Comparing value-at-risk and tail conditional expectation in shortfall-constrained portfolio selection

D. Akume
Mathematics Department, University of Buea, Cameroonian
e-mail: d_akume@yahoo.ca

We compare value-at-risk (VaR) and tail conditional expectation (TCE) as risk bounds in determining optimal portfolio strategies, in a Black — Scholes market. Our numerical procedure leads to an approximate solution to the problem, which enables us to verify that, be it tail conditional expectation or value-at-risk, the imposition of the constraint curbs investment in risky assets in much the same way, despite TCE being a coherent risk measure and value-at-risk not being coherent. Our numerical simulation also enables us to confirm that TCE takes a bigger numerical value than VaR to produce the the same limiting effect.

Keywords: financial market, market model, optimal portfolio choice, risky assets, numerical procedure, approximate solution.

Introduction

According to Artzner et al. [5], Tail Conditional Expectation (TCE) is a coherent risk measure[1] whereas value-at-risk (VaR) is not. In a simulation framework, we seek to show that inspite of this assertion, VaR is as good a risk measure as TCE in our Black — Scholes market.

In this paper we conduct our experiment by comparing the dynamic portfolio choice of a trader subject to a risk limit specified on the one hand in terms of TCE, and on the other hand, in terms of VaR. We achieve this by maximizing the agent's utility over wealth throughout the investment horizon. Both VaR and TCE are re-calculated and re-imposed at short intervals of time, throughout the investment horizon. The portfolio is assumed constant over each such short interval (re-evaluation horizon), as is the case in practice.

This paper is a follow-up on two earlier ones, Akume et al. [2, 3]. Here, we show through numerical simulations, by applying an algorithm similar to that in Yiu [17], that by limiting VaR or TCE, investment in risky assets is controlled in practically the same way. Our numerical experiments use two risky assets, as opposed to just one in other existing literature. Moreso, unlike other authors working on the same subject, we simulate both risk measures in a unified framework.

The rest of this paper is structured as follows. In Section 1, we model the financial market and describe the portfolio dynamic. In Section 2 we derive the VaR and TCE formulas for the market model under study, while Section 3 makes precise the optimal control problem to be solved. Section 4 develops the solution of the problem by using the Lagrange technique

[1]TCE is a coherent risk measure for an underlying continuous distribution [16].

to combine the Hamilton — Jacobi — Bellman (HJB) equation and the shortfall constraints. In Section 5, a numerical algorithm is presented to obtain an approximate solution to the constrained problem. Section 6 presents some simulation results with ensuing discussion, while Section 7 concludes the paper.

1. The model

We consider a standard Black — Scholes type market just like Akume et al. [3].

Uncertainty in the financial market is modeled by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, equipped with a filtration that is a non-decreasing family $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ of sub-σ-fields of $\mathcal{F}$

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad \forall \quad 0 \le s < t < \infty.$$

It is assumed throughout this paper that all inequalities as well as equalities hold $\mathbf{P}$-almost surely. Moreover, it is assumed that all stated processes are well defined without giving any regularity conditions ensuring this. The risk-free rate $r = r_t$ of the risk-free asset (bond) S^0 is supposed to evolve according to $dS_t^0 = rS_t^0dt,\ S_0^0 = s^0$. For the risky assets (stocks), for which the prices will be denoted by $S_t = (S_t^1, \dots, S_t^n)$ for some $n \in \mathbb{N}$, the basic evolution model is that of a log-normal diffusion process:

$$\frac{dS_t^i}{S_t^i} = \mu^i dt + \sum_{j=1}^{k} \sigma^{ij} dW_t^j \quad \forall\, t \in [0,T], \quad S_0^i = s^i, \quad i = 1, \dots, n,$$

where, for some $k \in \mathbb{N}$, $k > n$, $W_t = [W_t^1, \dots, W_t^k]^T$, with the symbol $(^T)$ standing for transpose, is a k-dimensional standard Wiener process, i. e., a vector of k independent one-dimensional Wiener processes. The n-vector $\mu = (\mu^1, \dots, \mu^n)^T$, contains the expected instantaneous rates of return and the $n \times k$-matrix $\sigma = \sigma^{ij}$ $(i = 1, \dots, n,\ j = 1, \dots, k)$ measures the instantaneous sensitivities of the risky asset prices with respect to exogenous shocks so that the $(n\times n)$-matrix $\sigma\sigma^T$ contains the variance and covariance rates of instantaneous rates of return. An agent invests according to an investment strategy that can be described by the $(n+1)$-dimensional, $\mathcal{F}_t$-predictable process $x_t = (x_t^0, x_t^1, \dots, x_t^n)$, where x_t^i $(i = 1, \dots, n)$ denotes the number of shares of asset i held in the portfolio at time t $(i = 0$ refers to the bond). The process x describes an investor's portfolio as carried forward through time. The value of the investor's wealth at time t is then

$$V_t^x = x_t^0 S_t^0 + \sum_{i=1}^{n} x_t^i S_t^i,$$

where $x_t^i S_t^i$ represents the amount invested in asset i at time t.

Equivalently, one may consider the vector

$$\theta_t = (\theta_t^1, \dots, \theta_t^n), \quad \theta_t^i = \frac{x_t^i S_t^i}{V_t^x}\ (i = 1, \dots, n),$$

with θ_t^i denoting the fraction of wealth invested in the risky asset i at time t, whereby the remaining fraction $1 - \sum\limits_{j=1}^{n} \theta_t^i$ of the agent's wealth is invested in the risk-free asset. It is assumed that $\theta_t^1, \dots, \theta_t^n$ are admissible and $\mathcal{F}_t$- adapted control processes. That is, θ_t^i is a non-anticipative function that satisfies the condition of integrability $\int_0^T \sum_{1=1}^n (\theta_t^i)^2 dt < \infty$, for an investment time horizon $T < \infty$. The corresponding portfolio value process reads

$$dV_t^\theta = V_t^\theta \left[\left(1 - \sum_{i=1}^n \theta_t^i\right) \frac{dS_t^0}{S_t^0} + \sum_{i=1}^n \theta_t^i \frac{dS_t^i}{S_t^i} \right], \quad V_0^\theta = v_0. \tag{1}$$

To have a better exposition, we adopt a matrix expression: denote $\sigma = [\sigma^{i,j}]$, $\theta_t = [\theta_t^1, \ldots, \theta_t^n]^T$, $\mu = [\mu^1, \ldots, \mu^n]^T$, $1_n = [1, \ldots, 1]^T$ and $W_t = [W_t^1, \ldots, W_t^k]^T$, so that σ is an $n \times k$ matrix, $\mu - r1_n$ and θ_t are n-dimensional column vectors and W_t is a k-dimensional column vector. Hence equation (1) can be rewritten as

$$dV_t^\theta = V_t^\theta \left[\left(r + \theta_t^T(\mu - r1_n)\right) dt + \theta_t^T \sigma_t dW_t \right], \quad V_0^\theta = v_0. \tag{2}$$

2. Risk measures

Here, like in Akume et al. [3], we formally define and compute value-at-risk and Tail Conditional Expectation, the two risk measures of interest.

Definition 1 (value-at-risk). *Given some probability level $\alpha \in (0, 1)$, a time t wealth benchmark Υ_t and horizon Δt, the value-at-risk (VaR_t^α) of time t wealth V_t at the confidence level $1-\alpha$ is given by the smallest number L such that the probability that the loss $G_{t+\Delta t} := \Upsilon_{t+\Delta t} - V_{t+\Delta t}^\theta$ exceeds L is no larger than α*

$$VaR_t^\alpha = \inf \{L \geq 0 : \mathbf{P}(G_{t+\Delta t} \geq L | \mathcal{F}_t) \leq \alpha\} := (Q_t^\alpha)^-,$$

where

$$Q_t^\alpha = \sup \left\{L \in \mathbb{R} : \mathbf{P}((V_{t+\Delta t}^\theta - \Upsilon_{t+\Delta t}) \leq L | \mathcal{F}_t) \leq \alpha\right\}$$

is the quantile of the projected wealth surplus at the horizon $t + \Delta t$ and $x^- = \max[0, -x]$.

Thus, $VaR_t^\alpha = 0$ for $Q_t^\alpha \geq 0$. VaR_t^α is therefore the loss of wealth with respect to a benchmark $\Upsilon_{t+\Delta t}$ at the horizon Δt which could be exceeded only with a small conditional probability α if the current portfolio θ_t were kept unchanged. Typical values for the probability level α are $\alpha = 0.05$ or $\alpha = 0.01$. In market risk management the time horizon Δt is usually one or ten days.

Proposition 1 (computation of value-at-risk). *We have*

$$VaR_t^\alpha = (Q_t^\alpha)^- = \Bigg(V_t^\theta \exp\Bigg[\Phi^{-1}(\alpha) \|\theta_t^T \sigma\| \sqrt{\Delta t} + \\ + \left(\theta_t^T(\mu - r1_n) + r - \frac{1}{2}\|\theta_t^T \sigma\|^2\right) \Delta t\Bigg] - \Upsilon_{t+\Delta t}\Bigg)^-,$$

where $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ denote the normal distribution and the inverse distribution functions respectively, and $\|\cdot\|$ stands for the Euclidean norm.

We refer to [2] for the proof.

TCE is closely related to the VaR concept, but overcomes some of the conceptual deficiencies of VaR (Rockafellar and Uryasev [16]). In particular, it is a coherent risk measure for continuous distributions (Artzner et al. [4]).

Definition 2 (tail conditional expectation). *Consider the loss distribution* $G_{t+\Delta t} := \Upsilon_{t+\Delta t} - V^{\theta}_{t+\Delta t}$ *represented by a continuous*[2] *distribution function* $F_{G_{t+\Delta t}}$ *with*

$$\int_{\mathbb{R}} |G_{t+\Delta t}| dF(G_{t+\Delta t}) < \infty.$$

Then the TCE^{α}_{t} *at confidence level* $1-\alpha$ *is defined as*

$$TCE^{\alpha}_{t} = \mathbb{E}_t \left\{ (\Upsilon_{t+\Delta t} - V^{\theta}_{t+\Delta t}) \geq VaR^{\alpha}_{t} | \mathcal{F}_t \right\},$$

$$TCE^{\alpha}_{t} = \frac{\mathbb{E}_t \left\{ (\Upsilon_{t+\Delta t} - V^{\theta}_{t+\Delta t}) I((\Upsilon_{t+\Delta t} - V^{\theta}_{t+\Delta t}) \geq -Q^{\alpha}_{t}) | \mathcal{F}_t \right\}^{+}}{\alpha},$$

where I(A) is the indicator function of the set A *and* $x^{+} = \max[0, x]$.

In other words, the TCE of wealth V_t at time t is the conditional expected value of the loss exceeding $(Q^{\alpha}_{t})^{-}$. Again, given the log-normal distribution of asset returns, the TCE^{α}_{t} can be explicitly computed as can be seen in the following proposition.

Proposition 2 (computation of tail conditional expectation). *We have*

$$TCE^{\alpha}_{t} = \frac{1}{\alpha} \left(\alpha \Upsilon_{t+\Delta t} - V^{\theta}_{t} \left[\exp\left((\theta^{T}_{t}(\mu - r 1_n) + r)\Delta t \right) \Phi \left(\Phi^{-1}(\alpha) - \|\theta^{T}_{t}\sigma\| \sqrt{\Delta t} \right) \right] \right),$$

where $\Phi(\cdot)$ *and* $\Phi^{-1}(\cdot)$ *denote the normal distribution and the inverse distribution functions.*

We refer to [2] for the proof.

3. Statement of the problem

We seek the optimal asset allocation that maximizes (over all allowable $\{\theta_t\}$) the expected utility of discounted wealth over the entire horizon $[0, T]$, for a risk averse investor who limits his risk by imposing an upper bound on VaR or TCE. We shall determine whether using the one risk measure presents any advantage over the use of the other.

The choice of this problem is motivated by the income drawdown option in defined contribution pension schemes. Such an option allows the member who retires not to convert the accumulated capital into annuity immediately at retirement, but to defer the purchase of the annuity until a certain point in time after retirement. The period of time can be limited to time T. Usually, freedom is given for a fixed number of years after retirement and at a certain age the annuity is bought.

Here, we consider the income drawdown option and investigate, by means of stochastic optimal control techniques, what should be the optimal investment and consumption allocation of the fund after retirement until the purchase of the annuity. The reason the pensioner chooses the drawdown option is the hope of being able to invest the accumulated capital at retirement and increase its value in order to buy a better annuity in the future than the one he otherwise could have bought at retirement.

Thus, our objective is to maximize the expected utility of wealth from retirement until the interruption of the income drawdown option. Therefore, in mathematical terms the final optimal control problem with constraint is

---

[2]Under the Black—Scholes model (μ, σ constant) and for fixed θ_t, the conditional distribution of $G_{t+\Delta t}$ given $\mathcal{F}_t$ is continuous (since it is lognormal).

$$\max_{\{\theta\}\in A(v)} \mathbb{E}_{0,V_0}\left\{\int_0^T e^{-\rho s}U^1(V_s,s)ds + e^{-\rho T}U^2(V_T,T)\right\}, \tag{3}$$

subject to the wealth dynamic

$$dV_t^\theta = \left[V_t^\theta(\theta_t^T(\mu - r1_n) + r)\right]dt + V_t^\theta \theta_t^T \sigma dW_t, \quad V_0^\theta = v$$

and, on the one hand, the TCE constraint

$$TCE_t^\alpha \le \varepsilon(v,t), \quad \forall\, t \in [0, T - \Delta t), \tag{4}$$

where for fixed $\Delta t > 0$

$$\begin{aligned} TCE_t^\alpha = TCE_t^\alpha(V_t^\theta, \theta_t) = \frac{1}{\alpha}\left(\alpha \Upsilon_{t+\Delta t} - V_t^\theta\left[\exp\left((\theta_t^T(\mu - r1_n) + r)\Delta t\right) \times \right.\right. \\ \left.\left. \times\, \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t^T\sigma\|\sqrt{\Delta t}\right)\right]\right) \end{aligned} \tag{5}$$

or, on the other hand, the VaR constraint

$$VaR_t^\alpha \le \varepsilon(v,t), \quad \forall\, t \in [0, T - \Delta t), \tag{6}$$

where for fixed $\Delta t > 0$

$$\begin{aligned} VaR_t^\alpha = VaR_t^\alpha(V_t^\theta, \theta_t) = \left(V_t^\theta \exp\left[\Phi^{-1}(\alpha)\|\theta_t^T\sigma\|\sqrt{\Delta t} + \right.\right. \\ \left.\left. + \left(\theta_t^T(\mu - r1_n) + r - \frac{1}{2}\|\theta_t^T\sigma\|^2\right)\Delta t\right] - \Upsilon_{t+\Delta t}\right)^-, \end{aligned}$$

for all $t \in [0, T)$, where $\mathbb{E}_{t,v}$ denotes the expectation operator at time t, given $V_t^\theta = v$ (and given the chosen investment strategies), U^1 and U^2 are twice differentiable, concave utility functions, $\varepsilon(v,t)$ is an upper bound on TCE and $\rho > 0$ is the rate at which wealth is discounted. Take note that we give room for the running utility function to differ (be weighted differently) from the terminal utility function. Now, we let $U^1(x) = U^2(x) = U(x) = Kx - (1-K)(x-\psi)^2$ to denote a quadratic utility which privileges with a weight $K \in (0,1)$, a large wealth whereas it penalizes with a weight $(1-K)$ the square of the spread between the current wealth, x and a target one, denoted ψ.

Noteworthy is the fact that despite exhibiting increasing absolute risk aversion, quadratic utilities are still widely used in the literature (Haberman et al. [8]), for reasons of tractability.

4. Optimality conditions

Just like in Akume et al. [2], we adopt a dynamic programming approach to solve the HJB equation associated with the utility maximization problem (3). Following Fleming & Rishel [7], the corresponding HJB equation is given by

$$J_t(v,t) - \rho J(v,t) + U(v) + \sup_{\theta_t}\left\{D^{\theta_t}J(v,t)\right\} = 0, \tag{7}$$

where $t \in [0,T]$ is the horizon, $V_t^\theta = v$ is any admissible state and $D^{\theta_t}J(v,t)$ is

$$D^{\theta_t}J(v,t) = J_v(v,t)\left(v\left[\theta_t^T(\mu - r1_n) + r\right]\right) + \frac{1}{2}v^2\|\theta_t^T\sigma\|^2 J_{vv}(v,t),$$

subject to the terminal condition $J(v,T) = U(v)$, and where J, the value function is given by

$$J(v,t) = \sup_{\{\theta\}\in A(v)} \mathbb{E}_{t,V_t}\left\{\int_t^T e^{-\rho(s-t)}U(V_s,s)ds + e^{-\rho(T-t)}U(V_T,T)\right\},$$

with subscripts on J denoting partial derivatives and $V_t^\theta = v$, the wealth realization at time t.

4.1. First-order conditions for the TCE constraint

In solving the HJB equation (7), the static optimization problem

$$\max_{\theta_t}\left\{D^{\theta_t}J(v,t)\right\},$$

subject to the TCE constraint (4) can be tackled separately to reduce the HJB equation (7) to a nonlinear partial differential equation of J only. We introduce the Lagrange function $\mathcal{L}(\theta,\lambda) = \mathcal{L}(\theta(v,t),\lambda(v,t))$ as

$$\mathcal{L}(\theta,\lambda) = J_v(v,t)\left(v\left[\theta^T(\mu - r1_n) + r\right]\right) + \frac{1}{2}v^2\theta^T\sigma\sigma^T\theta J_{vv}(v,t) + \\ + U(v) - \lambda(v,t)\left(\alpha TCE_t^\alpha(v,\theta) - \varepsilon_1\right), \quad (8)$$

where λ is the Lagrange multiplier, $\varepsilon_1 = \varepsilon\cdot\alpha$ and TCE_t^α is given in (3). The first-order necessary conditions with respect to θ and λ respectively of the static optimization problem (4.1) are given by

$$\nabla_\theta\mathcal{L} = vJ_v(\mu - r1_n) + \frac{1}{2}J_{vv}v^2\sigma\sigma^T\theta + \\ + \lambda v\Big[(\mu - r1_n)\Delta t\exp\left((\theta^T(\mu - r1_n) + r)\Delta t\right)\cdot\Phi\left(\Phi^{-1}(\alpha) - \|\theta^T\sigma\|\sqrt{\Delta t}\right) - \\ - \exp\left((\theta^T(\mu - r1_n) + r)\Delta t\right)\cdot\frac{\sqrt{\Delta t}}{2}\frac{\sigma\sigma^T\theta}{\|\theta^T\sigma\|}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(\Phi^{-1}(\alpha) - \|\theta^T\sigma\|\sqrt{\Delta t})^2\right)\Big] = 0,$$

and

$$\frac{\partial\mathcal{L}}{\partial\lambda} = H(v,t) = -\alpha\Upsilon_{t+\Delta t} + v\Big[\exp\left((\theta^T(\mu - r1_n) + r)\Delta t\right)\times \\ \times\Phi\left(\Phi^{-1}(\alpha) - \|\theta^T\sigma\|\sqrt{\Delta t}\right)\Big] + \varepsilon_1 = 0,$$

while the complimentary slackness condition is given as

$$\lambda(v,t)H(v,t) = 0 \quad\text{and}\quad \lambda(v,t)\geq 0.$$

4.2. First-order conditions for the VaR constraint

In analogy to Section 4.1 above, in solving the HJB equation (7), the static optimization problem

$$\max_{\theta_t} \left\{ D^{\theta_t} J(v,t) \right\},$$

subject to the VaR constraint (6) can be tackled separately to reduce the HJB equation (7) to a nonlinear partial differential equation of J only. We introduce the Lagrange function $\mathcal{L}\left(\theta,\lambda\right) = \mathcal{L}\left(\theta(v,t),\lambda(v,t)\right)$ as

$$\begin{aligned} \mathcal{L}\left(\theta,\lambda\right) &= J_v(v,t)\left(v\left[\theta_t^T(\mu-r)+r\right]\right)+ \\ &+\frac{1}{2}v^2\|\theta_t^T\sigma\|^2 J_{vv}(v,t)+U(v)-\lambda(v,t)\left(-v\exp\left[\Phi^{-1}(\alpha)\|\theta_t^T\sigma\|\sqrt{\Delta t}+\right.\right. \\ &\left.\left.+\left(\theta_t^T(\mu-r)+r-\frac{1}{2}\|\theta_t^T\sigma\|^2\right)\Delta t\right]+\Upsilon_{t+\Delta t}-\varepsilon(v,t)\right), \end{aligned} \tag{9}$$

and the first-order necessary conditions with respect to θ and λ respectively of the static optimization problem (4.2) are given by

$$\begin{aligned} \nabla_\theta \mathcal{L} &= vJ_v(\mu-r)+\frac{1}{2}J_{vv}v^2\sigma\sigma^T\theta_t+\lambda(v,t)v\left[\Phi^{-1}(\alpha)\frac{1}{2}\frac{\sigma\sigma^T\theta_t}{\|\theta_t^T\sigma\|}\sqrt{\Delta t}+\right. \\ &\left.+(\mu-r)\Delta t-\frac{1}{2}\sigma\sigma^T\theta_t\Delta t\right]\cdot\left(\exp\left[\Phi^{-1}(\alpha)\|\theta_t^T\sigma\|\sqrt{\Delta t}+\right.\right. \\ &\left.\left.+\left(\theta_t^T(\mu-r)+r-\frac{1}{2}\|\theta_t^T\sigma\|^2\right)\Delta t\right]\right)=0, \\ \frac{\partial\mathcal{L}}{\partial\lambda} &= H(v,t) = v\exp\left[\Phi^{-1}(\alpha)\|\theta_t^T\sigma\|\sqrt{\Delta t}+\right. \\ &\left.+\left(\theta_t^T(\mu-r)+r-\frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t\right]-\Upsilon_{t+\Delta t}+\varepsilon(v,t)=0, \end{aligned}$$

while the complimentary slackness condition is given as

$$\lambda(v,t)H(v,t)=0 \quad \text{and} \quad \lambda(v,t)\geq 0.$$

Simultaneous solution of the first-order conditions of either problem yields the optimal solutions θ^{opt} and λ^{opt}. Substituting these into (7) gives the partial differential equation

$$\begin{aligned} &-\rho J(v,t)+J_t(v,t)+J_v(v,t)\left(v[(\theta^{\mathrm{opt}}(v,t))^T(\mu-r1_n)+r]\right)+ \\ &+Kv-(1-K)(v-\psi)^2+\frac{1}{2}J_{vv}(v,t)v^2(\theta^{\mathrm{opt}}(v,t))^T\sigma\sigma^T(\theta^{\mathrm{opt}}(v,t))=0, \end{aligned} \tag{10}$$

with terminal condition

$$J(v,T)=Kv-(1-K)(v-\psi)^2,$$

which can then be solved for the optimal value function $J^{\mathrm{opt}}(v,t)$.

Due to the non-linearity in θ^{opt}, the first-order conditions together with the HJB equation are a non-linear system so the differential equation (4.2) has no analytic solution and numerical methods such as Newton's method or Sequential Quadratic Programming (SQP) (see, e. g., Nocedal & Wright [15]) are required to solve for $\theta^{\mathrm{opt}}(v,t)$, $\lambda^{\mathrm{opt}}(v,t)$ and $J^{\mathrm{opt}}(v,t)$ iteratively.

5. Numerical solution

Again as in Akume et al. [2] we use an iterative algorithm similar to that of Yiu [17] which yields a $\mathcal{C}^{2,1}$ approximation $\widehat{J}$ of the exact solution J, while $\widehat{\theta}_t$ is the investment strategy related to $\widehat{J}$. When the optimal solution strictly satisfies the constraint[3], the Lagrange multiplier $\lambda(v,t)$ is zero. If the constraint is active, the multiplier is positive. First, we divide the domain of resolution into a grid of $n_v \times n_t$ mesh points. Iterations are indexed by k.

1. For each point (t,v), with $t \in \{0, \Delta t, \ldots, n_t\Delta t\}$, $v \in \{0, \Delta v, \ldots, n_v \Delta v\}$, we compute the value function $\widehat{J}^{k=0} = J(v,t)$ and the optimal strategy θ_t^{opt} of the unconstrained problem. All Lagrange multipliers are set to zero, $\lambda_{t,v}^{k=0} = 0$. This solution is the starting point of the algorithm.

2. For all points of the grid, the constraint is checked. If the constraint is not active ($TCE_t^\alpha < \varepsilon$), the multiplier is zero $\lambda_{t,v}^{k+1} = 0$ and θ_t^{k+1} is the solution of a similar equation to that of the unconstrained case

$$\lambda_{t,v}^{k+1} = 0, \quad \theta_t^{k+1} = -\frac{\widehat{J}_v}{v\widehat{J}_{vv}}(\mu - r1_n)(\sigma^T\sigma)^{-1}.$$

If the TCE_t^α constraint is active ($TCE_t^\alpha \geq \varepsilon$), we solve a nonlinear system in $\lambda_{t,v}^{k+1}$ and $\widehat{\theta}_t^{j+1}$. This nonlinear system is composed of the first-order necessary conditions of the static optimization problem (4.1). That system is numerically solved by the sequential quadratic programming method (Nocedal and Wright [15]).

3. The last stage consists in the calculation of the value function $\widehat{J}^{k+1}$ according to the investment strategy $\widehat{\theta}_t^{k+1}$ as detailed in Section 7 in Akume et al. [3].

4. Return to step 2 with $k = k+1$ until the error at time t from wealth level v, $\epsilon_{t,v}$, satisfies $|\epsilon_{t,v}| < \delta$ with some small $\delta > 0$, where

$$\epsilon_{t,v} = \hat{J}_t - \rho\hat{J}(v,t) + \hat{J}_v\left(v[(\hat{\theta}_t^{\text{opt}})^T(\mu - r1_n) + r]\right) + \frac{1}{2}v^2\|(\hat{\theta}_t^{\ \text{opt}})^T\sigma\|^2\hat{J}_{vv} + U(v).$$

6. Simulation results and discussion

We have, in a MATLAB program, implemented the above algorithm to illustrate the optimal portfolios of the preceding section with examples. We assume that $n = 2$. That is, the market is composed of two risky stocks and a risk-free bond. Table shows the parameters for the portfolio optimization problem and the underlying Black — Scholes model of the financial market. We achieve convergence in 300 seconds after three iterations.

We consider the constraint (TCE or VaR) of the wealth surplus ($V_{t+\Delta t} - \Upsilon_{t+\Delta t}$) with respect to the benchmark $\Upsilon_{t+\Delta t}$ such that it satisfies

$$TCE_t^\alpha(V_{t+\Delta t} - \Upsilon_{t+\Delta t}) \leq \varepsilon(V_t, t) = 0.1 \quad \text{or} \quad VaR_t^\alpha(V_{t+\Delta t} - \Upsilon_{t+\Delta t}) \leq \varepsilon(V_t, t) = 0.05.$$

We obtain both bounds above after several simulations. This confirms the fact that TCE takes a bigger numerical value than VaR to produce the the same limiting effect. The constraint is then re-evaluated at each discrete time step (constraint horizon) Δt and kept

[3]This refers to either VaR (expression (6)) or TCE (expression (4)), depending on the risk measure being applied. However, for ease of exposition, we shall use just TCE to describe the algorithm, bearing in mind that it can always be replaced with VaR if need be.

Author's parameters for portfolio optimization problem

Parameter	Value
Stock (S^1)	$\mu = 4\,\%$, $\sigma^{11} = 5\,\%$, $\sigma^{12} = 5\,\%$
Stock (S^2)	$\mu = 6\,\%$, $\sigma^{21} = 5\,\%$, $\sigma^{22} = 20\,\%$
Bond (S^0)	$r = 3\,\%$
Investment horizon	$t \in [0, 1]$
State of wealth	$v \in [0, 20]$
Shortfall probability	$\alpha = 1\,\%$
Value-at-risk horizon	$\Delta t = 1/48 \approx 7$ days
No. of wealth mesh points	$N_v = 81$
Mesh size for wealth	$\Delta v = 20/80 = 0.25$
Quadratic utility function	$U(v) = Kv - (1-K)(v-\psi)^2$ $K = 0.6$, $\psi = 5$
Discounting factor	$\rho = 0.3$

below the upper bound $\varepsilon(V_t, t)$, by making use of conditioning information. Figure 1 plots, in the right panel, the investment in risky assets with TCE constraint (red), with VaR constraint (green), without constraint (black) against the possible wealth realization at different times. Here, the shortfall benchmark is taken to be the conditional expected wealth $\Upsilon_{t+\Delta t} = \mathbb{E}_t\{V_{t+\Delta t}\} = V_t \exp\left[(\theta_t'(\mu - r) + r)\,\Delta t\right]$, at each constraint horizon before the terminal date T.

We observe that as the wealth level increases, the amount of wealth invested in stocks diminishes. This results from the "increasing absolute risk aversion" that characterizes the quadratic nature of the utility function. Furthermore, it can be observed that when the wealth is smaller than the utility target $\psi = 5$ of the quadratic utility function, the optimal solution consists of increasing the position of stocks. The exposure to the market is however limited by the constraint which bounds the amount invested in stocks. On the other hand, as wealth increases more than ψ, the position in stocks is reduced, but again limited by the constraint.

This, nonetheless, is a rather counter-intuitive investment strategy, but directly results the quadratic nature of the utility function, whereby utility rises up to the satiation level ψ and falls thereafter. We would expect that as the wealth level falls we might shift into lower-risk assets to protect our position. The strategy we have found does the opposite. The reason for this is because of the quadratic form of the objective function. This, in a sense, defines an ideal wealth level ψ. If wealth is below this, then we invest in high-return, high-risk assets to increase the chance of getting quickly back to the ideal level. Conversely if the wealth level is too high then we are prepared to invest in an inefficient low-return investment strategy in order to get back to the ideal level. Owing to this shortcoming of quadratic utility, one could instead consider applying power utility with its more intuitive constant relative risk aversion property (see, e. g., Akume et al. [2]).

Like we noted at the end of Section 3, quadratic utilities are still widely used in the literature despite exhibiting increasing absolute risk aversion (Haberman, Sung [8]).

The left panel in Figure 1 depicts the value function, indicating that for the constrained problem, it is identical with that of the unconstrained problem when the Lagrange multipliers are null (the optimal constrained portfolio follows the unconstrained one), whereas it is inferior when the constraint is active (allocation to risky asset is controlled).

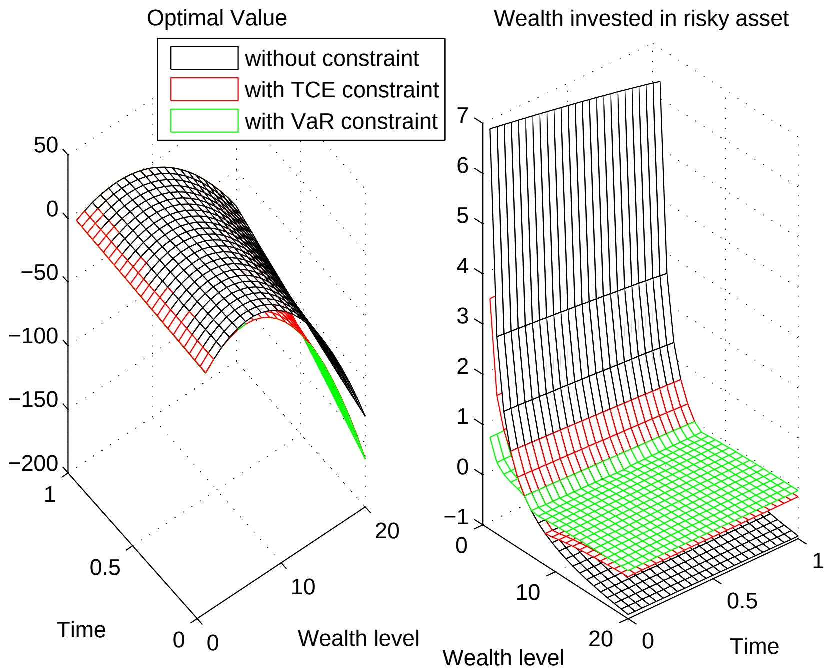

Fig. 1. Effect of constraint on the proportion of risky investment

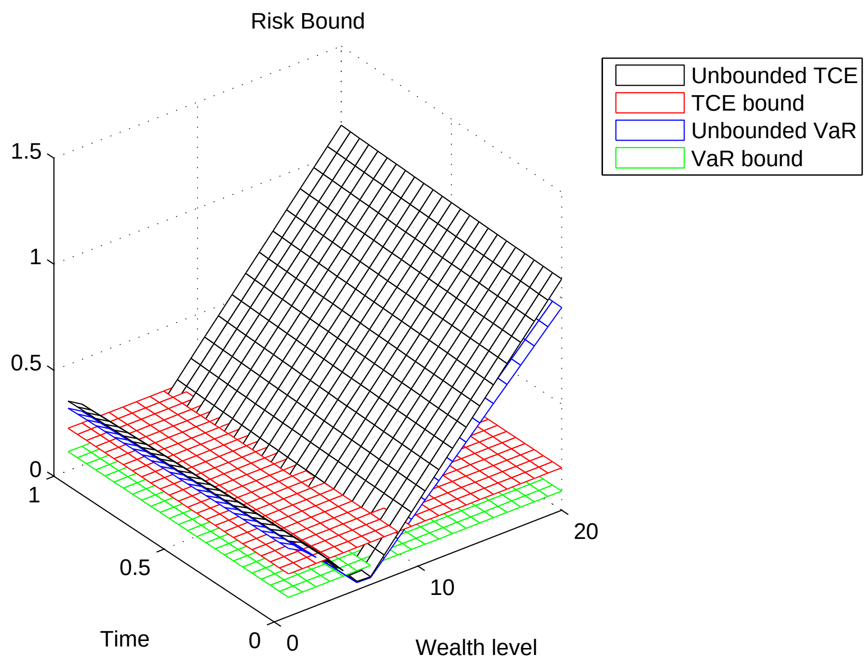

Fig. 2. Constraints and bounds, plotted against wealth at various times of the investment horizon

This explanation is also evidenced by Figure 2 which displays the constraints — TCE (black), VaR (blue) and bounds — VaR bound (green), TCE bound (red) in wealth/time space. The constrained and unconstrained strategies coincide as long as the risk measure lies below the bound. As expected, TCE (black) lies above VaR (blue), as it is by definition, conditional expectation of the loss exceeding VaR.

7. Concluding remarks

Using a quadratic utility function, we have investigated how a bound imposed on TCE or VaR affects the optimal portfolio choice. In so doing, we have taken for wealth benchmark — conditional expected wealth, whereby the constraint was re-evaluated at short intervals along the investment horizon. We deduce from our observations that the constraint controls risky investment.

The value function of the constrained problem is identical to that of the unconstrained one when the Lagrange multipliers are null (the optimal constrained portfolio follows the unconstrained one), whereas it is inferior when the constraint is active (allocation to risky asset is controlled).

On the whole we observe that similar to VaR, the TCE bound limits risky investment as well, albeit on a different scale.

Our simulation results therefore suggest that in a Black — Scholes market, where wealth is lognormally distributed, TCE is similar to VaR in controlling investment in risky assets despite the fact that TCE is coherent and VaR is not.

References

[1] Acerbi C., Tasche D. On the coherence of expected shortfall // J. of Banking & Finance. 2002. Vol. 26, No. 7. P. 1487–1503.

[2] Akume D., Luderer B., Wunderlich R. Optimal dynamic portfolios under a tail conditional expectation constraint // Studia Univ. Babes-Bolyai Math. 2008. Vol. LIII, No. 4. P. 3–21.

[3] Akume D., Luderer B., Wunderlich R. Optimal portfolio strategies under a shortfall constraint // J. of the South African Operat. Res. Soc. (ORION). 2009. Vol. 25, No. 1. P. 31–44.

[4] Artzner P., Delbaen F., Eber J.M., Heath D. Thinking coherently // Risk. 1997. Vol. 10. P. 68–71.

[5] Artzner P., Delbaen F., Eber J.M., Heath D. Coherent measures of risk // Math. Finance. 1999. Vol. 9, No. 3. P. 203–228.

[6] Cuoco D., He H., Isaenko S. Optimal dynamic trading strategies with risk limits // Operat. Res. 2008. Vol. 56, No. 2. P. 358–368.

[7] Fleming W.H., Rishel R.W. Deterministic and Stochastic Optimal Control. Berlin: Springer-Verlag, 1975.

[8] Haberman S., Sung J.H. Dynamic approaches to pension funding // Insurance Math. and Econom. 1994. Vol. 15. P. 151–162.

[9] He H., Pearson N. Consumption and portfolio policies with incomplete markets and short-sale constraints: The finite dimensional case // Math. Finance. 1991. Vol. 1. P. 1–10.

[10] Isaacson E., Keller H.B. Analysis of Numerical Methods. New York: Dover Publ., 1994.

[11] Jorion P. Value at Risk: The New Benchmark for Controlling Derivative Risk. McGraw-Hill Publ., 1997.

[12] Korn R. Optimal Portfolios. Singapore: World Sci., 1997.

[13] Markowitz H.M. Portfolio Selection // The J. Finance. 1952. Vol. 7, No. 1. P. 77–91.

[14] McNeil A.J., Frey R., Embrechts P. Quantitative Risk Management: Concepts, Techniques and Tools. Princeton Univ. Press, 2006.

[15] Nocedal J., Wright S. Martingale Methods in Financial Modeling. Heidelberg: Springer-Verlag, 1999.

[16] Rockafellar R.T., Uryasev S. Conditional value-at-risk for general loss distributions // J. of Banking and Finance. 2002. Vol. 26, No. 7. P. 1443–1471.

[17] Yiu K.F.C. Optimal portfolios under a value at risk constraint // J. of Econom. Dynamics and Control. 2004. Vol. 28. P. 1317–1334.

Received for publication 24 November 2010