}^{\mathbf{t},\mathbf{x}}[F-\mathbf{w},G]$ from (3.4), but also the ones given in (1.9), (1.10) (and a forthcoming one in part 2), to be *topological manifolds* of a positive dimension. In so far, on the one hand these restrictions, being well known in more topological or parametric optimization [9, 20, 22], reflect our approach in using optimization over $I\!R^n$ and $I\!R^q$, respectively, to analyze $\mathcal{P}(\ell,L,F,E,G)$. On the other hand, for example the following problem given by Maurer [28], has to be excluded. Namely, in

short notation we state

$$(\mathcal{P}_1) \quad \begin{cases} \text{Min } \int_0^1 u^2(t)\,dt, & \text{where} \\ \dot{x}_1 = x_2,\ \dot{x}_2 = u, & x_1(0) = x_2(1) = 0,\ x_2(0) = 1,\ x_2(1) = -1. \end{cases}$$

Here, we have $n = 2,\ q = 1,\ m = 4,\ s = 0,$ hence, $m > n > q,$ and $M^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}] = \{\mathbf{w}_2\}$ (0-dimensional) if $\mathbf{x}_2 = \mathbf{w}_1,$ while $M^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}] = \emptyset$ otherwise. Hence, $M^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}]$ is highly unstable with respect to small perturbations of the parameters $\mathbf{x}, \mathbf{w}$ (for the case $\mathbf{x}_2 = \mathbf{w}_1$). This instability, however, comes from the irregularity of the feasible set $M^{\mathbf{t},\mathbf{x}}[F - \mathbf{w}],$ while the violation of $m < q < n$ can be overcome by means of introducing further (dummy) variables, e. g., by adding in a quadratic way further controls into the integrand. This opportunity may be considered as some unfolding.

Let us therefore, and in order finally to illustrate some meaning of the problems $\mathcal{P}_{\mathcal{F}}^{\mathbf{t},\mathbf{x},\mathbf{w}}(L, F, G)$ for $\mathcal{P}(\ell, L, F, E, G),$ suitably modify and rigorously simplify our example. We study the problem

$$(\mathcal{P}_2) \quad \begin{cases} \text{Min} \int_0^1 (u_1^2(t) + u_2^2(t))\,dt, & \text{where} \\ \dot{x} = x, \quad -1 \le x, u_1, u_2 \le 1. \end{cases}$$

Then, we have, $m = 0 < n = 1 < q = 2,\ F = f_1,\ f(t, x, u) = x$ and, hence (shortly writing $t,\ x,\ w$ for $\mathbf{t},\ \mathbf{x},\ \mathbf{w}$):

$$M_{\mathcal{F}}^{t,x}[F - w, G] \ = \ \begin{cases} [-1, 1]^2, & \text{if} \quad x = w, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$(t, x, w) \in M_{pr}^{\eta_0}[F, G] = [0, 1] \times \{(\tilde{x}, \tilde{w}) \in [-1, 1] \times I\!R \mid |\tilde{x} - \tilde{w}| \le \eta_0\}.$ In this example, $M_{pr}^{\eta_0}[F, G]$ obviously consists of one single path component.

Here, the instability of $M_{\mathcal{F}}^{t,x}[F - w, G]$ with respect to parametric variations of x (or w) reflects that (MFCQ) is violated everywhere, namely because of $D_{\mathbf{u}}^T(F - w) \equiv \mathbf{0}_2\, (\in I\!R^2).$ For theoretical foundations (even in the semi-infinite case) we refer to [22], again, and to our explanations given in part 2. We remark that by introducing a third control and a second state coordinate it is easy completely to guarantee (COMP), i. e., also due to $M[E],$ while the relations $m < n < q$ remain preserved. Therefore, e. g., the equality constraint $x_1^2(0) + x_1^2(1) + x_2^2(0) + x_2^2(1) = 1$ establishes that compactness.

Modifying $(\mathcal{P}_2)$ a bit, we come to

$$(\mathcal{P}_3)\quad \begin{cases} \text{Min} \int_0^1 \, (u_1^2(t) + u_2^2(t))\, dt, \quad \text{where} \\ \\ \dot{x} = x + u_1, \quad -1 \le x, u_1, u_2 \le 1, \end{cases}$$

such that we really get a projective set $M_{pr}^{\eta_0}[F,G]$, which still consists of one component, and the optimization problems

$$(\mathcal{P}_{3,\mathcal{F}}^{t,x,w})\quad \begin{cases} \text{Min } \mathbf{u}_1^2 + \mathbf{u}_2^2 \quad \text{on} \quad M_{\mathcal{F}}^{t,x}[F-w,G], \ \text{ where} \\ \\ M_{\mathcal{F}}^{t,x}[F-w,G] \ = \ \{\, \mathbf{u} \in I\!R^2 \mid \ -1 \le \mathbf{u}_1, \mathbf{u}_2 \le 1, \ \ x + \mathbf{u}_1 = w \,\} \end{cases}$$

$(t,x,w) \in M_{pr}^{\eta_0}[F,G]$); see Figure 4.

It is not hard to realize now that, if and only if $\mathbf{u}_1 \neq -1$ $(\mathbf{u} \in M_{\mathcal{F}}^{t,x}[F-w,G])$, then (MFCQ) holds, and that the (global) minimum $\mathbf{u^0} = (-x+w, 0)$ is *the only Kuhn — Tucker point* $((t,x,w) \in M_{pr}^{\eta_0}[F,G])$. Actually, the (global) maxima $\mathbf{u}^1 = (-x+w,-1)$, $\mathbf{u}^2 = (-x+w,1)$ do not fulfill the Kuhn — Tucker condition $(\mathrm{KT}_{\mathcal{F}\,2}^{t,x,w})$. *Moreover, our* $\mathbf{u}^0$ (where $\mathbf{u}_1^0 = -x+w \neq -1$) *is strongly stable.* Consequently, in view of the characterization theorem from [22] (cf. also our Characterization Theorem $(\mathcal{A})$), if $\mathbf{u}_1 \neq -1$ for all $\mathbf{u} \in M_{\mathcal{F}}^{t,x}[F-w,G]$, then $(\mathcal{P}_{3,\mathcal{F}}^{t,x,w})$ *is structurally stable.*

Let us for all parameters $(t,x,w) \in M_{pr}^{\eta_0}[F,G]$ concentrate on our minima. This gives rise to study the unique Kuhn — Tucker function $u_\vee(t,x,w) = (-x+w,0)$. The unique control solving $(\mathcal{P}_3)$ is, of course, $u^0 \equiv \mathbf{0}_2$, which is also the pointwise limit of $u_\vee(t,x,w)$ for $-x+w \to 0$. The latter approach can be forced by means of the parametric approach $\eta_0 \to 0$. With the choice $w^0 := 0$ and from a uniform viewpoint (with respect to t) we arrive at $x^0 \equiv 0$ which together with u^0 is a solution of $(\mathcal{P}_3)$. However, together with u^0 *all* state variables $x_c^0(t) := ce^t$, $|c|$ being sufficiently small, are solutions of $(\mathcal{P}_3)$, too.

Hence, in particular, here we do not have local uniqueness of the solutions of the **minimum principle**. This principle will be stated in part 2, where it will stronger be incorporated into the structure (and stability) of our composite model than in the final *decomposite* one. In part 2, our example may also be continued.

We resume (and announce for part 2): After our regularity conditions in $\mathbf{u}$, regularity conditions have for completeness also to be imposed with respect to $\mathbf{x}$ and, from the *composite*

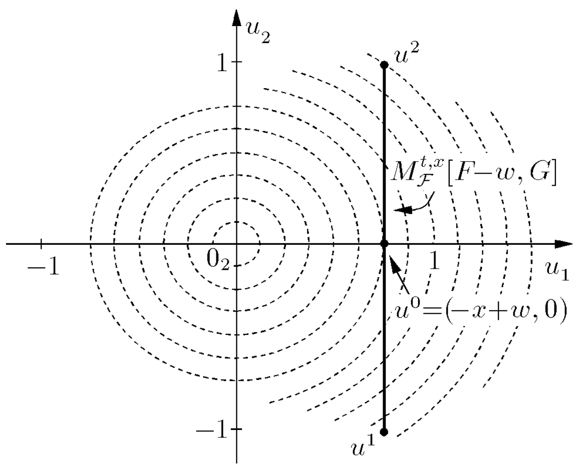

Fig. 4. On the examples $(\mathcal{P}_3)$, $(\mathcal{P}_{3,\mathcal{F}}^{t,x,w})$ of optimal control and optimization, respectively.

viewpoint, with respect to solutions (x^0, u^0) of the minimum principle. Hereby, $\mathbf{x}$ and $\mathbf{u}$ become treated in a *balanced* manner, and for each t (or $\mathbf{t}$) the relaxing parameter $\mathbf{w}$ should pointwise precisely become (re)specified by the derivative $\dot{x}(t)$ of the states $x(t)$. Consequently, the corresponding parameters $(\mathbf{t}, \mathbf{x}, \mathbf{w})$ get some explicit path connectedness. Moreover, the minimum principle gives us also the further opportunity to let the *boundary functions* ℓ and E enter into our structural investigation.

In this way, the unfolding of the composite structure will be continued in part 2. So, in particular, we have to give a structure for each of the Kuhn—Tucker functions, and a corresponding regularity condition which is defined by a certain transversality. The case of $C^0_{\mathrm{pw}2}$-controls will be explained first. Then, jumps are admitted within the general case of $F_{\mathrm{pw}\,2}$-controls.

Those structures and, similarly as in Section 2, the *structures* of the $\mathcal{F}$ problems $\mathcal{P}^{\mathbf{t},\mathbf{x},\mathbf{w}}_{\mathcal{F}}(L, F, G)$ and of further ($\mathcal{F}$ or $\mathcal{GSI}$) optimization problems will be presented. From the static and dynamical viewpoints, the *completeness* of the whole *structure*, a suitable concept of *global stability* and all the possible *irregularities* will be incorporated into the composite structure, too.

Part 2 [43] ends up with an evaluation of the less analytical-causal *decomposite structure*, which treats the different *steps* of its unfolding in a separate way.

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